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AN EXTENSION OF BESICOVITCH'S THEOREM TO THE CLASS OF WEAKLY ALMOST PERIODIC AND PSEUDO ALMOST PERIODIC FUNCTIONS

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Abstract: In this paper we are concerned with the Eberlein's weakly almost periodic functions. First, we shall use The Ryll-Nardzewski fixed point theorem to prove that $\mathcal{M}\{f(t)\}$, the generalized mean in time, of a weakly almost periodic function f is the unique common fixed point of the operators of translation. Next, we give a generalization of a Besicovitch's theorem for the class of weakly almost periodic functions and the class of pseudo almost periodic functions. Next, we apply our result for the study of some abstract differential equation.

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1. Introduction

Let H be a Hilbert space. A function $f \in C(\mathbb{R}, H)$ is almost periodic if and only if the orbit $\tau(f) = \{\tau_r(f) = f(\cdot + r),\ r \in \mathbb{R}\}$ of f is relatively compact

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in $C(\mathbb{R}, H)$. If we require the orbit is relatively weakly compact then we will get the concept of weakly almost periodicity another generalization of almost periodicity. This type of functions was first introduced by W.F. Eberlein [14] in 1949. This category of functions is natural to define, since it permits to obtain a more adequate description of the physical phenomena as turbulence, noise.... It is well known that things will be much more complicated if one changes compactness to weak compactness. The complication lies in the obscure structure of the weak topology and the fact that the operators, although equicontinuous in the norm topology, are not equicontinuous in the weak topology.

By studying the almost periodic functions, John Von Newmann showed that closed convex hull of the translators of f contains only one single constant function: that constant function is considered as the average of f

$$\mathcal{M}\{f\} = \mathcal{M}\{f(t)\}_t = \left(\lim_{T \longrightarrow +\infty} \frac{1}{2T} \int_{-T}^{+T} f(t)dt \right).$$

The techniques used by Von Newmann are related to the elementary analysis. Some years later, Loomis [16] gives another proof of this result using the techniques of harmonic analysis.

By another way, Blot [5] followed Smart's Step which consists in making apparent the generalized average of an almost periodic function f as a common fixed point of a family of operators on a compact convex set. The methods of Blot and Smart are based on the Markov-Kakutani theorem [18] which assures the existence (but not the uniqueness) of the constant function.

Our approach is similar to Blot's, except that we're going to work on a bigger spaces, that to say $WAP(\mathbb{R}, H)$ and $PAP(\mathbb{R}, H)$. Consequently the Markov-Kakutani theorem can't be applied to our situation, which leads us to use the Ryll-Nardzewski theorem [18]. On the other hand, in 1963 Besicovitch proves that for almost periodic function f and for each non negative real T there exists a unique $T-$periodic strongly continuous function f^T such that

$$\lim_{\nu \to +\infty} \left\| \frac{1}{\nu} \sum_{k=0}^{\nu-1} f(\cdot + kT) - f^T \right\|_\infty = 0.$$

Another proof and some refinements of the above result are given in [5]. In this paper, we give a counterpart of the besicovitch's theorem for the class of weakly almost periodic functions and for the space of pseudo almost periodic functions. As an application, we study the existence of periodic solutions of a class of differential equations. In fact, recently Diagana [11] discussed the abstract differential equations of the form:

$$\frac{d}{ds}w(s) = Aw(s) + Bw(s) + p(s) \qquad (*)$$

where A,B are densely defined closed linear operators acting in a the Hilbert space H, and $p : \mathbb{R} \longrightarrow H$ is a H valued pseudo almost periodic function. Using the theory of the invariant subspaces for unbounded linear operators, he showed that every bounded solution to $(*)$ is a pseudo almost periodic function. Motivated by our result we will prove that if the equation $(*)$ possess a bounded solution then it admits a periodic solution.

This paper is organized as follows: In Section 2, we start to collect some basic tools needed for the investigation of $AP(\mathbb{R}, H)$, $WAP(\mathbb{R}, H)$ and $PAP(\mathbb{R}, H)$ spaces. We also give some interesting examples of weakly almost periodic functions.

In Section 3, we show the main results of this paper. First, by using suitable fixed point theorem, we prove that generalized average of a weakly almost periodic function f as a common fixed point of a family of operators on a compact convex set. Next, we study the the Besicovitch theorem for the class of weakly almost periodic functions and for the space of the pseudo almost periodic functions respectively. Finally, in Section 4 we apply our main result to study the periodic solutions of some abstract differential equation. It should be mentioned that the notion of weakly almost periodic functions dealt with is Eberlein's. It is different from the one of Amerio and Prousse [1].

2. Preliminaries: The Functions Spaces

In this section we fix some of the notations used throughout this paper, recall some properties and Fourier Analysis of the spaces which we use in this paper. The proofs and further properties can be found in [14]. Throughout the paper H will denote an arbitrary complex Hilbert space equipped with the norm $\|\cdot\|$ and H^* the topological dual space of H.

Let $BC(\mathbb{R}, H)$ denote the set of bounded continued functions from $\mathbb{R}$ to H. Note that $(BC(\mathbb{R}, H), \|.\|_\infty)$ is a Banach space where $\|.\|_\infty$ denotes the sup norm

$$\|f\|_\infty := \sup_{t\in\mathbb{R}} \|f(t)\|.$$

For $x \in H$ and $\mu \in H^*$, $\prec \mu, x \succ$ will denote the value of μ in x. Here $\prec .,. \succ$ denotes duality $\prec H^*, H \succ$.

A function $f : \mathbb{R} \longrightarrow H$ is called *periodic* if there exists a real number s such that $f(t+s) = f(t)$ for every $t \in \mathbb{R}$. A real number $\delta \in \mathbb{R} \setminus \{0\}$ is called an $\varepsilon-$period for $f : \mathbb{R} \longrightarrow H$ if

$$\|f(\cdot + \delta) - f(\cdot)\|_\infty \leq \varepsilon.$$

The function f is called *almost periodic* if, for every $\varepsilon > 0$, the set of all $\varepsilon-$periods is relatively dense in $\mathbb{R}$. In other words, an almost periodic function f must satisfy the following property:

$$\forall \varepsilon > 0, \exists l_\epsilon > 0, \forall \alpha \in \mathbb{R}, \exists \delta \in [\alpha, \alpha + l_\varepsilon[\,, \|f(\cdot + \delta) - f(\cdot)\|_\infty \leq \epsilon.$$

Consequently, a Bohr almost periodic function is a continuous function which possesses very much almost periods. When the fuction $f \in BC(\mathbb{R}, H)$, the $r-$translate of f is defined by $\tau_r(f)$ and

$$\tau(f) = \{\tau_r(f) = f(\cdot + r),\ r \in \mathbb{R}\} \subset BC(\mathbb{R}, H)$$

will denote the set of all translates of f.

S. Bochner [6] characterized continuous almost periodic functions defined on $\mathbb{R}$:

Theorem 1. *Let $f : \mathbb{R} \longrightarrow H$ be a continuous function. For f to be almost periodic if its necessary and sufficient that $\tau(f)$ is relatively compact in $BC(\mathbb{R}, H)$.*

It is well known (see [4], [6]) that f is almost periodic if and only if for every sequence of real numbers $(s'_n)_{n\in\mathbb{N}}$ there exists a subsequence $(s_n)_{n\in\mathbb{N}}$ such that $(f(t+s_n))_{n\in\mathbb{N}}$ is uniformly convergent in $t \in \mathbb{R}$. On the other hand, since weak compactness is equivalent to sequential weak compactness in a Hilbert space we will say:

Definition 2. A function $f : \mathbb{R} \longrightarrow H$ is said to be weakly almost periodic (w.a.p.) if for every sequence of real numbers $(s'_n)_{n\in\mathbb{N}}$ there exists a subsequence $(s_n)_{n\in\mathbb{N}}$ such that $(f(t+s_n))_{n\in\mathbb{N}}$ is convergent in the weak sense; uniformly in $t \in \mathbb{R}$. In other words, for every $\mu \in H^*$; the sequence $(\prec \mu, f(t+s_n) \succ)_n$ is uniformly convergent in $t \in \mathbb{R}$.

Denote by $AP(\mathbb{R}, H)$ and $WAP(\mathbb{R}, H)$ the collection of almost periodic function from $\mathbb{R}$ into H and the collection of weakly almost periodic function from $\mathbb{R}$ into H, respectively. For $A \subset H$ we denote $\overline{A}$ the closure of A for the strong topology and $\overline{A}^w$ the closure of A for the weak topology. Roughly speaking we have

$$f \in WAP(\mathbb{R}, H) \Longleftrightarrow \left\{ f \in BC(\mathbb{R}, H) / \ \overline{\tau(f)}^{\,w} \text{ is compact} \right\}$$

$$f \in AP(\mathbb{R},H) \Longleftrightarrow \left\{ f \in BC(\mathbb{R},H) / \ \overline{\tau(f)} \text{ is compact} \right\}.$$

Among other things, weakly almost periodic functions satisfy the following properties.

Theorem 3. (see [14], [20]) *One has the following properties:*

1. $WAP(\mathbb{R},H)$ *is a Banach space for the norm* $\|\cdot\|_\infty$ *which contains the constant functions.*

2. $WAP(\mathbb{R},H)$ *is an algebra for the standard inner product.*

3. *A weakly almost periodic function is uniformly continuous.*

4. *If* $f \in WAP(\mathbb{R},H)$,*then*

$$\mathcal{M}\{f\} = \mathcal{M}\{f(t)\}_t = \left(\lim_{T \longrightarrow +\infty} \frac{1}{2T} \int_{-T}^{+T} f(t)dt \right) \in H$$

5. *If* $f \in WAP(\mathbb{R},H)$, *then* $\forall r \in \mathbb{R}, \ \tau_r(f) \in WAP(\mathbb{R},H)$.

Remark 4. For $f \in WAP(\mathbb{R},H)$ and $\lambda \in \mathbb{R}$, we asset formally the Bohr-Fourier series

$$f(t) \sim \sum_{\lambda \in \mathbf{R}} a(f,\lambda)e^{i\lambda t}$$

where

$$a(f,\lambda) = \mathcal{M}\left\{ f(t)e^{-i\lambda t} \right\}_t = \left(\lim_{T \longrightarrow +\infty} \frac{1}{2T} \int_{-T}^{+T} f(t)e^{-i\lambda t}dt \right).$$

As is known there is an at most countable set of values λ (called the Bohr-Fourier exponents or frequencies) such that the above limit differs from zero. This set will be denoted by

$$\wedge(f) = \{\lambda \in \mathbb{R}, a(f,\lambda) \neq 0\}$$

and called Bohr spectrum of f. The approximation Theorem says that for every almost periodic function f there exists a sequence of trigonometric polynomials

$$P_n(t) = \sum_{k=1}^{N_n} a_{k,n} \exp(\lambda_{k,n} t)$$

where $\lambda_{k,n} \in \wedge(f)$ for all k, n, that converges uniformly in $t \in \mathbb{R}$ to f as $n \to +\infty$. Hence, as in the case of almost periodic, an $f \in WAP(\mathbb{R}, H)$ is also completely defined by its Fourier series. Furthermore, It is still an open question whether or not the Parseval's equality also holds true for Hilbert-valued weakly almost periodic functions. However the answer is affirmative for the Bessel's inequality

$$\sum_{\lambda \in \mathbf{R}} \|a(f,\lambda)\|^2 \leq \mathcal{M}\left\{\|f\|^2\right\}.$$

Example 5. The set of pseudo-random functions developed by [3] to give a model of a some physical phenomena, as turbulence, is a subset of the weakly almost periodic function.

Example 6. Let

$$f_0(t) = \begin{cases} e^{2i\pi\alpha n^2} & \text{if} \quad 0 \leq n < t < n+1 \\ 0 & \text{if} \qquad\qquad t < 0 \end{cases}$$

where $\alpha \in \mathbb{R} - \mathbb{Q}$. Let $c \in \mathbb{N}$, then the convolution product

$$f = \frac{\sin(4\pi\alpha c t)}{t} * f_0 \in WAP(\mathbb{R}, \mathbb{R})$$

is weakly almost periodic [2] and not almost periodic.

Besides, the concept of pseudo almost periodicity (pap) was introduced by Zhang (see for example [12] and [20]) in the early nineties It is a natural generalization of the classical almost periodicity. Define the class of functions $PAP_0(\mathbb{R}, \mathbb{R}^n)$ as follows:

$$PAP_0(\mathbb{R}, X) = \left\{ f \in BC(\mathbb{R}, \mathbb{R}^n) \,/ \lim_{T \to +\infty} \frac{1}{2T} \int_{-T}^{T} \|f(t)\|\, dt = 0 \right\}.$$

Definition 7. A function $f \in BC(\mathbb{R}, X)$ is called pseudo almost periodic if it can be expressed as

$$f = h + \varphi,$$

where $h \in AP(\mathbb{R}, \mathbb{R}^n)$ and $\varphi \in PAP_0(\mathbb{R}, \mathbb{R}^n)$. The collection of such functions will be denoted by $PAP(\mathbb{R}, \mathbb{R}^n)$.

Remark 8. The functions h and φ in above definition are respectively called the almost periodic component and the ergodic perturbation of the pseudo almost periodic function f. Further, The decomposition given in Definition above is unique.

For some preliminary results on hierarchy of almost periodic functions, we refer the reader to ([4],[8],[9],[12] [14], [20] and the references therein).

3. Main Results

For the proofs we need the following lemmas. The first lemma is trivial.

Lemma 9. $(WAP(\mathbb{R},H),((.,.)))$ *is prehilbertian space where the scalar product* $((.,.))$ *is defined by*

$$((u,v)) := \mathcal{M}\left\{< u(t)/\overline{v(t)} >\right\}_t .$$

One can remark that this space is not complete and hence not a Hilbert space.

Lemma 10. *Let* $f \in WAP(\mathbb{R},H)$*. Then the set* $\mathcal{K}_f = \overline{co}\,\{\tau_r(f), \quad r \in \mathbb{R}\}$ *is convex non empty and weakly compact. Furthermore, we have*

$$\forall r \in \mathbb{R},\ \tau_r\left(\mathcal{K}_f\right) \subset \mathcal{K}_f,$$

where $\overline{co}(A)$ *is the closure of the convex hull of* A.

Proof. It is clear that $f \in \mathcal{K}_f$, then $\mathcal{K}_f \neq \varnothing$ and for each $\Psi \in \mathcal{K}_f,\ \tau_r(\Psi) \in \mathcal{K}_f$ and then $\tau_r\left(\mathcal{K}_f\right) \subset \mathcal{K}_f$. By hypothesis the function $f \in WAP(\mathbb{R},E)$, then

$$\overline{\tau(f)}^{w} = \overline{\{f(.+r),\ r \in \mathbb{R}\}}^{w} \text{ is compact.}$$

It follows by using the Krein-Smulian Theorem (see for example [10]), that

$$\overline{co}\left(\overline{\{f(.+r),\ r \in \mathbb{R}\}}^{w}\right) \text{ is compact.}$$

Moreover

$$\overline{co}\left(\{f(.+r),\ r \in \mathbb{R}\}\right) \subset \overline{co}\left(\overline{\{f(.+r),\ r \in \mathbb{R}\}}^{w}\right) \text{ is compact.}$$

Since $\overline{co}\left(\{f(.+r),\ r \in \mathbb{R}\}\right)$ is closed convex set (with respect to the strong topology), then we deduce by Mazur lemma [7] that $\overline{co}\left(\{f(.+r),\ r \in \mathbb{R}\}\right)$ is weakly closed; and so that the set $\mathcal{K}_f$ is weakly closed and contained in a weakly compact set, then it is weakly compact. Thus our lemma holds. □

Lemma 11. *Let* $h \in \mathcal{K}_f = \overline{co}\{\tau_r(f), \quad r \in \mathbb{R}\}$ *then* ,

$$\forall T \in \mathbb{R}, \forall\ \upsilon \in \mathbb{N}^*, \quad C_{T,\upsilon}(h) = \frac{1}{\upsilon}\sum_{k=0}^{\upsilon-1} \tau_{kT}(h) \in \mathcal{K}_f$$

Proof. It is clear that $\forall T \in \mathbb{R}, \forall\ \upsilon \in \mathbb{N}^*,\ \forall t \in \mathbb{R}$

$$C_{T,\upsilon}(h)(t) = \frac{1}{\upsilon}\sum_{k=0}^{\upsilon-1} \tau_{kT}(h)(t) = \frac{1}{\upsilon}\sum_{k=0}^{\upsilon-1} h(t + kT)$$

is a convex combination of $\mathcal{K}_f$ elements. And so the convexity of $\mathcal{K}_f$ permits us to conclude. □

Let us recall Ryll-Nardzeweski theorem, which is the fundamental tool of theorem (13).

Theorem 12. *[18]Let E a normed space, and F be a subset of E. Let $\mathcal{G}$ be a semi-group of affine functions of E. We assume the following conditions fulfilled:*

i. F is convex non empty and weakly compact.

ii. $\forall\ g \in \mathcal{G}, \quad g$ is continuous (for the norm topology) on F. Furthermore $g(F) \subset F$.

iii. $\forall x, y \in F$, $x \neq y$, $0 \notin \overline{\{gx - gy, \quad g \in \mathcal{G}\}}$.

Then the elements of $\mathcal{G}$ possess a common fixed point i.e.

$$\exists x_* \in F,\ \forall g \in \mathcal{G},\ g(x_*) = x_*.$$

Theorem 13. *Let* $f \in WAP(\mathbb{R}, H)$, *then*

$$Fix_f(\mathbb{R}) := \{\Psi \in \mathcal{K}_f / \ \forall r \in \mathbb{R},\ \tau_r(\Psi) = \Psi\} = \mathcal{K}_f \cap H = \{\mathcal{M}\{f\}\}.$$

In other words, $\mathcal{M}\{f\}$ is the unique common fixed point of the family $(\tau_r)_r$.

Proof. By Lemma 10, the set $\mathcal{K}_f$ is convex non empty and weakly compact in $WAP(\mathbb{R}, H)$. Then the condition i. of the Ryll-Nardzeweski theorem is verified. Let $\mathcal{G} := \{\tau_r / \ r \in \mathbb{R}\}$. Then, for all $r \in \mathbb{R}$, τ_r is continuous on $\mathcal{K}_f$ for the uniform since for all φ, $h \in \mathcal{K}_f$, $\forall\ r \in \mathbb{R}$ we have :

$$\|\tau_r(\varphi) - \tau_r(h)\|_\infty = \|\varphi(\cdot + r) - h(\cdot + r)\|_\infty = \|\varphi(\cdot) - h(\cdot)\|_\infty.$$

Moreover, by Lemma 11 we obtain

$$\forall r \in \mathbb{R}, \tau_r(\mathcal{K}_f) \subset \mathcal{K}_f$$

thus the condition ii. of the Ryll-Nardzeweski theorem is satisfied.

Let φ, $h \in \mathcal{K}_f$ such that $\varphi \neq h$. We suppose that

$$0 \in \overline{\{\tau_r(\varphi) - \tau_r(h), \quad r \in \mathbb{R}\}},$$

then there exists a sequence $(r_n)_{n\in\mathbb{N}}$, such that, for all $n \in \mathbb{N}$, we have :

$$\lim_{n\longrightarrow+\infty} \|\tau_{r_n}\varphi(\cdot) - \tau_{r_n}h(\cdot)\|_\infty = \lim_{n\longrightarrow+\infty} \|\varphi(\cdot + r_n) - h(\cdot + r_n)\|_\infty = 0.$$

But, for all $n \in \mathbb{N}$, we have

$$\lim_{n\longrightarrow+\infty} \|\tau_{r_n}\varphi(\cdot) - \tau_{r_n}h(\cdot)\|_\infty = \|\varphi(\cdot) - h(\cdot)\|_\infty > 0$$

which is a contradiction since $\varphi \neq h$. In other words, the set $\mathcal{K}_f$ satisfy iii) as well as i) and ii). So, by the Ryll-Nardzeweski theorem we obtain $Fix_f(\mathbb{R}) \neq \varnothing$.

We prove now the uniqueness of the fixed point.

The space H is assimilated to the space of constant functions from $\mathbb{R}$ to H. Since the constant functions from $\mathbb{R}$ to H are weakly almost periodic, we assimilate H to a vector subspace of $WAP(\mathbb{R}, H)$. By construction of the inner product of $WAP(\mathbb{R}, H)$, one has

$$\forall z, w \in H, \quad < z, w >_H = ((z, w)).$$

Consequently H is a Hilbert subspace of $(WAP(\mathbb{R}, H), ((.,.)))$. Since H is complete, by the theorem of the Orthogonal Projection on a complete vector space, there exists an orthogonal projector $P_H \in \mathcal{L}(WAP(\mathbb{R}, H), WAP(\mathbb{R}, H))$ such that

$$P_H (WAP(\mathbb{R}, H)) \subset H.$$

Otherwise, for all $z \in H$, we have :

$$\begin{aligned} ((f - \mathcal{M}\{f\}, z)) &= ((f, z)) - ((\mathcal{M}\{f\}, z)) \\ &= \mathcal{M}\{< f(t), z >_H\}_t - < \mathcal{M}\{f\} / z >_H . \end{aligned}$$

Since the integral is permeable to linear continuous operators, we obtain

$$\mathcal{M}\{< f(t), z >_H\}_t = < \mathcal{M}\{f\}, z >_H .$$

This is why we get

$$((f - \mathcal{M}\{f\}, z)) = 0.$$

Finally, by using the characterization and uniqueness of orthogonal projection, we can deduce

$$P_H(f) = \mathcal{M}\{f\}.$$

Otherwise, by theorem 3

$$\forall r \in \mathbb{R}, \quad \tau_r(f) \in WAP(\mathbb{R}, H),$$

and then

$$P_H(\ \tau_r(f)) = \mathcal{M}\{\tau_r(f)\} = \mathcal{M}\{f\}.$$

Let $\phi \in co\{\tau_r(f), \quad r \in \mathbb{R}\}$ then there exist $s \in \mathbb{N}$ and real numbers $\lambda_1, \lambda_2, \cdots, \lambda_s$, $r_1, r_2, \cdots, r_s$ such that

$$\forall i, \quad 1 \leq i \leq s, \quad \lambda_i \geq 0, \quad \sum_{i=1}^{s} \lambda_i = 1 \quad \text{and} \quad \phi = \sum_{i=1}^{s} \lambda_i \tau_{r_i}(f).$$

By using the linearity of the orthogonal projector we obtain

$$\begin{aligned} P_H(\phi) &= P_H\left(\sum_{i=1}^{s} \lambda_i \tau_{r_i}(f)\right) \\ &= \sum_{i=1}^{s} \lambda_i P_H(\tau_{r_i}(f)) \\ &= \sum_{i=1}^{s} \lambda_i \mathcal{M}\{f)\} \\ &= \mathcal{M}\{f\} \sum_{i=1}^{s} \lambda_i = \mathcal{M}\{f\}. \end{aligned}$$

Since P_H is continuous, then for all $\psi \in \mathcal{K}_f$, we have $P_H(\psi) = \mathcal{M}\{f\}$. Otherwise for all

$$\theta \in Fix_f(\mathbb{R}) = H \cap \mathcal{K}_f,$$

we have

$$P_H(\theta) = \mathcal{M}\{f\}.$$

Moreover we have

$$P_H(\theta) = \theta$$

then

$$Fix_f(\mathbb{R}) \subset \{\mathcal{M}\{f\}\},$$

and since $Fix_f(\mathbb{R}) \neq \varnothing$, then

$$Fix_f(\mathbb{R}) = \mathcal{M}\{f\}. \qquad \square$$

Remark 14. It should be mentioned that the above result is cited in [14] without proof.

Now, we give an extension of Besicovitch theorem for weakly almost periodic functions. In his book "Almost periodic Functions" Besicovitch [4] gives an approximation of $C_{T,\upsilon}(f) = \dfrac{1}{\upsilon}\displaystyle\sum_{k=0}^{\upsilon-1} \tau_{kT}(f)$ of an almost periodic function by a $T-$periodic function. Our approach, for this generalization, will be different from Besicovitch's. However, we obtain only the weak convergence.

Theorem 15. *Let $f \in WAP(\mathbb{R}, H)$ and $T \in \mathbb{R}^*$, there exists a function f^T with period T such that*

$$C_{T,\nu}(f) \rightharpoonup f^T \quad \text{when } \nu \longrightarrow +\infty.$$

Here $\rightharpoonup$ denotes the weak convergence. Otherwise, the Fourier-Bohr series of f^T is the $T-$periodic part of Fourier-Bohr series of the function f i.e.

$$f^T(t) \sim \sum_{\lambda \in \frac{2\pi}{T}\mathbb{Z}} a(f,\lambda) e^{i\lambda t}.$$

Proof. By using Lemma 2, we have

$$\forall r \in \mathbb{R},\ \tau_r(\mathcal{K}_f) \subset \mathcal{K}_f,$$

and by using Lemma 3, we obtain

$$C_{T,\upsilon}(f) = \frac{1}{\upsilon}\sum_{k=0}^{\upsilon-1} \tau_{kT}(f) \in \mathcal{K}_f.$$

Furthermore, the set $\mathcal{K}_f$ is weakly compact, so we can extract a subsequence $(C_{T,\upsilon_s}(f))_s$ of $(C_{T,\upsilon}(f))_\nu$ which converges weakly to the function θ.

Let $X := WAP(\mathbb{R}, H)$ and we denote $\widehat{X}$ a Hilbertian completion of $(X, \|.\|)$, where $\|.\|$ is the norm associated to $((.,.))$.

For the operator $\tau_T : X \longrightarrow X$, we denote as in [10]

$$Inv(\tau_T) = \{\psi \in X \ \ / \ \ \tau_T(\psi) = \psi\} \subset X.$$

and $\widehat{\tau_T} \in \mathcal{L}\left(\widehat{X}, \widehat{X}\right)$ by the unique extension of τ_T.

Let $\widehat{f}$ be the orthogonal projection of f on $Inv(\widehat{\tau_T})$. Then by Von Newmann ergodic theorem [10] we get

$$\lim_{\nu \longrightarrow +\infty} \left\| C_{T,\nu}(f) - \widehat{f} \right\|_\infty = 0.$$

Then $(C_{T,\upsilon}(f))_\nu$ converges weakly to $\widehat{f}$ i.e.

$$\forall \mu \in \left(\widehat{X}\right)^*, \quad \lim_{\nu \longrightarrow +\infty} < \mu, C_{T,\nu}(f) - \widehat{f} >= 0.$$

By uniqueness of the limit we obtain $\widehat{f} = \theta$. Then $\theta \in \mathcal{K}_f$. Furthermore, $\widehat{f} = \theta$ is $T-$periodic since $\widehat{f} = \theta \in Inv(\widehat{\tau_T})$, where $\widehat{\tau_T}$ is the unique extension of τ_T . So,

$$\tau_T(\widehat{f}) = \widehat{\tau_T}(\widehat{f}) = \widehat{f}.$$

Thus θ is $T-$periodic, and consequently

$$\widehat{f}(t) = \theta(t) \sim \sum_{\lambda \in \frac{2\pi}{T}\mathbb{Z}} c_k e^{i\lambda t}.$$

Otherwise, since

$$f - \widehat{f} \perp Inv(\widehat{\tau_T})$$

and

$$Inv(\tau_T) \subset Inv(\widehat{\tau_T})$$

then

$$f - \widehat{f} \perp Inv(\tau_T).$$

and so

$$\forall k \in \mathbb{Z}, \qquad \mathcal{M}\left\{ < f - \widehat{f} / e^{-i\frac{2\pi}{T}kt} > \right\} = 0,$$

consequently

$$\forall k \in \mathbb{Z}, \qquad \mathcal{M}\left\{ < f \ / \ e^{-i\frac{2\pi}{T}kt} > \right\} = \qquad \mathcal{M}\left\{ < \widehat{f} \ / \ e^{-i\frac{2\pi}{T}kt} > \right\}$$

then we obtain

$$f^T(t) = \widehat{f}(t) = \theta(t) \sim \sum_{k\in\mathbb{Z}} a\left(\widehat{f}, \frac{2k\pi}{T}\right) e^{i\frac{2k\pi}{T}t}$$

with

$$a\left(f^T, \frac{2k\pi}{T}\right) = a\left(f, \frac{2k\pi}{T}\right). \qquad \square$$

Remark 16. If $dim\,(H) < +\infty$ then the above convergence is pointwise i.e.

$$C_{T,\nu}(f) \longrightarrow f^T \quad \text{when } \nu \longrightarrow +\infty.$$

Theorem 17. *Let* $dim\,(H) < +\infty$, *then*

$$\begin{array}{ccc} WAP(\mathbb{R}, H) & \longrightarrow & WAP(\mathbb{R}, H) \\ f & \longmapsto & f^T \end{array}$$

is Lipschitz function.

Proof. Let $f_1, f_2 \in WAP(\mathbb{R}, H)$. For all $T \in \mathbb{R}$ and for all $\upsilon \in \mathbb{N}^*$, by the convexity of the norm and the fact that

$$\|\tau_{kT}\|_{\mathcal{L}(WAP(\mathbb{R},H))} = 1,$$

we get

$$\begin{aligned} \|C_{T,\nu}(f_1) - C_{T,\nu}(f_2)\| &= \left\| \frac{1}{\upsilon}\sum_{k=0}^{\upsilon-1} \tau_{kT}(f_1) - \frac{1}{\upsilon}\sum_{k=0}^{\upsilon-1} \tau_{kT}(f_2) \right\| \\ &= \left\| \frac{1}{\upsilon}\sum_{k=0}^{\upsilon-1} \tau_{kT}(f_1) - \tau_{kT}(f_2) \right\| \\ &\leq \left\| \frac{1}{\upsilon}\sum_{k=0}^{\upsilon-1} \tau_{kT}(f_1 - f_2) \right\| \\ &\leq \sum_{k=0}^{\upsilon-1} \frac{1}{\upsilon} \|\tau_{kT}\|_{\mathcal{L}(X,X)} \|f_1 - f_2)\| \\ &\leq \|f_1 - f_2)\|. \end{aligned}$$

Then, by setting $\nu \longrightarrow +\infty$, we obtain by using theorem 5 the inequality

$$\left\|f_1^T - f_2^T\right\| \leq \|f_1 - f_2)\|. \qquad \square$$

Theorem 18. *Let $f \in PAP(\mathbb{R}, H)$ and $T \in \mathbb{R}^*$, there exists a function f^T with period T such that*

$$C_{T,\nu}(f) \longrightarrow f^T \quad \text{when } \nu \longrightarrow +\infty.$$

Otherwise, the Fourier-Bohr series of f^T is the $T-$periodic part of Fourier-Bohr series of the function f i.e.

$$f^T(t) \sim \sum_{\lambda \in \frac{2\pi}{T}\mathbb{Z}} a(f, \lambda) e^{i\lambda t}$$

Proof. By definition of the pseudo almost periodic function, we can write $f = g + \varphi$, where $g \in AP(\mathbb{R}, H)$ and $\varphi \in PAP_0(\mathbb{R}, H)$. Obviously, for all $\nu \in \mathbb{N}$

$$C_{T,\nu}(f) = C_{T,\nu}(g) + C_{T,\nu}(\varphi).$$

By using the Besicovitch Theorem [4], there exists a periodic function f^T with period T such that

$$C_{T,\nu}(g) \longrightarrow f^T \quad \text{when } \nu \longrightarrow +\infty.$$

Consequently

$$C_{T,\nu}(f) \longrightarrow f^T \quad \text{when } \nu \longrightarrow +\infty$$

since $C_{T,\nu}(\varphi) \longrightarrow 0 \quad$ when $\nu \longrightarrow +\infty$. On the other hand, the invariance of the mean time by translation gives us

$$\begin{aligned}
a(C_{T,\nu}(f), \frac{2\pi}{T}) &= \mathcal{M}\left\{C_{T,\nu}(f)(t)\, e^{-\frac{2i\pi}{T}t}\right\}_t \\
&= \mathcal{M}\left\{\frac{1}{\upsilon}\sum_{k=0}^{\upsilon-1} f(kT+t)e^{-\frac{2i\pi}{T}t}\right\}_t \\
&= \frac{1}{\upsilon}\sum_{k=0}^{\upsilon-1} \mathcal{M}\left\{f(kT+t)e^{-\frac{2i\pi}{T}t}\right\}_t \\
&= \frac{1}{\upsilon}\sum_{k=0}^{\upsilon-1} \mathcal{M}\left\{f(t)e^{-\frac{2i\pi}{T}(t-kT)}\right\}_t \\
&= \frac{1}{\upsilon}\sum_{k=0}^{\upsilon-1} \mathcal{M}\left\{f(t)e^{-\frac{2i\pi}{T}t}e^{2i\pi k}\right\}_t \\
&= \frac{1}{\upsilon}\sum_{k=0}^{\upsilon-1} \mathcal{M}\left\{f(t)e^{-\frac{2i\pi}{T}t}\right\}_t
\end{aligned}$$

$$\begin{aligned} &= \frac{1}{\upsilon}\nu\mathcal{M}\left\{f(t)e^{-\frac{2i\pi}{T}t}\right\}_t \\ &= \mathcal{M}\left\{f(t)e^{-\frac{2i\pi}{T}t}\right\}_t \\ &= a(f,\frac{2\pi}{T}). \end{aligned}$$

Consequently,

$$\begin{aligned} a(f,\frac{2\pi}{T}) &= a(C_{T,\nu}(f), \frac{2\pi}{T}) = a(C_{T,\nu}(g), \frac{2\pi}{T}) + a(C_{T,\nu}(\varphi), \frac{2\pi}{T}) \\ &= a(g,\frac{2\pi}{T}) + a(\varphi,\frac{2\pi}{T}) \\ &= a(f^T,\frac{2\pi}{T}). \end{aligned}$$

□

4. Application

Let us consider the abstract differential equations of the form:

$$\frac{d}{ds}w(s) = Aw(s) + Bw(s) + p(s) \qquad (*),$$

where A,B are densely defined closed linear operators acting in a the Hilbert space H, and $p : \mathbb{R} \longrightarrow H$ is a H valued pseudo almost periodic function. Using the theory of the invariant subspaces for unbounded linear operators, Diagana [11] showed that every bounded solution to $(*)$ is a pseudo almost periodic function. Here we will prove that if the equation $(*)$ possess a bounded solution the it admits also a periodic solution.

Theorem 19. *Under the following assumptions:*

(H_1) *the function $p : \mathbb{R} \longrightarrow H$ is a H-valued pseudo almost periodic,*

(H_2) *there exists $S \subset H$, a closed subspace that reduces both A and B. We denote by P_S, $Q_S = (I - P_S) = P_{H\ominus S}$, the orthogonal projections onto S and $H \ominus S$, respectively;*

(H_3) *A and B are the infinitesimal generators of c_0-groups of bounded operators $(T(s))_{s\in\mathbb{R}}$, $(R(s))_{s\in\mathbb{R}}$, respectively, such that, there exist $M, K, c, d > 0$ with*

$$\|T(s-\sigma)P_S\| \leq Me^{-c(s-\sigma)} \text{ if } s \geq \sigma,$$

$$\|T(s-\sigma)Q_S\| \leq Me^{-c(s-\sigma)} \text{ if } s \leq \sigma,$$

and

$$\begin{aligned}\|R(s-\sigma)P_S\| &\leq Ke^{-d(s-\sigma)} \text{ if } s \geq \sigma,\\ \|R(s-\sigma)P_S\| &\leq Ke^{-d(s-\sigma)} \text{ if } s \leq \sigma;\end{aligned}$$

(H_4) $R(A) \subset R(P_S) = N(Q_S)$;

(H_5) $R(B) \subset R(Q_S) = N(P_S)$;

(H_6) *A, B bounded operators, if the equation $(*)$ admits a bounded solution then it possess a $T-$ periodic solution for all $T \in \mathbb{R}^*$.*

Proof. First, let us prove that

$$S_* = \left\{ w \in PAP(\mathbb{R}, H), \frac{d}{ds}w(s) = Aw(s) + Bw(s) + p(s), s \in \mathbb{R} \right\}$$

is closed subset of $PAP(\mathbb{R}, H)$. Let $(w_n)_n$ a sequence of functions of S_* which converges to some function w. Then, for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n > N$ then $\|w_n(t) - w(t)\| < \frac{\epsilon}{2(\|A\|+\|B\|)}$ for all $t \in \mathbb{R}$. Pose

$$\alpha_n(s) = \frac{d}{ds}w_n(s).$$

The sequence $(\alpha_n(s))_n$ converges uniformly since it is a Cauchy sequence. In fact, for all $p > q > N$, one has

$$\begin{aligned}\|\alpha_p(s) - \alpha_q(s)\| &= \left\| \frac{d}{ds}w_p(s) - \frac{d}{ds}w_q(s) \right\|\\ &= \|Aw_p(s) + Bw_p(s) - Aw_q(s) - Bw_q(s)\|\\ &\leq (\|A\| + \|B\|)\|w_p(s) - w_q(s)\|\\ &\leq (\|A\| + \|B\|)(\|w_p(s) - w(s)\| + \|w_q(s) - w(s)\|)\\ &\leq \epsilon.\end{aligned}$$

By passing to the limit in the formula

$$\alpha_n(s) = \frac{d}{ds}w_n(s) = Aw_n(s) + Bw_n(s) + p(s)$$

we get

$$\frac{d}{ds}w(s) = Aw(s) + Bw(s) + p(s),$$

which proves that the function w belongs to S_* and consequently S_* is closed subset. Now, by hypothesis the equation $(*)$ admits a bounded solution, then by [11] $S_* \neq \emptyset$. Fix an arbitrary $T \in \mathbb{R}^*$, and $x(\cdot) \in S_*$. Since S_* is convex set then

$$C_{T,\nu}(x) = \frac{1}{\upsilon} \sum_{k=0}^{\upsilon-1} \tau_{kT}(x) = \frac{1}{\upsilon} \sum_{k=0}^{\upsilon-1} x(kT + \cdot)$$

is also a solution of $(*)$ for all $\nu \in \mathbb{N}^*$. By theorem 7, there exists a function f^T with period T such that

$$C_{T,\nu}(x) \longrightarrow x^T \quad \text{when } \nu \longrightarrow +\infty.$$

So $x^T(\cdot) \in S_*$ which imply the existence of the periodic solution of $(*)$. □

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