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SOME IDENTITIES OF THE TWISTED (h, q)-GENOCCHI NUMBERS AND POLYNOMIALS ASSOCIATED WITH BERNSTEIN POLYNOMIALS

Cheon Seoung Ryoo

Department of Mathematics
Hannam University
Daejeon, 306-791, KOREA

Abstract: In this paper, we give some interesting identities on the twisted (h, q)-Genocchi polynomials and Bernstein polynomials.

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1. Introduction

Throughout this paper, let p be a fixed odd prime number. The symbol, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of algebraic closure of $\mathbb{Q}_p$. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. As well known definition, the p-adic absolute value is given by $|x|_p = p^{-r}$ where $x = p^r \frac{t}{s}$ with $(t, p) = (s, p) = (t, s) = 1$. When one talks of q-extension, q is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or a p-adic number $q \in \mathbb{C}_p$. In this paper we assume that $q \in \mathbb{C}_p$ with $|1 - q|_p < 1$ (see [1-9]).

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We assume that $UD(\mathbb{Z}_p)$ is the space of the uniformly differentiable function on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the fermionic p-adic invariant integral on $\mathbb{Z}_p$ is defined as follows(see [1, 2, 3, 4]):

$$I_{-1}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x. \tag{1.1}$$

For $n \in \mathbb{N}$, let $f_n(x) = f(x+n)$ be translation. As well known equation, by (1.1), we have

$$\begin{aligned} I_{-1}(f_n) &= \int_{\mathbb{Z}_p} f(x+n) d\mu_{-1}(x) \\ &= (-1)^n \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) + 2\sum_{l=0}^{n-1} (-1)^{n-1-l} f(l). \end{aligned} \tag{1.2}$$

The Genocchi polynomials are defined by the generating function as follows:

$$\frac{2t}{e^t+1} e^{xt} = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!}, \text{ see [1-9]}. \tag{1.3}$$

In the special case, $x = 0$, $G_n(0) = G_n$ are called the n-th Euler numbers(see [1-9]).

Let $C_{p^n} = \{w | w^{p^n} = 1\}$ be the cyclic group of order p^n and let

$$T_p = \lim_{n\to\infty} C_{p^n} = \cup_{n\geq 1} C_{p^n}.$$

For $w \in T_p$ and $h \in \mathbb{Z}$, we defined the twisted (h, q)-Genocchi polynomials as follows:

$$\frac{2t}{wq^h e^t + 1} e^{xt} = \sum_{n=0}^{\infty} G_{n,q,w}^{(h)}(x) \frac{t^n}{n!}. \tag{1.4}$$

In the special case, $x = 0$, $G_{n,q,w}^{(h)}(0) = G_{n,q,w}^{(h)}$ are called the n-th twisted (h, q)-Genocchi numbers.

From (1.4), we note that

$$G_{n,q,w}^{(h)}(x) = \sum_{l=0}^{n} \binom{n}{l} G_{l,q,w}^{(h)} x^{n-l}. \tag{1.5}$$

From (1.2) and (1.4), for $n = 1$, we have

$$t\int_{\mathbb{Z}_p} q^{hy} w^y e^{(x+y)t} d\mu_{-1}(y) = \frac{2t}{wq^h e^t + 1} e^{xt} = \sum_{n=0}^{\infty} G_{n,q,w}^{(h)}(x) \frac{t^n}{n!}. \tag{1.6}$$

By (1.6), we obtain

$$\begin{aligned} &G_{0,q,w}^{(h)}(x)=0, \\ &\int_{\mathbb{Z}_p} q^{hy}w^y(x+y)^n d\mu_{-1}(y)=\frac{G_{n+1,q,w}^{(h)}(x)}{n+1}, \text{ for } n\in\mathbb{N}. \end{aligned} \tag{1.7}$$

As well known definition, Bernstein polynomials of degree n are given by

$$B_{k,n}(x)=\binom{n}{k}x^k(1-x)^{n-k}, \text{ where } x\in[0,1], n,k\in\mathbb{Z}_+. \tag{1.8}$$

In [1], Kim introduced p-adic extension of Bernstein polynomials as follows:

$$B_{k,n}(x)=\binom{n}{k}x^k(1-x)^{n-k}, \text{ where } x\in\mathbb{Z}_p \text{ and } n,k\in\mathbb{Z}_+. \tag{1.9}$$

In this paper, we investigate some properties for the twisted (h,q)-Genocchi numbers and polynomials. By using these properties, we give some interesting identities on the twisted (h,q)-Genocchi polynomials and Bernstein polynomials.

2. Some Identities on the Bernstein and Twisted (h,q)-Genocchi Polynomials

From (1.6), we can derive the following recurrence formula for the twisted (h,q)-Genocchi numbers:

$$G_{0,q,w}^{(h)}=0, \text{ and } wq^h(G_{q,w}^{(h)}+1)^n+G_{n,q,w}^{(h)}=\begin{cases}2, & \text{if } n=1,\\ 0, & \text{if } n>1,\end{cases} \tag{2.1}$$

with usual convention about replacing $(G_{q,w}^{(h)})^n$ by $G_{n,q,w}^{(h)}$.

By (1.4), we easily get

$$\begin{aligned} \sum_{n=0}^{\infty} G_{n,q^{-1},w^{-1}}^{(h)}(1-x)(-1)^n\frac{t^n}{n!} &= (-1)\frac{2twq^h}{wq^he^t+1}e^{xt} \\ &= (-1)wq^h\sum_{n=0}^{\infty}G_{n,q,w}^{(h)}(x)\frac{t^n}{n!}. \end{aligned} \tag{2.2}$$

By (2.2), we obtain the following theorem.

Theorem 1. *Let $n \in \mathbb{Z}_+$. For $w \in T_p$, we have*

$$G_{n,q,w}^{(h)}(x) = (-1)^{n-1} w^{-1} q^{-h} G_{n,q^{-1},w^{-1}}^{(h)}(1-x).$$

From (1.7), we note that

$$G_{0,q,w}^{(h)} = 0, \quad \int_{\mathbb{Z}_p} q^{hx} w^x x^n d\mu_{-1}(x) = \frac{G_{n+1,q,w}^{(h)}}{n+1}, \text{ for } n \in \mathbb{N}. \tag{2.3}$$

By (2.1), for $n \in \mathbb{N}$ with $n > 1$, we have

$$\begin{aligned} G_{n,q,w}^{(h)}(2) &= (G_{q,w}^{(h)} + 1 + 1)^n = \sum_{l=0}^{n} \binom{n}{l} G_{l,q,w}(1) \\ &= \frac{1}{wq^h} \sum_{l=1}^{n} \binom{n}{l} wq^h G_{l,q,w}^{(h)}(1) \\ &= \frac{1}{wq^h}(nwq^h G_{1,q,w}^{(h)}(1)) + \frac{1}{wq^h} \sum_{l=2}^{n} \binom{n}{l} G_{l,q,w}^{(h)}(1) \\ &= \frac{2n}{wq^h} - \frac{1}{(wq^h)^2} wq^h G_{n,q,w}(1) \\ &= \frac{2n}{wq^h} + \frac{1}{(wq^h)^2} G_{n,q,w}^{(h)}. \end{aligned} \tag{2.4}$$

Therefore, by (2.4), we obtain the following theorem.

Theorem 2. *For $n \in \mathbb{N}$ with $n > 1$, we have*

$$wq^h G_{n,q,w}^{(h)}(2) = 2n + \frac{1}{wq^h} G_{n,q,w}^{(h)}.$$

By (2.3) and Theorem 2, we obtain the following corollary.

Corollary 3. *For $n \in \mathbb{N}$ with $n > 1$, we have*

$$\frac{1}{wq^h} \int_{\mathbb{Z}_p} q^{-hx} w^{-x} (x+2)^n d\mu_{-1}(x) = 2 + wq^h \frac{G_{n+1,q^{-1},w^{-1}}^{(h)}}{n+1}.$$

By (1.7), (2.3) and Corollary 3, we know that

$$\begin{aligned}
\int_{\mathbb{Z}_p} q^{hx}w^x(1-x)^n d\mu_{-1}(x) &= (-1)^n \int_{\mathbb{Z}_p} q^{hx}w^x(x-1)^n d\mu_{-1}(x) \\
&= (-1)^n \frac{G^{(h)}_{n+1,q,w}(-1)}{n+1} \\
&= \frac{1}{wq^h}\frac{G^{(h)}_{n+1,q^{-1},w^{-1}}(2)}{n+1} \\
&= \frac{1}{wq^h}\int_{\mathbb{Z}_p} q^{-hx}w^{-x}(x+2)^n d\mu_{-1}(x) \\
&= 2 + wq^h \frac{G^{(h)}_{n+1,q^{-1},w^{-1}}}{n+1} \\
&= 2 + wq^h \int_{\mathbb{Z}_p} q^{-hx}w^{-x}x^n d\mu_{-1}(x).
\end{aligned}$$

Therefore, we have the following theorem.

Theorem 4. *For $n \in \mathbb{N}$ with $n > 1$, we have*

$$\int_{\mathbb{Z}_p} q^{hx}w^x(1-x)^n d\mu_{-1}(x) = 2 + wq^h \int_{\mathbb{Z}_p} q^{-hx}w^{-x}x^n d\mu_{-1}(x).$$

In (1.9), we take the fermionic p-adic invariant integral on $\mathbb{Z}_p$ for one Bernstein polynomials as follows:

$$\begin{aligned}
&\int_{\mathbb{Z}_p} q^{hx}w^x B_{k,n}(x) d\mu_{-1}(x) \\
&= \binom{n}{k}\sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^{n-k-l}\int_{\mathbb{Z}_p} q^{hx}w^x x^{n-l} d\mu_{-1}(x) \\
&= \binom{n}{k}\sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^{n-k-l}\frac{G^{(h)}_{n-l+1,q,w}}{n-l+1} \\
&= \binom{n}{k}\sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^{l}\frac{G^{(h)}_{k+l+1,q,w}}{k+l+1}, \text{ where } n,k \in \mathbb{Z}_+.
\end{aligned} \tag{2.5}$$

From the reflection symmetric properties of Bernstein polynomials, we note that

$$B_{k,n}(x) = B_{n-k,n}(1-x), \text{ where } n,k \in \mathbb{Z}_+ \text{ and } x \in \mathbb{Z}_p. \tag{2.6}$$

For $n, k \in \mathbb{Z}_+$ with $n > k+1$, we have

$$\begin{aligned}
&\int_{\mathbb{Z}_p} q^{hx} w^x B_{k,n}(x) d\mu_{-1}(x) \\
&= \int_{\mathbb{Z}_p} q^{hx} w^x B_{n-k,n}(1-x) d\mu_{-1}(x) \\
&= \binom{n}{k} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \int_{\mathbb{Z}_p} q^{hx} w^x (1-x)^{n-l} d\mu_{-1}(x) \\
&= \binom{n}{k} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \left(2 + wq^h \int_{\mathbb{Z}_p} q^{-hx} w^{-x} x^{n-l} d\mu_{-1}(x) \right).
\end{aligned}$$

Therefore, we have the following theorem.

Theorem 5. *For $n, k \in \mathbb{Z}_+$ with $n > k+1$, we have*

$$\begin{aligned}
&\int_{\mathbb{Z}_p} q^{hx} w^x B_{k,n}(x) d\mu_{-1}(x) \\
&= \binom{n}{k} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \left(2 + wq^h \frac{G^{(h)}_{n-l+1,q^{-1},w^{-1}}}{n-l+1} \right).
\end{aligned}$$

By (2.5) and Theorem 5, we have the following theorem.

Theorem 6. *Let $n, k \in \mathbb{Z}_+$ with $n > k+1$. Then we have*

$$\begin{aligned}
&\sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l \frac{G^{(h)}_{k+l+1,q,w}}{k+l+1} \\
&= \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \left(2 + wq^h \frac{G^{(h)}_{n-l+1,q^{-1},w^{-1}}}{n-l+1} \right).
\end{aligned}$$

Let $n_1, n_2, k \in \mathbb{Z}_+$ with $n_1 + n_2 > 2k+1$. Then we get

$$\begin{aligned}
&\int_{\mathbb{Z}_p} B_{k,n_1}(x) B_{k,n_2}(x) q^{hx} w^x d\mu_{-1}(x) \\
&= \binom{n_1}{k}\binom{n_2}{k} \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{l+2k} \int_{\mathbb{Z}_p} q^{hx} w^x (1-x)^{n_1+n_2-l} d\mu_{-1}(x) \\
&= \binom{n_1}{k}\binom{n_2}{k} \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{l+2k} \frac{1}{wq^h} \int_{\mathbb{Z}_p} q^{-hx} w^{-x} (x+2)^{n_1+n_2-l} d\mu_{-1}(x) \\
&= \binom{n_1}{k}\binom{n_2}{k} \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{l+2k} \left(2 + wq^h \int_{\mathbb{Z}_p} q^{-hx} w^{-x} x^{n_1+n_2-l} d\mu_{-1}(x)\right).
\end{aligned}$$

Therefore, we obtain the following theorem.

Theorem 7. *For $n_1, n_2\, k \in \mathbb{Z}_+$ with $n_1 + n_2 > 2k+1$, we have*

$$\begin{aligned}
&\int_{\mathbb{Z}_p} B_{k,n_1}(x) B_{k,n_2}(x) q^{hx} w^x d\mu_{-1}(x) \\
&= \binom{n_1}{k}\binom{n_2}{k} \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{l+2k} \left(2 + wq^h \frac{G^{(h)}_{n_1+n_2-l+1,q^{-1},w^{-1}}}{n_1+n_2-l+1}\right).
\end{aligned}$$

By simple calculation, we easily see that

$$\begin{aligned}
&\int_{\mathbb{Z}_p} B_{k,n_1}(x) B_{k,n_2}(x) q^{hx} w^x d\mu_{-1}(x) \\
&= \binom{n_1}{k}\binom{n_2}{k} \sum_{l=0}^{n_1+n_2-2k} (-1)^l \binom{n_1+n_2-2k}{l} \int_{\mathbb{Z}_p} q^{hx} w^x x^{l+2k} d\mu_{-1}(x) \\
&= \binom{n_1}{k}\binom{n_2}{k} \sum_{l=0}^{n_1+n_2-2k} (-1)^l \binom{n_1+n_2-2k}{l} \frac{G^{(h)}_{l+2k+1,q,w}}{l+2k+1}.
\end{aligned}$$

where $n_1, n_2, k \in \mathbb{Z}_+$.

Therefore, by the above equation and Theorem 7, we obtain the following theorem.

Theorem 8. *Let $n_1, n_2, k \in \mathbb{Z}_+$ with $n_1+n_2 > 2k+1$. Then we have*

$$\sum_{l=0}^{2k} \binom{2k}{l} (-1)^{l+2k} \left(2 + wq^h \frac{G^{(h)}_{n_1+n_2-l+1,q^{-1},w^{-1}}}{n_1+n_2-l+1} \right)$$
$$= \sum_{l=0}^{n_1+n_2-2k} (-1)^l \binom{n_1+n_2-2k}{l} \frac{G^{(h)}_{l+2k+1,q,w}}{l+2k+1}.$$

For $n_1, n_2, n_3, k \in \mathbb{Z}_+$ with $n_1+n_2+n_3 > 3k+1$, by the symmetry of Bernstein polynomials, we see that

$$\int_{\mathbb{Z}_p} B_{k,n_1}(x) B_{k,n_2}(x) B_{k,n_3}(x) q^{hx} w^x d\mu_{-1}(x)$$
$$= \binom{n_1}{k}\binom{n_2}{k}\binom{n_3}{k} \sum_{l=0}^{3k} \binom{3k}{l} (-1)^{l+3k} \int_{\mathbb{Z}_p} q^{hx} w^x (1-x)^{n_1+n_2+n_3-l} d\mu_{-1}(x)$$
$$= \binom{n_1}{k}\binom{n_2}{k}\binom{n_3}{k} \sum_{l=0}^{3k} \binom{3k}{l} (-1)^{l+3k} \frac{1}{wq^h}$$
$$\int_{\mathbb{Z}_p} q^{-hx} w^{-x} (x+2)^{n_1+n_2+n_3-l} d\mu_{-1}(x)$$
$$= \binom{n_1}{k}\binom{n_2}{k}\binom{n_3}{k} \sum_{l=0}^{3k} \binom{3k}{l} (-1)^{l+3k}$$
$$\left(2 + wq^h \int_{\mathbb{Z}_p} q^{-hx} w^{-x} x^{n_1+n_2+n_3-l} d\mu_{-1}(x) \right).$$

Therefore, we have the following theorem.

Theorem 9. *For $n_1, n_2, n_2, k \in \mathbb{Z}_+$ with $n_1+n_2+n_3 > 3k+1$, we have*

$$\int_{\mathbb{Z}_p} B_{k,n_1}(x) B_{k,n_2}(x) B_{k,n_3}(x) q^{hx} w^x d\mu_{-1}(x)$$
$$= \binom{n_1}{k}\binom{n_2}{k}\binom{n_3}{k} \sum_{l=0}^{3k} \binom{3k}{l} (-1)^{l+3k} \left(2 + wq^h \frac{G^{(h)}_{n_1+n_2+n_3-l,q^{-1},w^{-1}}}{n_1+n_2+n_3-l+1} \right).$$

In the same manner, multiplication of three Bernstein polynomials can be given by the following relation:

$$\int_{\mathbb{Z}_p} B_{k,n_1}(x)B_{k,n_2}(x)B_{k,n_3}(x)q^{hx}w^x d\mu_{-1}(x)$$
$$= \binom{n_1}{k}\binom{n_2}{k}\binom{n_3}{k}\sum_{l=0}^{n_1+n_2+n_3-3k}(-1)^l\binom{n_1+n_2+n_3-3k}{l}$$
$$\int_{\mathbb{Z}_p} q^{hx}w^x x^{l+3k} d\mu_{-1}(x)$$
$$= \binom{n_1}{k}\binom{n_2}{k}\binom{n_3}{k}\sum_{l=0}^{n_1+n_2+n_3-3k}(-1)^l\binom{n_1+n_2+n_3-3k}{l}\frac{G^{(h)}_{l+3k+1,q,w}}{l+3k+1},$$

where $n_1, n_2, n_3, k \in \mathbb{Z}_+$ with $n_1+n_2+n_3 > 3k+1$.

Therefore, by Theorem 9, we obtain the following theorem.

Theorem 10. *Let $n_1, n_2, n_3, k \in \mathbb{Z}_+$ with $n_1+n_2+n_3 > 3k+1$. Then we have*

$$\sum_{l=0}^{3k}\binom{3k}{l}(-1)^{l+3k}\left(2+wq^h\frac{G^{(h)}_{n_1+n_2+n_3-l+1,q^{-1},w^{-1}}}{n_1+n_2+n_3-l+1}\right)$$
$$= \sum_{l=0}^{n_1+n_2+n_3-3k}(-1)^l\binom{n_1+n_2+n_3-3k}{l}\frac{G^{(h)}_{l+3k+1,q,w}}{l+3k+1}.$$

Using the above theorem and mathematical induction, we have the following theorem.

Theorem 11. *Let $m \in \mathbb{N}$. For $n_1, n_2, \ldots, n_m, k \in \mathbb{Z}_+$ with $n_1+\cdots+n_m > mk+1$, the multiplication of the sequence of Bernstein polynomials $B_{k,n_1}(x), \ldots, B_{k,n_m}(x)$ with different degrees under fermionic p-adic invariant integral on $\mathbb{Z}_p$ can be given as*

$$\int_{\mathbb{Z}_p}\left(\prod_{i=1}^{m} B_{k,n_i}(x)\right)q^{hx}w^x d\mu_{-1}(x)$$
$$= \left(\prod_{i=1}^{m}\binom{n_i}{k}\right)\sum_{l=0}^{mk}\binom{mk}{l}(-1)^{l+mk}\left(2+wq^h\frac{G^{(h)}_{n_1+\cdots+n_m-l+1,q^{-1},w^{-1}}}{n_1+\cdots+n_m-l+1}\right).$$

We also easily see that

$$\begin{aligned}&\int_{\mathbb{Z}_p}\left(\prod_{i=1}^{m}B_{k,n_i}(x)\right)q^{hx}w^x d\mu_{-1}(x)\\&=\left(\prod_{i=1}^{m}\binom{n_i}{k}\right)\sum_{l=0}^{n_1+\cdots+n_m-mk}\binom{n_1+\cdots+n_m-mk}{l}(-1)^l\frac{G^{(h)}_{l+mk+1,q,w}}{l+mk+1}.\end{aligned}$$

By using Theorem 11 and the above equation, we have the following corollary.

Corollary 12. *Let* $m\in\mathbb{N}$. *For* $n_1,n_2,\ldots,n_m,k\in\mathbb{Z}_+$ *with* $n_1+\cdots+n_m>mk+1$, *we have*

$$\begin{aligned}&\sum_{l=0}^{mk}\binom{mk}{l}(-1)^{l+mk}\left(2+wq^h\frac{G^{(h)}_{n_1+\cdots+n_m-l+1,q^{-1},w^{-1}}}{n_1+\cdots+n_m-l+1}\right)\\&=\sum_{l=0}^{n_1+\cdots+n_m-mk}\binom{n_1+\cdots+n_m-mk}{l}(-1)^l\frac{G^{(h)}_{l+mk+1,q,w}}{l+mk+1}.\end{aligned}$$

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