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TURNING THINGS INSIDE OUT

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Abstract: This article presents 3 puzzles which turn things inside out: the cube puzzle, turning a paper strip inside out, and hexaflexagons. Readers who wish to find the solutions for themselves are advised not to read the full explanations.

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1. Cube Puzzle

I spent a year conducting research overseas at Cambridge University in England from April 2005. While enrolled as a visiting fellow at Saint Edmund's College I performed my job at a research laboratory in the university department known as the Centre for Mathematical Sciences. This is like an amalgamation of what would be the physics department and the mathematics department in the

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science faculty at a Japanese university, and Stephen Hawking has his office in this very building.

I had never experienced such a period of stay overseas, and I was interested in everything there was to see. Also, having a hopeless sense of direction, I often got my routes mixed up, took the wrong bus, or guessing incorrectly set off in the wrong direction, but after a month I eventually settled down. Once when I lost my way I thought to myself, “This is no good, I’d better get a map”, and so I went into a tourist information office. There I found an interesting cube made of wood (Figure 1).

Figure 1: Cube puzzle

Figure 2: Folding up the cube

Cambridge is a small university town with a population of 100,000, but with 31 colleges, various university departments and research buildings, the university’s breadth from edge to edge is beyond walking. There are many structures that have existed since the middle ages such as King’s College, Saint John’s College, Round Church, Saint Mary’s Church, the statue of Henry VIII, the Fitzwilliam Museum, the Mathematical bridge over the river Cam, the Bridge of Sighs, and many more. There are not only students attending university, on non-working days it is bustling with tourists. Each of the colleges has a vast grass compound known as The Backs where students go strolling or may be found lounging about. I felt that time moved so slowly there!

The cube had photographs of these scenic attractions attached to its faces.

It did not simply have one picture attached to each of its faces. It could be folded up horizontally and vertically in various directions, and different photographs could me viewed one after the other. There were 6 square photographs, and 3 photographs formed by connecting two squares which is a total of 9 scenic images. The mechanics of its folding are interesting, so I bought one and took it back to my lodgings.

9 scenic photographs is a strange number, but I soon understood the reason. Recalculating the sizes of the photographs yields $6 + 3 \times 2 = 12$ squares. The number of faces on a cube is 6, and doubling this yields 12. This is equal to the number of photographs. I realized that 12 seemed to be a number related to the construction of this puzzle. I investigated what kind of structure was behind the cube. Firstly, the smaller sub-cubes, as basic units, had an edge length of 3.5cm. There are a total of 8 such sub-cubes, and they are packed together in 3 dimensions as $2 \times 2 \times 2 = 8$ cubes (Figure 2, Figure 3).

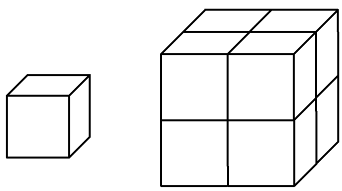

Figure 3: The 8 sub-cubes

The wooden sub-cubes were made with unusual precision, and since they did not reveal a crack there was no sense of a join within the photographs. It was also made of a hard and heavy wooden material, so it made a satisfying 'clock clock' sound while it was being manipulated. The 8 sub-cubes were overlaid with what seemed like high quality film photographs. Maybe it was laminated - it didn't seem like it would be easily damaged.

Well, the most intriguing aspect is where and how the 8 sub-cubes are attached to each other. On the day that I bought it, I enjoyed analyzing its mechanism while manipulating it repeatedly. The scenic photographs were squares composed of four parts. As shown above there were a total of 12 faces covered with parts of these scenes. The basic unit is a sub-cube with 6 faces, and there were 8 such units, so the total number of faces is 48. Each scenic image is composed of sets of 4 faces, and dividing the 48 faces by 4 yields 12 sets. This means the puzzle is constructed in a way that makes expert use of all the sub-cube faces.

While manipulating it over and over again, I discovered that the 8 sub-cubes are attached at the places marked with thick lines in Figure 4. There are 8 places where the cubes are attached, and their positional relationship is interesting. If they are attached in this way the 8 sub-cubes do not fall apart separately, and all 48 of the faces can be revealed. I wonder who thought it up. A cube that can be turned inside out! It really is a wonderful idea.

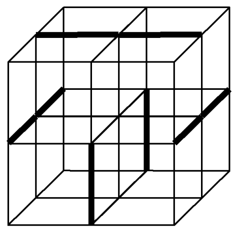

Figure 4: The attachments between the cubes

Investigating this far made me want to take the next step and make one for myself. I thought about making a wooden model in the same way as the commercial item, but being abroad I couldn't easily lay hands on a saw and wood. I wondered whether it wouldn't be possible to make one out of drawing paper. I bought some paper, adhesive tape and glue from a stationary shop, but I was surprised at how unusually expensive the price was. In Japan I was able to use paper in profuse quantities, but in England stationary was a precious commodity. I was further troubled by the complete absence of B series copy paper. In Japan, paper of all sizes - A4, A3, B4, B5 and so on - is readily available, but because Cambridge is a small city there was no B series to be found, only A series.

In the event, I made 8 cubes out of thick drawing paper, and while comparing them with the real thing, I sticky-taped them together. It took a while but I produced something resembling the original. It was a good thing I made it out of paper after all. Assembling it was a pleasure, and making it out of paper has the advantage that some degree of mismatch in size is permissible. Paper is soft, and the 8 sub-cubes kindly fit together harmoniously (Figure 5).

The attachments are shown in Figure 4, and 24 of the faces of the small unit cubes are connected as a single surface. Let's take this as the outer surface. The 24 hidden faces (the inner surface) are also connected as a single surface. The 24 faces of the outer surface and the 24 faces of the inner surface are connected too, so the 48 faces of the whole body are thus connected. It can be manipulated in such a way that it is turned completely inside out, revealing

Figure 5: Turning the home-made cube inside out

that this is a cube reversal puzzle which involves turning a cube inside out.

For such a cube reversal puzzle there are surely only $2 \times 2 \times 2 = 8$ possible cases. For $3 \times 3 \times 3 = 27$, the manipulation would be impossible so the number of faces would not match up. This aspect is an interesting mathematical point. Interested readers please make one for yourself.

2. Turning a Strip Inside Out

Talking about turning things inside out, I previously investigated another puzzle which involves turning things inside out in a different sense, and I'd like to introduce it here. The puzzle also involves cubes, but they are not the kind of cubes that are filled out completely like that in Figure 1, rather, it is a puzzle which involves connecting a paper strip into a cubic form. By making a model like that shown in Figure 6(1), and manipulating it repeatedly, it can be turned inside out into the form shown in Figure 6(2). The manipulation process is altogether like looking at a chienowa puzzle ring.

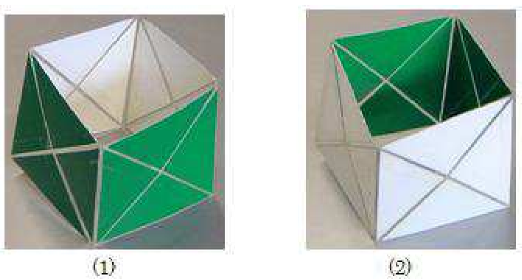

Figure 6: Turning the strip inside out

The real thrill of a puzzle is not looking at the solution, but rather the experience of solving it oneself, so I'll withhold the solution here. I'd like to encourage you to make this puzzle yourself. At the present time I have discovered two solution methods for turning it inside out. The fact that I am deliberately not supplying these solutions embodies my hope that a third solution might be found!

Now then, regarding how to make it, there is one method which involves making it from drawing paper and subsequently applying color, but I don't suggest this. The reason is that applying color using magic markers and so on may cause surface irregularities yielding a poor end result. Rather, art stores sell different types of art paper and it's better to buy these to use. It's best to choose your preferred colors from among clearly different colors of blue, green, red *etc.* Then, taking care not to damage the colored side, measures can be marked on the white reverse side to guide construction. As shown in Figure 7, 4 squares with an edge length of 7 cm are connected horizontally. Diagonal lines are drawn to produce 16 right-angled isosceles triangles.

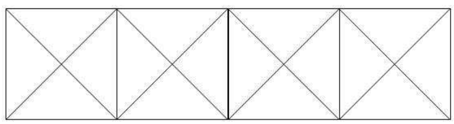

Figure 7: Construction diagram for the strip puzzle

Hasty people may think that this completes the puzzle, but this is not the case. This crucial part is what follows. The 16 triangles are neatly cut apart with a pair of scissors. Next, the triangles are attached using sticky tape, with a gap between the triangles as shown in Figure 8, so that it can be operated smoothly. Since the edge of each square is 7 cm, a gap of about 2 mm should be suitable. If this gap is too large the visual appearance suffers, but if it is too small it cannot be easily manipulated.

It's best to apply the sticky tape to the white side first, and if it goes well, then apply it to the colored side. This is because the colored side will not tolerate a mistake. The sticky tape is applied to both sides, extending beyond the edge of the squares. After applying the sticky tape to every part, the bits which stick out can be cut away. That is to say, I don't recommend reapplying or applying multiple layers of sticky tape.

In this way, 4 squares connected by the sticky tape are produced. Connecting these horizontally into a strip completes the puzzle.

There is pleasure in solving this puzzle oneself. One may solve the puzzle

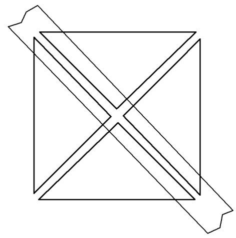

Figure 8: Sticking it together with gaps using sticky tape

while trying this and that, or alternatively find oneself stuck. There is definitely a route to a solution, and it must not be forced inside out. It is a peculiar thing, but knowing the route, the puzzle can be turned inside out easily. The interesting thing about this puzzle is that it is a strip which forms a cube. Increasing the number of faces in the strip to 5 or 6 faces for example, would allow it to be turned inside out easily and it would cease to be a puzzle. The fact that the key to the puzzle is that there are 4 faces in the band which form a cube is strangely similar to the fact that the cube reversal puzzle shown in Figure 1 involves a cube.

For those readers who simply have to know a solution, there is Gardner's orthodox method (see Gardner, 1968) [1]. The procedure has 12 steps and is a little on the long side, but it is a well-known process for turning the band inside out. Another method was taught to me by an acquaintance, and appears in 'Mathematics Seminar' (Nishiyama, 1994) [2]. This method involves around 7 steps and is significantly shorter. It's a method for turning the band inside out that is brilliant enough to make one think, "Wow! This manipulation is possible!" I couldn't turn it inside out until I heard the solution.

Some people may find that they just can't turn the band inside out and seek a hint. In such cases I always say something like the following. Puzzles have a front and rear. This means that there must be a mid-point. The mid-point is when half the outer surface and half the inner surface are visible. If the mid-point is found, it is possible to get to the outer or the inner surfaces. This is just like climbing a mountain, where the mid-point is the peak, from which it is possible to return, or to descend to the other side. This means that if one can just find the mid-point, it is the same as having solved the problem.

3. Hexaflexagons

I have dealt with turning things inside out twice now, but the following hexaflexagon is also, in a sense, a puzzle which involves turning things inside out. The ‘hexa’ in hexaflexagon means 6, and the ‘flexagon’ refers to something which is flexible, i.e, something that is possible to fold up. Figure 9 shows a hexaflexagon that I made myself. It is hexagonal, and as shown in Figure 10(1), by pinching two pairs of triangles, each in a diamond shape, a new face may be revealed from the center. Opening it out completely reveals a different face, as shown in Figure 10(2). There are 3 faces that can be revealed, and technically speaking it is not turned inside out, but rather, it is in the same class of puzzles as the other two because they all involve revealing different faces.

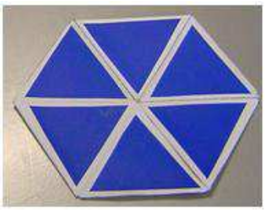

Figure 9: Hexaflexagon

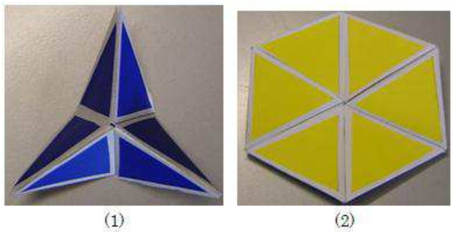

Figure 10: Pinching in a diamond shape reveals a new face

There number of faces is 3 and, for example, by applying colors, blue, yellow and red faces can be revealed alternately. This is another peculiar puzzle, and for those readers who have never tried it, I’d like to explain how to make it.

10 equilateral triangles with an edge length of 6 cm are lined up side by side as shown in Figure 11(1). The triangle on the right hand end marked ‘glue’ is not related to the puzzle, it’s just for gluing the hexaflexagon together. A diagram like this can be made with a pair of compasses and a ruler, so everyone should draw it up for themselves. For the paper, standard copy paper is better than drawing paper. Copy paper is easy to fold and hard to tear.

The folding process begins by folding towards oneself along a-b as shown in Figure 11(2). Next a valley fold is made along c-d, leaving the part marked ‘glue’ exposed (Figure 11(3)). Then it is folded towards oneself again along e-f, and completed by gluing.

A Möbius strip is a normal strip glued into a loop with a 180° twist, and the hexaflexagon is a strip glued with three 180° twists, i.e., a 540° twist. Gluing

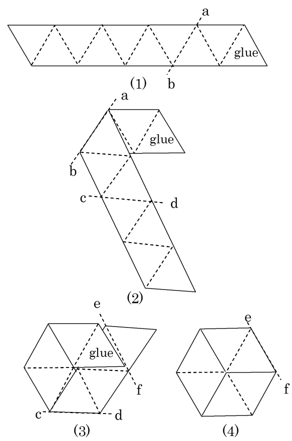

Figure 11: 3-face folding template and folding method

together with an odd number of twists results in a surface for which the inside and outside cannot be discriminated. This puzzle is an application of topology.

Once the puzzle is made, be sure to try manipulating it. If two pairs of triangles are pinched into diamond shapes, as shown in Figure 10(1), a new surface is unexpectedly revealed from the center. This is surely a delight that can only be experienced by people who make and manipulate the hexaflexagon for themselves. This puzzle is not exactly a 'turning inside out' puzzle, but it is in the same class in the sense that different faces are revealed.

After completing the 3-face-folding hexaflexagon in which 3 faces may be revealed, it's also fun to attempt the challenge of the 4-face-folding hexaflexagon. If you think about what kind of paper template, and what kind of folding are needed to make the 4-face-folding model, I think you'll enjoy this puzzle too. After achieving the 4-face-folding, and while attempting a 5-face-folding, a kind of principle may be found, leading from the world of puzzles to the realm of mathematics.

I completed templates for the 4-face-folding up to the 8-face-folding by referring to templates appearing in Gardner's book. I learned that the templates and folding methods for the 3-face-folding, 6-face-folding, 12-face-folding, etc., and in general 3×2^n face-foldings, follow a principle. My interest was captured and I went as far as making a 24-face-folding. I also learned the method behind the templates and folding patterns for the other numbers of faces. It was really quite an interesting experience. The topic is fully introduced in under the title 'The Mathematics of Pleated Folding' in this book. Interested readers please take a look. However, this article includes the solution, so those readers who wish to find the solution for themselves might be advised not to read it yet!

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