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ABOUT k TO k INSERTION WORDS OF STURMIAN WORDS

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Abstract: In an infinite word u we insert a foreign letter steadly with step of length k. The word obtained is called k to k insertion word of u. In this paper, we are interested in studying some combinatorial properties of k to k insertion words of Sturmian words: recurrence, erasure of letter and balance property. We have also given palindrome complexity of these words.

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1. Introduction

Sturmian words are identified among infinite words as aperiodic words which have a minimal complexity. The first significant studies of these words go back to the works of Morse and Hedlund [19, 20]. Over the past three decades, these words have been intensively studied (see Berstel's surveys [3, 4]). The numerous studies have established various characterizations of these words, see [22, 18] for more details. Many authors have also undertaken the study of the generalizations of Sturmian words [8, 9, 2, 17, 14, 4, 11, 12].

Our work falls within the general framework of the study of Sturmian words and their extensions.

In [16], Tapsoba and the author have introduced k to k insertion words of Sturmian words. Here, we are interested in studying some combinatorial properties of these words.

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Durand, Guerziz and Koskas have studied in [13] ternary words u having the following property: for any letter a in u, the word obtained by erasing all a in u is a Sturmian word. They said "u is a word with Sturmian erasures". In this paper, we show that k to k insertion words of Sturmian words are words with Sturmian erasures (see Section 3).

The palindrome complexity, **Pal**, which counts the number of palindromes of given length in the word, is one of the important tool in combinatorial study of infinite words. For instance, it is used to characterize Sturmian words (voir [10]). For a general study of palindrome complexity see [1]. In this paper, we are also interested in studying palindrome complexity of k to k insertion words of Sturmian words (see Section 4).

2. Preliminaries

2.1. Definitions and Notations

In all the sequel, except express mention, the alphabet $\mathcal{A}$ is the binary alphabet $\{a, b\}$. The set of finite words over $\mathcal{A}$ is denoted $\mathcal{A}^*$ and ε is the empty word. For any $u \in \mathcal{A}^*$, $|u|$ denotes the length of u ($|\varepsilon| = 0$) and for each $x \in \mathcal{A}$, $|u|_x$ is the number of occurrences of the letter x in u. A word u of length n written with a single letter x is simply denoted $u = x^n$. The n-th power of a finite word w denoted by w^n is the word corresponding to the concatenation $(www \dots w)$ n times of w. By extension, $w^0 = \varepsilon$.

Let $u = u_1 u_2 \cdots u_n$ be a finite word such that $u_i \in \mathcal{A}$ for all $i \in \{1, 2, \dots, n\}$. The image of u by the reversal map is the word denoted $\overline{u}$ and defined by $\overline{u} = u_n u_{n-1} \cdots u_1$. The word $\overline{u}$ is simply called reversal image of u.

A finite word u is said to be a palindrome if $\overline{u} = u$. For instance, the words ¡¡refer¿¿, ¡¡civic¿¿, ¡¡level¿¿ and ¡¡noon¿¿ are palindromes in English language.

An infinite word is a sequence of letters of $\mathcal{A}$. The set of infinite words over $\mathcal{A}$ is denoted by $\mathcal{A}^\omega$.

A finite word v is a factor of u if there exist two words u_1 and u_2 over $\mathcal{A}$ such that $u = u_1 v u_2$; we also say that u contains v. The factor v is said to be a prefix (resp. a suffix) if u_1 (resp. u_2) is the empty word.

An infinite word w is ultimately periodic if there exist two words u and v such that $w = uv^\omega$ where v^ω is the infinite concatenation of v. It is periodic if u is the empty word. If the infinite word w does not have any of the previous forms, we say that it is aperiodic.

Let u be an infinite word over $\mathcal{A}$. The set of factors of u of length n is written $\mathcal{L}_n(u)$ and the set of all factors of u is denoted by $\mathcal{L}(u)$. The set $\mathcal{L}(u)$ is usually called the language of u.

For all infinite word u over $\mathcal{A}$ we shall write $u = u_0u_1u_2\cdots$, where $u_i \in \mathcal{A}$, $i \geq 0$.

A factor v of length n of an infinite word $u = u_0u_1u_2\cdots$ appears in u in the position k if $v = u_ku_{k+1}\cdots u_{k+n-1}$.

A non empty factor v of an infinite word u is said to be a right (resp. a left) special factor of u if va and vb (resp. av and bv) are factors of u.

An infinite word u is recurrent if each of its factors appears infinitely many times.

The complexity of an infinite word u is the map of $\mathbb{N}$ to $\mathbb{N}^*$ defined by $\mathbf{p}_u(n) = \#\mathcal{L}_n(u)$, where $\#\mathcal{L}_n(u)$ is the cardinality of $\mathcal{L}_n(u)$. For more developments on this map we refer reader to Chap. 4 in [7] for instance.

The palindrome complexity of an infinite word u is the map of $\mathbb{N}$ to $\mathbb{N}^*$ defined by $\mathbf{Pal}(n) = \#\{v \in \mathcal{L}_n(u) : \overline{v} = v\}$.

Definition 2.1. An infinite word $u = u_0u_1u_2\cdots$ is said to be modulo-recurrent if, for all $k \geq 1$, every factor w of u appears in u in every position modulo k, *i.e.*,

$$\forall i \in \{0,\, 1,\, \ldots,\, k-1\}\,,\ \exists l_i \in \mathbb{N} : w = u_{kl_i+i}u_{kl_i+i+1}\cdots u_{kl_i+i+|w|-1}\ .$$

This notion has been introduced in [15] (see also [6]).

A morphism f is a map from $\mathcal{A}^*$ to itself such that $f(uv) = f(u)f(v)$ for all $u,\, v \in \mathcal{A}^*$. A morphism is entirely determined by the knowledge of the images of letters.

It is said that an infinite word u is generated by a morphism f if, there exists a letter $x \in \mathcal{A}$ such that the words x, $f(x)$, $f^2(x)$, $\cdots$, $f^n(x)$, $\cdots$ are longer and longer prefixes of u. We denote $u = f^\omega(x)$.

Let u be an infinite word over an alphabet $\mathcal{A}$ and v a factor of u. The Parikh vector of v is $\chi(v) = \begin{pmatrix} |v|_a \\ |v|_b \end{pmatrix}$.

Definition 2.2. A word u is said to be δ-balanced if for every pair $(v,\, w)$ of factors of u with the same length and every letter a we have:

$$\Big|\, |v|_a - |w|_a \Big| \leq \delta.$$

If a word u is 1-balanced we say that u is balanced. In the opposite case we say that u is unbalanced.

The following property is useful.

Property 2.1. [9] An infinite word u over the alphabet $\mathcal{A}$ is unbalanced if and only if, there exists a unique word t with minimal length such that u contains ata and btb.

2.2. Sturmian Words

We recall in this subsection some basic results on Sturmian words which will be useful in Section 3 and Section 4.

Definition 2.3. An infinite word u over $\mathcal{A}$ is Sturmian if for all $n \geq 0$, $\mathbf{p}(n) = n + 1$.

The most known Sturmian word is the famous Fibonacci word generated by the morphism Φ defined by $\Phi(a) = ab$ and $\Phi(b) = a$. Its first few terms are:

$$F = abaababaabaababaababaabaababaaba \cdots .$$

Every Sturmian word is recurrent. An infinite word u over $\mathcal{A}$ is Sturmian if and only if, for all $n \geq 0$, u possesses a unique right (resp. left) special factor of length n.

The following theorem displays two useful characterizations of Sturmian words.

Theorem 2.2. *Let u be an infinite word over $\mathcal{A}$. The following assertions are equivalent.*

1. *u is Sturmian.*
2. *[9] u is aperiodic and balanced.*
3. *[10]* $\forall n \in \mathbb{N} : \ \mathbf{Pal}(n) = \begin{cases} 1 & \text{if } n \text{ is even} \\ 2 & \text{otherwise} \end{cases}$.

For any Sturmian word u, Coven and Hedlund showed in [9] that:

$$\forall n \geq 1, \ \#\{\chi(v) : v \in L_n(u)\} = 2.$$

Theorem 2.3. *[6] Every Sturmian word is modulo-recurrent.*

For all letter $x \in \{a, b, c\}$, we define the morphism e_x over $\{a, b, c\}^*$ by

$$\forall y \in \{a, b, c\}, \ e_x(y) = \begin{cases} \varepsilon & \text{if } y = x \\ y & \text{if } y \neq x \end{cases}.$$

Remark 2.1. By considering a ternary word u over $\{a, b, c\}$ and $x \in \{a, b, c\}$, $e_x(u)$ is the word obtained from u by erasing the letter x.

Definition 2.4. A ternary word u over $\{a, b, c\}$ is said to be a word with Sturmian erasures if $e_x(u)$ is Sturmian for all $x \in \{a, b, c\}$.

The following theorem is shown in [13].

Theorem 2.4. *Every word with Sturmian erasures is* 2*-balanced.*

3. k to k Insertion Words of Sturmian Words and Erasures of Letters

In all the sequel, we consider $k \geq 1$. Let u be a Sturmian word. The k to k insertion word of u is $\mathcal{I}_k^c(u) = cm_1cm_2cm_3c \cdots cm_ic \cdots$ where $u = m_0m_1m_2 \cdots m_i \cdots$ with $m_i \in L_k(u),\ i \in \mathbb{N}$.

Example. Let us consider the Fibonacci word

$$F = abaababaabaababaababaabaababaaba \cdots \ .$$

We have

$$\mathcal{I}_3^c(F) = cabacabacbaacbaacbabcaabcabacabacabacbaacba \cdots \ .$$

$$\mathcal{I}_4^c(F) = cabaacbabacabaacbabacababcaabacababcaabacaba \cdots \ .$$

The complexity of k to k insertion words of Sturmian words is determined in [16].

Proposition 3.1. *Let u be a Sturmian word.*

1. *If ps is a factor of u such that p and s are words satisfying $|p| + |s| \leq k$ then, pcs is in $\mathcal{I}_k^c(u)$.*

2. *If $pv_1 \cdots v_qs$ is a factor of u such that p, s and v_i are words satisfying $|p| + |s| < k$ and $|v_i| = k$, $i = 1, \cdots, q$ then, $pcv_1c \cdots v_qcs$ is in $\mathcal{I}_k^c(u)$.*

Proof. Let u be a Sturmian word.

Let v be a factor of u such that $|v| \leq k$. Consider p and s two words such that $v = ps$. Thus, since u is modulo-recurrent v appears in u in some position $\equiv -|p| \bmod k$. Therefore, pcs is in $\mathcal{I}_k^c(u)$.

Let v be a factor of u such that $|v| > k$. Consider the words p, s, $v_1, \cdots, v_q$ such that $v = pv_1 \cdots v_qs$, $|p|+|s| < k$ and $|v_i| = k$, $i = 1, \cdots, q$. Then, as previously, v appears in u in some position $\equiv -|p| \bmod k$. Therefore, $pcv_1c \cdots v_qcs$ is in $\mathcal{I}_k^c(u)$. □

A consequence of Proposition 3.1 is the following corollary.

Corollary 3.1. *Every k to k insertion word of any Sturmian word is recurrent.*

Lemma 3.1. *Let u be a Sturmian word. Then, for all $x \in \{a, b\}$ there exists an integer n_x such that*

$$e_x(\mathcal{I}_k^c(u)) = cy^{n_x+\epsilon_1}cy^{n_x+\epsilon_2}cy^{n_x+\epsilon_3}cy^{n_x+\epsilon_4}\cdots$$

where y is a letter such that $\{x, y\} = \{a, b\}$ and $(\epsilon_i)_i$ is a sequence in $\{0, 1\}$.

Proof. Let u be a Sturmian word and $x \in \{a, b\}$. Then

$$\mathcal{I}_k^c(u) = cv_1cv_2cv_3\cdots$$

where $u = v_1v_2v_3\cdots$ with $v_i \in L_k(u)$, $i \geq 1$.

Observe that $e_x(v_i) = y^{|v_i|_y}$, where y is a letter such that

$$\{x, y\} = \{a, b\}.$$

Let us notice that

$$\left\{|v_i|_y : i \geq 0\right\} = \{\chi(v_i) : i \geq 1\}.$$

Now, we have $\{v_i : i \geq 1\} = L_k(u)$ since u is modulo-recurrent. Consequently, $\#\left\{|v_i|_y : i \geq 0\right\} = 2$ because u is Sturmian. Thus, there exists n_x such that $|v_i|_y \in \{n_x, n_x + 1\}$ since u is balanced. Therefore

$$e_x(\mathcal{I}_k^c(u)) = cy^{n_x+\epsilon_1}cy^{n_x+\epsilon_2}cy^{n_x+\epsilon_3}cy^{n_x+\epsilon_4}\cdots$$

where $(\epsilon_i)_i$ is a sequence over $\{0, 1\}$. □

Lemma 3.2. *Let u be a Sturmian word. Then, for all $x \in \{a, b\}$, $e_x(\mathcal{I}_k^c(u))$ is aperiodic.*

Proof. Suppose that $e_x(\mathcal{I}_k^c(u))$ is ultimately periodic. By Corollary 3.1, we know that $\mathcal{I}_k^c(u)$ is recurrent. Hence, $e_x(\mathcal{I}_k^c(u))$ is also recurrent. Consequently, $e_x(\mathcal{I}_k^c(u))$ is periodic. Thus, there exists a word w such that $e_x(\mathcal{I}_k^c(u)) = w^\omega$. Let us write $e_x(\mathcal{I}_k^c(u))$ in the form

$$e_x(\mathcal{I}_k^c(u)) = ct_1ct_2ct_3ct_4\cdots$$

with $t_i = y^{n_x+\epsilon_i}$, $i \geq 1$. Since $|t_i| \in \{n_x, n_x + 1\}$, then there exists an integer q such that w ends by t_q, *i.e.*, $w = ct_1\cdots ct_q$. Therefore, for all $1 \leq i \leq q$ and all

$l \geq 0$, we have $t_{i+lq} = t_i$. Furthermore, t_{i+lq} comes from a factor v_{i+lq} of u of length k. Therefore, in particular, for $i = 1$ we have:

$$\forall l \in \mathbb{N} : |v_{1+lq}|_x = |v_1|_x .$$

Hence,

$$\# \left\{ |v_{1+lq}|_x : l \geq 0 \right\} = 1.$$

Therefore,

$$\# \left\{ \chi\left(v_{1+lq}\right) : l \geq 0 \right\} = 1.$$

Furthermore, we have $\{v_{1+lq} : l \geq 0\} = L_k(u)$ since u is modulo-recurrent. So,

$$\# \left\{ \chi\left(v_{1+lq}\right) : l \geq 0 \right\} = 2.$$

Thus, we have a contradiction. So, $e_x(\mathcal{I}_k^c(u))$ is aperiodic. □

Theorem 3.2. *Let u be a Sturmian word. Then, $\mathcal{I}_k^c(u)$ is a word with Sturmian erasures.*

Proof. Since $e_c\left(\mathcal{I}_k^c(u)\right) = u$, we only need to prove that $e_x\left(\mathcal{I}_k^c(u)\right)$ is Sturmian for all $x \in \{a, b\}$. From Lemma 3.1, we have

$$e_x\left(\mathcal{I}_k^c(u)\right) = cy^{q+\epsilon_1}cy^{q+\epsilon_2}cy^{q+\epsilon_3}cy\cdots$$

where y is a letter such that $\{x, y\} = \{a, b\}$ and $(\epsilon_i)_i \geq 1$ is a sequence over the alphabet $\{0, 1\}$. From Lemma 3.2, $e_x\left(\mathcal{I}_k^c(u)\right)$ is aperiodic. Thus, to prove that $e_x\left(\mathcal{I}_k^c(u)\right)$ is Sturmian, we only have to prove that $e_x\left(\mathcal{I}_k^c(u)\right)$ is balanced. By contradiction, assume that $e_x\left(\mathcal{I}_k^c(u)\right)$ is unbalanced. Then, $\mathcal{I}_k^c(u)$ admits two factors of the form cTc and yTy. This implies that T begins and ends by y^q. If T did not contain c then we would have $T = y^q$ and this is impossible because $cy^{q+2}c$ is not in $e_x\left(\mathcal{I}_k^c(u)\right)$. Therefore, T contains c and can be written in the form $T = y^qT_0y^q$. Thus, the words $cy^qT_0y^qc$ and $cy^{q+1}T_0y^{q+1}c$ are in $e_x\left(\mathcal{I}_k^c(u)\right)$. Hence, we can get back to the factors $cv_icT_1cv_fc$ and $cv'_icT_1cv'_fc$ in $\mathcal{I}_k^c(u)$. Then, by erasing the letter c we obtain the two words v_itv_f and $v'_itv'_f$ as factors of u, where v_i, v_f, v'_i and v'_f are words of common length k such that

$$|v_i|_x = |v_f|_x \text{ and } |v'_i|_x = |v'_f|_x = |v_i|_x + 1.$$

So, taking $x = b$ and $y = a$ we can write

$$v_i = vav_0,\ v_f = w_0aw,\ v'_i = v'bv_0 \text{ and } v'_f = w_0bw'.$$

In other words, av_0tw_0a and bv_0tw_0b are in u. This contradicts the fact that u is Sturmian. Therefore, $e_x\left(\mathcal{I}_k^c(u)\right)$ is balanced. □

Corollary 3.2. *Let u be a Sturmian word. Then, $\mathcal{I}_k^c(u)$ is 2-balanced.*

Proof. It suffices to apply successively Theorem 3.2 and Theorem 2.4. $\square$

4. Palindrome Complexity of k to k Insertion Words of Sturmian Words

It is well known that the language of any Sturmian word u is stable by the reversal map [5], *i.e.*,

$$\forall v \in \mathcal{L}(u) : \overline{v} \in \mathcal{L}(u).$$

Proposition 4.1. *Let u be a Sturmian word. Then, $\mathcal{L}(\mathcal{I}_k^c(u))$ is stable by the reversal map.*

Proof. Let $v \in \mathcal{L}(\mathcal{I}_k^c(u))$.

Case 1. $|v|_c = 0$. In this case, we have $v \in \mathcal{L}(u)$ and $|v| \leq k$. So, we have $\overline{v} \in \mathcal{L}(u)$ and $|\overline{v}| \leq k$. Thus, from Proposition 3.1, taking $\overline{v} = ps$ with $p = \varepsilon$ and $s = \overline{v}$, we deduce that $c\overline{v} \in \mathcal{L}(\mathcal{I}_k^c(u))$. So, $\overline{v} \in \mathcal{L}(\mathcal{I}_k^c(u))$.

Case 2. $|v|_c = 1$. Then, v can be written in the form $v = pcs$ with p, $s \in \mathcal{L}(u)$ such that $|p| \leq k$ and $|s| \leq k$. Hence, we have $w = ps \in \mathcal{L}(u)$. So, $\overline{w} = \overline{s}\,\overline{p} \in \mathcal{L}(u)$ since u is Sturmian. From Proposition 3.1, it follows that $\overline{v} = \overline{s}c\overline{p} \in \mathcal{L}(\mathcal{I}_k^c(u))$.

Case 3. $|v|_c > 1$. Then, v can be written in the form $v = pcv_1 \cdots v_q cs$ with p, s, $v_i \in \mathcal{L}(u)$ such that $|p| \leq k$ and $|s| \leq k$ and $|v_i| = k$, $i = 1, \cdots, q$. We carry on the reasoning similarly in Case 2. $\square$

Lemma 4.1. *Let u be a Sturmian word and P be a palindrome of $\mathcal{I}_k^c(u)$ such that $|P| \geq k+1$. Then, there exists a unique letter x in $\{a, b, c\}$ such that xPx is in $\mathcal{I}_k^c(u)$.*

Proof. First, assume that $Pc \in \mathcal{L}(\mathcal{I}_k^c(u))$. Then, since c succeeds a factor of u of length k, P can be factorized in the form $P = P_1cP_0$, where $P_0 \in \mathcal{L}_k(u)$. So, the letter c is the unique right extension of P since P ends with a factor of u of length k. Since P is a palindrome, we have $P = P_1cP_0 = \overline{P_0}c\overline{P_1}$. So, the letter c is the unique left extension of P since P begins with $\overline{P_0}$, a factor of u of length k. Thus, c is the unique letter such that $cPc \in \mathcal{L}(\mathcal{I}_k^c(u))$.

Now, assume that $Pc \notin \mathcal{L}(\mathcal{I}_k^c(u))$. We have two cases to consider: P is either a right special factor or it is not.

- If P is not a right special factor of u then, P extends by a unique letter $x \in \{a, b\}$. Using Proposition 4.1, we show that x is the unique left extension of P in $\mathcal{I}_k^c(u)$. Thus, x is the unique letter such that $xPx \in \mathcal{L}(\mathcal{I}_k^c(u))$.
- If P is a right special factor of $\mathcal{I}_k^c(u)$, then P extends by Pa and Pb in $\mathcal{I}_k^c(u)$. By Proposition 4.1, it follows that aP and bP are in $\mathcal{I}_k^c(u)$. Furthermore, we observe that $e_c(P)$ is the right special factor of length $|e_c(P)|$ of the Sturmian word $u = e_c(\mathcal{I}_k^c(u))$. So, the unique right special factor of the Sturmian word u of length $|e_c(P)| + 1$ is either $ae_c(P)$ or $be_c(P)$. It follows that, there exists a unique letter x in $\{a, b\}$ such that $xe_c(P)x \in \mathcal{L}(u)$. Now, by Proposition 3.1, we get from $xe_c(P)x \in \mathcal{L}(u)$ that $xPx \in \mathcal{L}(\mathcal{I}_k^c(u))$. □

Theorem 4.2. *Let u be a Sturmian word.*

1. *If k is even, the palindrome complexity of $\mathcal{I}_k^c(u)$, **Pal**, is given by:*

$$\forall n \in \mathbb{N} : \mathbf{Pal}(n) = \begin{cases} 1 & \textit{if} \quad n \textit{ is even}, \ n \leq k+1 \\ 3 & \textit{if} \quad n \ \textit{ is odd}, \ n \leq k+1 \\ 1 & \textit{if} \quad n > k+1 \end{cases}.$$

2. *If k is odd, the palindrome complexity of $\mathcal{I}_k^c(u)$, **Pal**, is given by:*

$$\forall n \in \mathbb{N} : \mathbf{Pal}(n) = \begin{cases} 1 & \textit{if} \quad n \textit{ is even}, \ n \leq k+1 \\ 0 & \textit{if} \quad n \ \textit{ is even}, \ n > k+1 \\ 3 & \textit{if} \quad n \ \textit{ is odd} \end{cases}.$$

Item 2 in this theorem extends Theorem 7 in [16] where $k = 1$.

Proof. Let u be a Sturmian word.

Consider $n \leq k+1$. Then, any factor of $\mathcal{I}_k^c(u)$ of length n contains at most one occurrence of c. So, the palindromes of even length of $\mathcal{I}_k^c(u)$ are those of u whereas the palindromes with odd length of $\mathcal{I}_k^c(u)$ are not only those of u but also those obtained from palindromes of even length of u by inserting the letter c in the middle. So, the palindromes of $\mathcal{I}_k^c(u)$ are either the palindromes of the Sturmian word u and the palindromes containing c in the middle. Since u possesses a unique palindrome if n is even and two otherwise, we deduce that $\mathcal{I}_k^c(u)$ possesses a unique palindrome if n is even and 3 otherwise.

It remains to treat the case $n \geq k+2$.

Case 1. k is even. Let us put $k = 2l_0$.

- For $l = l_0+1$. We know that u possesses a unique palindrome P_0 with (even) length $k = 2l_0$. Then, P_0 admits the letter c as unique right (resp. left) extension in $\mathcal{I}_k^c(u)$ and so cP_0c is a unique palindrome of length $2l_0 + 2$ which extends P_0. Thus, $\mathcal{I}_k^c(u)$ admits a unique palindrome $P = cP_0c$ of length $2l$ from unicity of P_0.

  Assume that u admits a unique palindrome P_1 of (even) length $2l$ for all $l \geq l_0 + 1$. Let us check that it is the same for $l + 1$. From Lemma 4.1, one of the three words aP_1a, bP_1b, cP_1c is in $\mathcal{I}_k^c(u)$. Let us denote it $x_1P_1x_1$. This word is a palindrome since P_1 is a palindrome. Thus, $\mathcal{I}_k^c(u)$ contains at least one palindrome of length $2(l + 1)$.

  Now, consider Q, a palindrome of length $2(l + 1)$ in $\mathcal{I}_k^c(u)$. Then, it can be written in the form $Q = xPx$, where x is a letter and P, a palindrome of $\mathcal{I}_k^c(u)$ of length $2l$. Thus, from the induction hypothesis, we deduce that $P = P_1$. Hence, $Q = xP_1x$. From Lemma 4.1, it follows that $x = x_1$. So, we have $Q = x_1P_1x_1$ and u admits a unique palindrome of length $2(l + 1)$.

- For $l = l_0 + 1$, $\mathcal{I}_k^c(u)$ possesses a unique palindrome of (odd) length $2l + 1$ which is $Q_0c\overline{Q_0}$, where $Q_0\overline{Q_0}$ is the unique palindrome of u of length k. By induction, we show as previously that for all $l \geq l_0 + 1$, $\mathcal{I}_k^c(u)$ possesses a unique palindrome of length $2l + 1$.

Case 2. k is odd. Let us put $k = 2l_0 + 1$.

- For $l = l_0 + 1$. Observe that $\mathcal{I}_k^c(u)$ does not possess any palindrome of (even) length $2l$. It follows that for all $l \geq l_0 + 1$, u does not admit any palindrome of length $2l$.

- For $l = l_0 + 1$. Observe that $\mathcal{I}_k^c(u)$ possesses exactly three palindromes of (odd) length $2l + 1$: cP_1c, cP_2c, $P_0c\overline{P_0}$, where P_1 and P_2 are the two palindromes of length k of u and $P_0\overline{P_0}$ is the unique palindrome of length $k+1$ of u. By induction, we show as previously that $\mathcal{I}_k^c(u)$ possesses exactly three palindromes of (odd) length $2l+1$ for all $l \geq l_0 + 1$. □

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