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ON COMPACTNESS AND CONNECTEDNESS OF INVERTIBLE FUZZY TOPOLOGICAL SPACES

Anjaly Jose[1], Sunil C. Mathew[2 §]

[1]Department of Mathematics
St. Joseph's College Devagiri
Calicut (Dt), 673008, Kerala, INDIA

[2]Department of Mathematics
St. Thomas College Palai
Kottayam (Dt), 686574, Kerala, INDIA

Abstract: In this paper we study the effect of invertibility on compactness and connectedness of a fuzzy topological space. We have obtained certain conditions for an invertible fuzzy topological space to be compact and fuzzy connected. If the invertible subspace is compact, then it is proved that the parent fuzzy topological space is also compact. It is also proved that in a type 2 completely invertible fuzzy topological space, fuzzy connectedness and fuzzy super connectedness are same.

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1. Introduction

The concept of invertible topological spaces was introduced by Doyle and Hocking [2] in 1961. In [8], Mathew extended the concept of invertibility to fuzzy topological spaces and examined the basic nature of such spaces. Later, we studied invertible fuzzy topological spaces in [10] and [9] and based on the structure of inverting pairs, introduced different types of invertible fuzzy topological spaces and derived certain characterizing properties in [11].

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[§]Correspondence author

In this paper we study the role of invertibility on compactness and connectedness of a fuzzy topological space. It is proved that every closed proper fuzzy subset of a type 2 completely invertible fuzzy topological space is compact. Also the compactness, α-compactness and α^*-compactness of an invertible subspace are carried over to the parent fuzzy topological space. Further, in a completely invertible strongly compact fuzzy topological space every open fuzzy subset contains a compact fuzzy subset.

Also a type 2 completely invertible fuzzy topological space is fuzzy connected if and only if it is fuzzy super connected and a finite completely invertible c-fts is not fuzzy connected. It is also observed that if a completely invertible fuzzy topological space (X, F) contains a fuzzy connected open subset, then (X, F) has at most two components. Finally if a completely invertible FT_1 fuzzy topological space contains a non-empty fuzzy connected open subset, then the space itself is fuzzy connected.

2. Preliminaries

In this section we include certain definitions and known results needed for the subsequent development of the study. Throughout this paper X stands for a non empty set with at least two elements and I stands for the unit interval $[0, 1]$. For any fuzzy subset g of X, by $\not{C}(g)$ we mean the complement of g in X. Fuzzy subset with a constant degree of membership α is denoted by $\underline{\alpha}$. The closure of a fuzzy subset g is denoted by $\overline{g}$. The identity map on X is denoted by e. A fuzzy subset g of X is said to be proper if $g \neq \underline{0}, \underline{1}$.

Definition 1. (see [1]) Let X and Y be two sets and let $\theta : X \to Y$ be a function. Then for any fuzzy subset g of X, $\theta(g)$ is a fuzzy subset of Y defined by

$$\theta(g)(y) = \begin{cases} \sup\{g(x) : x \in X, \ \theta(x) = y\}; & \theta^{-1}(y) \neq \phi \\ 0; & \theta^{-1}(y) = \phi \end{cases}$$

For a fuzzy subset h of Y, we define $\theta^{-1}(h)(x) = h(\theta(x)), \forall x \in X$.

Obviously $\theta^{-1}(h)$ is a fuzzy subset of X.

Definition 2. (see [6]) The fuzzy subset x_λ of X, with $x \in X$ and $0 < \lambda \leq 1$ defined by

$$x_\lambda(y) = \begin{cases} \lambda; & y = x \\ 0; & \text{otherwise} \end{cases}$$

is called a fuzzy point in X with support x and value λ.

Definition 3. (see [6]) A fuzzy point x_λ is called strong if $\lambda > \frac{1}{2}$.

Definition 4. (see [13]) Let f and g be fuzzy subsets of X. Then f is said to be quasi-coincident with g, denoted by $f\hat{q}g$ if there exists an $x \in X$ such that $f(x) + g(x) > 1$.

Definition 5. (see [6]) Let (X, F) be an fts. A fuzzy subset g of X is said to be nearly crisp if $\overline{g} \wedge \not{C}(\overline{g}) = \underline{0}$.

Definition 6. (see [7]) Let (X, F) be an fts. Then it is called a c-fts if the members of F are characteristic functions.

Definition 7. (see [5]) An fts (X, F) is said to be an FT_1 fts if for any two distinct fuzzy points x_λ and y_γ there exist open fuzzy subsets f and g such that $x_\lambda \in f \leq \not{C}(y_\gamma)$ and $y_\gamma \in g \leq \not{C}(x_\lambda)$.

Definition 8. (see [5]) An fts (X, F) is called regular if for any fuzzy point x_λ in X and any closed fuzzy subset h of X such that $x_\lambda \in \not{C}(h)$, there exist $f,\ g \in F$ such that $x_\lambda \in f$ and $h \leq g$ with $f \leq \not{C}(g)$.

Definition 9. (see [5]) An fts (X, F) is called normal if for every two closed fuzzy subsets h_1 and h_2 such that $h_1 \leq \not{C}(h_2)$, there exist two open fuzzy subsets f_1 and f_2 such that $h_1 \leq f_1$ and $h_2 \leq f_2$ with $f_1 \leq \not{C}(f_2)$.

Definition 10. (see [8]) An fts (X, F) is said to be invertible with respect to a proper open fuzzy subset g if there is a homeomorphism θ of (X, F) such that $\theta(\not{C}(g)) \leq g$. This homeomorphism θ is called an inverting map for g and g is said to be an inverting fuzzy subset of (X, F).

If an fts (X, F) is invertible, then there exists an inverting fuzzy subset g and an inverting map θ of (X, F). This g and θ together called an inverting pair of (X, F). Clearly there can be different inverting pairs for an invertible fts.

Definition 11. (see [8]) An fts (X, F) is said to be completely invertible if for every proper open fuzzy subset g of X, there is a homeomorphism θ of (X, F) such that $\theta(\not{C}(g)) \leq g$.

When we say (X, F) is invertible with respect to g, it is understood that $g \in F$. Further, an fts (X, F) is said to be invertible if it is invertible with respect to at least one proper open fuzzy subset of X. It should be noted that for a completely invertible fts every proper open fuzzy subset is an inverting fuzzy subset.

Definition 12. rm (see [5]) Let (X, F) be any fts and let M be any crisp subset of X. Then the induced fuzzy topology or relative fuzzy topology for M

is given by $F_M = \{M \wedge f : f \in F\}$ and the pair (M, F_M) is called a subspace of (X, F).

Theorem 13. (see [9]) *Let (X, F) be a completely invertible fts and A be a non-empty open crisp subset of X. If the subspace (A, F_A) is normal, then (X, F) is also normal.*

Definition 14. (see [11]) An invertible fts (X, F) is said to be type 1 if identity is an inverting map.

Definition 15. (see [11]) An invertible fts (X, F) is said to be type 2 if identity is an inverting map for all the inverting fuzzy subsets.

Theorem 16. (see [11]) *Let (X, F) be an invertible fts. Then (g, e) is an inverting pair of (X, F) if and only if $\underline{\frac{1}{2}} \leq g$.*

Theorem 17. (see [11]) *An fts (X, F) is type 2 completely invertible if and only if $\underline{\frac{1}{2}} \leq g$ for every $g \neq \underline{0}, \in F$.*

Theorem 18. (see [10]) *Let (X, F) be an fts invertible with respect to g where X is finite. Then $|X| \leq 2|supp\ g|$.*

Theorem 19. (see [10]) *An fts (X, F) is completely invertible iff for each closed proper fuzzy subset h and each $g \neq \underline{0}, \in F$, there is a homeomorphism θ of (X, F) such that $\theta(h) \leq g$.*

Theorem 20. (see [11]) *Let θ be a homeomorphism of an fts (X, F) and $g \neq \underline{0}, \underline{1}, \in F$. Then the invertibility of (X, F) with respect to the following pairs are equivalent.*

(i) (g, θ) *(ii)* (g, θ^{-1})

(iii) $(\theta^{-1}(g), \theta^{-1})$

(iv) $(\theta(g), \theta)$

(v) $(\theta(g), \theta^{-1})$

(vi) $(\theta^{-1}(g), \theta)$

Theorem 21. (see [11]) *Let (X, F) be an invertible fts and (g, θ) be an inverting pair of (X, F). Then the following statements are equivalent.*

(i) g and $\theta(g)$ are not quasi-coincident.

(ii) $\theta(g) = \not{C}(g)$.

Notations 22. Let (X, F) be an fts. Then for each $g \in F$ and $\alpha \in I$, let $S_\alpha(g) = \{x \in X : g(x) = \alpha\}$. If f and g are fuzzy subsets of X, then $f + g$ is the function, $f + g : X \to [0, 2]$ such that $(f + g)(x) = f(x) + g(x)$ for all $x \in X$.

3. Invertibility and Compactness

Definition 23. (see [1]) A family $\mathcal{U} = \{U_i : i \in \Delta\}$ of open fuzzy subsets of a fuzzy topological space (X, F) is an open covering of a fuzzy subset f if $f \leq \bigvee_{i \in \Delta} U_i$. A subcovering of an open covering $\mathcal{U}$ of f is a subfamily of $\mathcal{U}$ which is still an open covering of f.

Definition 24. (see [15]) A fuzzy subset g of a fuzzy topological space (X, F) is compact if each open covering of g has a finite subcovering. If $\underline{1}$ is compact, then (X, F) is said to be a compact fts.

Theorem 25. (see [15]) *A closed crisp subset of a compact fuzzy subset in a fuzzy topological space is compact.*

Definition 26. (see [15]) A fuzzy topological space (X, F) is strongly compact if every closed fuzzy subset of X is compact.

Theorem 27. (see [15]) *A regular and strongly compact fuzzy topological space is normal.*

Definition 28. (see [4]) Let (X, F) be an fts and let $\alpha \in I$. A collection $G \subset F$ will be called an α-shading (resp. α^*-shading) of X if, for each $x \in X$ there exists a $g \in G$ with $g(x) > \alpha$ (resp. $g(x) \geq \alpha$). A subcollection H of an α-shading (resp. α^*-shading) G of X that is also an α-shading (resp. α^*-shading) of G is called an α-subshading (resp. α^*-subshading) of G. (X, F) is called α-compact (resp. α^*-compact) if each α-shading (resp. α^*-shading) of X has a finite α-subshading (resp. α^*-subshading).

Theorem 29. (see [4]) *Let B be a closed crisp subset of an fts (X, F). If X is α-compact (resp. α^*-compact), then B is α-compact (resp. α^*-compact) as a subspace of (X, F).*

Theorem 30. (see [1]) *The continuous image of a compact fts is compact.*

Theorem 31. *The continuous image of an α-compact (resp. α^*-compact) fts is α-compact (resp. α^*-compact).*

Proof. Let (X, F) be an α-compact (resp. α^*-compact) space and $\theta : X \to Y$ be a continuous onto function from (X, F) to (Y, G). Let U be an α-shading (resp. α-shading) for Y. Then $\theta^{-1}(U)$ is an α-shading (resp. α^*-shading) for X. Since X is α-compact (resp. α-compact), it has a finite α-subshading (resp. α^*-subshading), say V. Then $\theta(V)$ is a finite α-subshading (resp. α^*-subshading) of U so that (Y, G) is α-compact (resp. α^*-compact). $\square$

Example 32. A type 2 completely invertible fts need not be compact. Let X be the set of all natural numbers. Then (X, F) where $F = \{\underline{0},\ \underline{\alpha} :\ \alpha \in [\frac{4}{5}, 1]\}$ is a type 2 completely invertible fts. The family $\{\underline{\alpha} :\ \alpha \in [\frac{8}{9}, 1)\}$ is an open cover of $\underline{1}$, but it has no finite sub cover. Hence $\underline{1}$ is not compact. But all other closed fuzzy subsets are compact as guaranteed by the following theorem.

Theorem 33. *Every closed fuzzy subset other than $\underline{1}$ of a type 2 completely invertible fts is compact.*

Proof. Let (X, F) be a type 2 completely invertible fts and $g \neq \underline{1}$, be any closed fuzzy subset of (X, F). From Theorem 17, it follows that $g \leq \underline{\frac{1}{2}}$. Let U be any open cover of g. Since $\underline{0}$ is always compact, we may assume that $g \neq \underline{0}$. Then again by Theorem 17, any $f \neq \underline{0}, \in U$ is such that $f \geq \underline{\frac{1}{2}}$ and hence $\{f\}$ covers g so that g is compact. □

Consequently under type 2 complete invertibility the distinction between compactness and strong compactness disappears.

Corollary 34. *If a compact fts is type 2 completely invertible, then it is strongly compact.*

Example 35. A strongly compact fts need not even be invertible. Let X be the set of all rational numbers. Then (X, F) where $F = \{\underline{0}, \underline{1},\ \underline{\alpha} :\ \alpha \in [\frac{1}{5}, \frac{1}{3}]\}$ is an fts. Clearly (X, F) is strongly compact, but is not invertible.

Theorem 36. *A type 2 completely invertible fts is α-compact (resp. α^*-compact) for each $\alpha \in [0, \frac{1}{2})$ (resp. $\alpha \in [0, \frac{1}{2}]$).*

Proof. Follows from Theorem 17. □

Remark 37. Converse of the above theorem is not true. Consider the fts (X, F) in Example 35. Clearly (X, F) is α-compact (resp. α^*-compact) for each $\alpha \in [\frac{1}{3}, 1)$ (resp. $\alpha \in [\frac{1}{3}, 1]$), but (X, F) is not invertible.

In [14] it is shown that an fts (X, F) containing an invertible fuzzy subset A such that $\overline{A}$ is compact, need not be compact. But the following theorem shows that an fts (X, F) containing an invertible crisp subset A such that $\overline{A}$ is compact, is compact.

Theorem 38. *If an fts (X, F) contains an invertible crisp subset A such that $(\overline{A}, F_{\overline{A}})$ is compact, then X is compact.*

Proof. Since A is an inverting subset, there exists a homeomorphism θ of (X, F) such that $\theta(\not{C}(A)) \leq A$. Since $\theta(\not{C}(A))$ is a closed crisp subset of $\overline{A}$, it is compact by Theorem 25. Consequently $\not C(A)$ is compact. Since $X = \overline{A} \vee \not C(A)$ it follows that (X, F) is compact. $\square$

Theorem 39. *If an fts (X, F) contains an invertible nearly crisp fuzzy subset A such that $(\overline{A}, F_{\overline{A}})$ is strongly compact, then X is compact.*

Proof. Let (A, θ) be an inverting pair of (X, F) so that $\theta(\not C(A)) \leq A$. Since $\theta(\not C(A))$ is a closed fuzzy subset of the strongly compact fts $(\overline{A}, F_{\overline{A}})$, it is compact. Consequently $\not C(A)$ is compact. Since $X = \overline{A} \vee \not C(A)$ it follows that (X, F) is compact. $\square$

Theorem 40. *If a completely invertible fts (X, F) has an open crisp subset A such that (A, F_A) is strongly compact and regular, then X is normal.*

Proof. Since (A, F_A) is strongly compact and regular, it is normal by Theorem 27. Now the result follows from Theorem 13. $\square$

Theorem 41. *If an fts (X, F) contains an invertible crisp subset A such that $(\overline{A}, F_{\overline{A}})$ is α-compact (resp α-compact), then X is α-compact (resp. α^*-compact).*

Proof. Let $\mathcal{U}$ be a finite α-shading (resp. α^*-shading) of A. Since A is an inverting subset, there exists a homeomorphism θ of (X, F) such that $\theta(\not C(A)) \leq A$. Since $\theta(\not C(A))$ is a closed crisp subset of $\overline{A}$, it is α-compact (resp. α^*-compact) by Theorem 29. Consequently $\not C(A)$ is α-compact (resp. α^*-compact). Since $X = \overline{A} \vee \not C(A)$ it follows that (X, F) is α-compact (resp. α^*-compact). $\square$

Remark 42. Converses of Theorem 38 and Theorem 41 are not true. For example, let X be the set of all real numbers and define $G = \{f \in I^X : f(x) = 1$ for some $x \in X$ and there exists an $\epsilon > 0$ such that $f(y) = 1$, for all $y \in (x - \epsilon, x + \epsilon) = B$ and $f(x) = 0, \forall x \in \not C(B)\}$. Let F be the fuzzy topology generated by $G \cup \{\frac{1}{2}\}$. Let $Y = [0, 1]$ and $F_Y = H$ be the subspace fuzzy topology on Y. Clearly (Y, F_Y) is compact and α- compact (resp. α^*-compact) for all $\alpha \in [0, 1]$. Let $A = [0, \frac{2}{3})$. Then A is an invertible fuzzy subset of (Y, H), but (A, H_A) is not compact. Also (A, H_A) is not α-compact(resp. α^*-compact) for any $\alpha \in [0, 1)$ (resp. $(0, 1]$).

The following example shows that compactness, α-compactness and α^*-compactness of $(\overline{A}, F_{\overline{A}})$ need not be transferred to (X, F) if (X, F) is not invertible with respect to A.

Example 43. Let X be the set of all natural numbers. Let $A, B \in I^X$ by

$$A(x) = \begin{cases} 1; & x \text{ is odd} \\ 0; & \text{otherwise} \end{cases}$$
$$B(x) = \begin{cases} 1; & x \text{ is even} \\ 0; & \text{otherwise} \end{cases}$$

For each $\alpha \in (0,1]$, let $f_\alpha \in I^X$, defined by

$$f_\alpha(x) = \begin{cases} 1; & x \text{ is odd} \\ \alpha; & \text{otherwise} \end{cases}$$

Let F be the fuzzy topology generated by the collection $\{A, B, \{f_\alpha : \alpha \in (0,1]\}, \{x_\lambda : \lambda = 1 \text{ and } x \text{ is even}\}\}$. Here $\overline{A} = A$ and clearly $(\overline{A}, F_{\overline{A}})$ is compact, α-compact, and α^*-compact for all $\alpha \in [0,1]$, but (X, F) is not compact. Also (X, F) is not α-compact (resp α^*-compact) for any $\alpha \in [0,1)$ (resp. $(0,1]$).

Theorem 44. *In a completely invertible strongly compact fts every open fuzzy subset contains a compact fuzzy subset.*

Proof. Since a closed fuzzy subset in a strongly compact fts is compact, the result follows from Theorem 19 and Theorem 30. □

4. Invertibility and Connectedness

Definition 45. (see [3]) An fts (X, F) is said to be fuzzy connected if it has no proper clopen fuzzy subset.

Theorem 46. (see [3]) *An fts (X, F) is fuzzy connected iff it has no non empty open fuzzy subsets f and g such that $f + g = \underline{1}$.*

Definition 47. (see [3]) Let (X, F) be an fts, then $A \subset X$ is said to be fuzzy connected subset if (A, F_A) is a fuzzy connected space.

Theorem 48. (see [3]) *If (X, F) is an fts and A is a fuzzy connected subset of X, and f and g are non empty open fuzzy subsets of X such that $f + g = \underline{1}$, then either $f \wedge A = \underline{1}_A$ or $g \wedge A = \underline{1}_A$.*

Definition 49. (see [12]) Let (X, F) be an fts and $f \in I^X$, f is called a fuzzy connected component of (X, F), if f is a maximally connected fuzzy subset of (X, F), i.e. for $g \in I^X$, g is fuzzy connected and $g \geq f \implies g = f$.

Definition 50. (see [3]) An fts (X, F) is fuzzy super connected if the closure of every non empty open fuzzy subset of X is $\underline{1}$.

Theorem 51. (see [3]) *A fuzzy super connected fts is fuzzy connected.*

Theorem 52. *A type 2 completely invertible fts is fuzzy super connected iff it is fuzzy connected.*

Proof. The necessary part follows from Theorem 51.

Sufficiency: Suppose (X, F) is fuzzy connected. Then (X, F) has no proper clopen fuzzy subset so that $\underline{\frac{1}{2}} \notin F$. Since (X, F) is type 2 completely invertible, by Theorem 17, $\underline{\frac{1}{2}} \le g$ for every $g \neq \underline{0}, \in F$. Thus for every $g \neq \underline{0}, \underline{\frac{1}{2}}, \in F$, $\overline{g} = \underline{1}$. Hence $\underline{\frac{1}{2}} \notin F \implies (X, F)$ is fuzzy super connected. □

Remark 53. In a type 2 invertible fts fuzzy connectedness need not imply fuzzy super connectedness. Let X be the set of real numbers and $F = \{\underline{1}, \underline{0}\} \cup G \cup H$ where $G = \{\underline{\alpha} \in I^X : \alpha \in [\frac{2}{3}, \frac{4}{5}]\}$ and $H = \{\underline{\beta} \in I^X : \beta \in [\frac{1}{7}, \frac{1}{6}]\}$. Then clearly (X, F) is a type 2 invertible fts and is fuzzy connected. Now consider $g = \underline{\frac{1}{6}}$. Then $\overline{g} = \underline{\frac{1}{5}}$, so that (X, F) is not fuzzy super connected.

Theorem 54. (see [11]) *Let (X, F) be an invertible fts and (g, θ) be an inverting pair of (X, F). If g and $\theta(g)$ are not quasi-coincident then (X, F) is not fuzzy connected.*

Theorem 55. *A finite c-fts (X, F) is completely invertible iff $|X|$ is even and $F = \{\underline{0}, \underline{1}, f, \not{C}(f)\}$.*

Proof. Suppose that (X, F) is completely invertible. Let (f, θ) be an inverting pair of (X, F). Then by Theorem 18, $|supp\ f| \ge \frac{|X|}{2}$. Let $g_0 = f$ and $\theta_0 = \theta$. Define $g_1 = g_0 \wedge \theta_0(g_0)$. Clearly $g_1 \in F$ and $|supp\ g_1| < |supp\ g_0|$. Let θ_1 be an inverting map of g_1. Let $g_2 = g_1 \wedge \theta_1(g_1)$. Clearly $g_2 \in F$ and $|supp\ g_2| < |supp\ g_1|$. Continuing this, after a certain stage there exists a $g_n \neq \underline{0}, \in F$ where $n \in N$ and $g_n = g_{n-1} \wedge \theta_{n-1}(g_{n-1})$ such that $|supp\ g_n| \le \frac{|X|}{2}$. Since (X, F) is completely invertible, $|supp\ g_n| = \frac{|X|}{2}$. Now consider $k = \theta_n(g_n) \wedge f \in F$, then $|supp\ k| < \frac{|X|}{2}$ so that by Theorem 18, $k = \underline{0}$. Since $\theta_n(g_n) = \not{C}(g_n)$, $k = \underline{0} \implies g_n = f$. Hence $|supp\ f| = \frac{|X|}{2}$. Thus $f \neq \underline{0}, \in F \implies |supp\ f| = \frac{|X|}{2}$.

Now if possible let $g \in F$ and $g \neq f$. Then from above we have $|supp\ g| = \frac{|X|}{2}$. Let $h = g \wedge f$, then $h \in F$ and $|supp\ h| < \frac{|X|}{2}$. Since (X, F) is completely invertible, $h = \underline{0}$ which implies that $g = \not{C}(f)$. Hence $F = \{\underline{0}, \underline{1}, f, \not{C}(f)\}$.

Conversely assume that $|X|$ is even and $F = \{\underline{0}, \underline{1}, f, \not{C}(f)\}$. Let *supp* $f = \{x_1, x_2, ..., x_n\}$ and *supp* $\not{C}(f) = \{y_1, y_2, ..., y_n\}$. Define $\theta : X \to X$ by $\theta(x_i) = y_i$ and $\theta(y_i) = x_i$ where $i = 1, 2, ..., n$. Clearly (f, θ) and $(\not{C}(f), \theta)$ are inverting pairs of (X, F). Consequently (X, F) is completely invertible. □

Theorem 56. *A fuzzy connected finite $c-fts$ is not completely invertible.*

Proof. Follows from Theorems 55 and 54. □

Remark 57. Converse of the above theorem is not true. For example, let X be the set of all real numbers and $F = \{\underline{\alpha};\ \alpha \in [0, 1]\}$. Then (X, F) is an fts which is not completely invertible, but since $\underline{\frac{1}{2}} \in F$, (X, F) is not fuzzy connected.

Theorem 58. *The continuous image of a fuzzy connected fts is fuzzy connected.*

Proof. Let (X, F) be a fuzzy connected fts and $\theta : X \to Y$ be a continuous onto function from (X, F) to (Y, G). If possible assume that (Y, G) is not fuzzy connected. Then there exists two non empty open fuzzy subsets f and g of Y such that $f + g = \underline{1}$. Hence $\theta^{-1}(f)$ and $\theta^{-1}(g)$ are two non empty open fuzzy subsets of X such that $\theta^{-1}(f) + \theta^{-1}(g) = \underline{1}$ so that (X, F) is not fuzzy connected, a contradiction. □

Theorem 59. *Let A be a fuzzy connected open subset of a completely invertible fts (X, F), which is not fuzzy connected. Then A and $\not{C}(A)$ are the homeomorphic components of (X, F).*

Proof. Let θ be an inverting homeomorphism for A. Then by Theorem 20, $\not{C}(A) \leq \theta(A)$. Since (X, F) is not fuzzy connected, there exists $f, g \neq \underline{0}, \in F$ such that $f + g = \underline{1}$. Let $f' = f \wedge A$ and $g' = g \wedge A$. Then by Theorem 48, one of them say $f' = \underline{1}_A$, so that $A \leq f$. Since $f + g = \underline{1}$, by Theorem 21, $\theta(f) = \not{C}(f) = g$. Now $A \leq f \implies \theta(A) \leq g$ and $g \leq \not{C}(A)$ so that $\theta(A) \leq \not{C}(A)$. Hence $\not{C}(A) = \theta(A)$. Consequently A and $\not{C}(A)$ are the components of (X, F) and they are homeomorphic. □

Theorem 60. *If an FT_1 fts (X, F) with a non-empty fuzzy connected open subset A, is completely invertible, then (X, F) is fuzzy connected.*

Proof. If (X, F) is not fuzzy connected, then by Theorem 59, A and $\not{C}(A)$ are the components of (X, F) and they are homeomorphic. Let $x \in A$, then $B = A \wedge \not{C}(x)$ is open. Let θ be an inverting homeomorphism for B. Then

$\theta(\not{C}(B)) \leq B$ so that $\theta(\not{C}(A)) \leq B$. But then $\theta(\not{C}(A))$ is a component of (X, F) properly contained in A, a contradiction. $\square$

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