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A NOTE ON THE TWISTED q-EULER NUMBERS AND POLYNOMIALS WITH WEIGHT (α, β)

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Abstract: In this paper we construct a new type of the twisted q-Euler numbers $E_{n,q,w}^{(\alpha,\beta)}$ and polynomials $E_{n,q,w}^{(\alpha,\beta)}(x)$ with weight (α, β). Some interesting results and relationships are obtained.

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Key Words: Euler numbers and polynomials, twisted Euler numbers and polynomials, twisted (h, q)-Euler numbers and polynomials, twisted q-Euler numbers and polynomials with weight (α, β)

1. Introduction

Recently, many mathematicians have studied in the area of the Euler numbers and polynomials, twisted Euler numbers and polynomials, q-Euler numbers and polynomials, and (h, q)-Euler numbers and polynomials(see [1-12]). These numbers and polynomials possess many interesting properties and arises in many areas of mathematics and physics. In this paper, we construct a new type of q-Euler numbers $E_{n,q,w}^{(\alpha,\beta)}$ and polynomials $E_{n,q,w}^{(\alpha,\beta)}(x)$ with weight (α, β).

Throughout this paper we use the following notations. By $\mathbb{Z}_p$ we denote the ring of p-adic rational integers, $\mathbb{Q}_p$ denotes the field of p-adic rational numbers, $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$, $\mathbb{N}$ denotes the set of

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natural numbers, $\mathbb{Z}$ denotes the ring of rational integers, $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{C}$ denotes the set of complex numbers, and $\mathbb{Z}^+ = \mathbb{N} \cup \{0\}$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of q-extension, q is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ one normally assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we normally assume that $|q-1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$. Throughout this paper we use the notation:

$$[x]_q = \frac{1-q^x}{1-q}, \quad [x]_{-q} = \frac{1-(-q)^x}{1+q} \text{ (cf. [1-11])} . \tag{1.1}$$

Hence, $\lim_{q\to 1}[x] = x$ for any x with $|x|_p \leq 1$ in the present p-adic case. For

$$f \in UD(\mathbb{Z}_p) = \{f | f : \mathbb{Z}_p \to \mathbb{C}_p \text{ is uniformly differentiable function}\},$$

the fermionic p-adic q-integral on $\mathbb{Z}_p$ is defined by Kim as follows:

$$\begin{aligned} I_{-q}(f) &= \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) \\ &= \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x)(-q)^x \text{ (cf. [3-6])} . \end{aligned} \tag{1.2}$$

If we take $f_1(x) = f(x+1)$ in (1.1), then we easily see that

$$qI_{-q}(f_1) + I_{-q}(f) = [2]_q f(0). \tag{1.3}$$

From (1.3), we obtain

$$q^n I_{-q}(f_n) + (-1)^{n-1} I_{-q}(f) = [2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} q^l f(l), \tag{1.4}$$

where $f_n(x) = f(x+n)$(cf. [3-6]).

As well known definition, the Euler polynomials are defined by

$$F(t) = \frac{2}{e^t+1} = e^{Et} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!},$$

$$F(t,x) = \frac{2}{e^t+1} e^{xt} = e^{E(x)t} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \tag{1.5}$$

with the usual convention of replacing $E^n(x)$ by $E_n(x)$.

In the special case, $x = 0$, $E_n(0) = E_n$ are called the n-th Euler numbers(cf. [1-11]).

These numbers and polynomials are interpolated by the Euler zeta function and Hurwitz-type Euler zeta function, respectively.

$$\begin{aligned} \zeta_E(s) &= 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n^s}, \\ \zeta_E(s,x) &= 2\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+x)^s}. \end{aligned} \tag{1.6}$$

Our aim in this paper is to define twisted q-Euler numbers $E_{n,q,w}^{(\alpha,\beta)}$ and polynomials $E_{n,q,w}^{(\alpha,\beta)}(x)$ with weight (α,β). We investigate some properties which are related to twisted q-Euler numbers $E_{n,q,w}^{(\alpha,\beta)}$ and polynomials $E_{n,q,w}^{(\alpha,\beta)}(x)$ with weight (α,β). We also derive the existence of a specific interpolation function which interpolate twisted q-Euler numbers $E_{n,q,w}^{(\alpha,\beta)}$ and polynomials $E_{n,q,w}^{(\alpha,\beta)}(x)$ with weight (α,β) at negative integers.

2. The Twisted q-Euler Numbers and Polynomials with Weight (α, β)

Our primary goal of this section is to define twisted q-Euler numbers $E_{n,q,w}^{(\alpha,\beta)}$ and polynomials $E_{n,q,w}^{(\alpha,\beta)}(x)$ with weight (α,β). We also find generating functions of twisted q-Euler numbers $E_{n,q,w}^{(\alpha,\beta)}$ and polynomials $E_{n,q,w}^{(\alpha,\beta)}(x)$ with weight (α,β).

Let $C_{p^n} = \{w | w^{p^n} = 1\}$ be the cyclic group of order p^n and let

$$T_p = \lim_{n\to\infty} C_{p^n} = C_{p^\infty} = \cup_{n\geq 0} C_{p^n}$$

be the locally constant space. For $w \in T_p$, we denote by $\phi_w : \mathbb{Z}_p \to \mathbb{C}_p$ the locally constant function $x \longmapsto w^x$.

For $\alpha \in \mathbb{Z}$, $\beta \in \mathbb{Z}$, $w \in T_p$, and $q \in \mathbb{C}_p$ with $|1-q|_p \leq 1$, twisted q-Euler numbers $E_{n,q,w}^{(\alpha,\beta)}$ with weight (α,β) are defined by

$$E_{n,q,w}^{(\alpha,\beta)} = \int_{\mathbb{Z}_p} \phi_w(x)[x]_{q^\alpha}^n d\mu_{-q^\beta}(x). \tag{2.1}$$

By using p-adic q-integral on $\mathbb{Z}_p$, we obtain,

$$\begin{aligned}
&\int_{\mathbb{Z}_p} \phi_w(x)[x]_{q^\alpha}^n d\mu_{-q^\beta}(x) \\
&= \lim_{N\to\infty} \frac{1}{[p^N]_{-q^\beta}} \sum_{x=0}^{p^N-1} [x]_{q^\alpha}^n w^x (-1)^x q^{\beta x} \\
&= [2]_{q^\beta} \left(\frac{1}{1-q^\alpha}\right)^n \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{1}{1+wq^{\alpha l+\beta}} \\
&= [2]_{q^\beta} \sum_{m=0}^{\infty} (-1)^m w^m q^{\beta m} [m]_{q^\alpha}^n .
\end{aligned} \tag{2.2}$$

By (2.1) and (2.2), we have

$$E_{n,q,w}^{(\alpha,\beta)} = [2]_{q^\beta} \sum_{m=0}^{\infty} (-1)^m w^m q^{\beta m} [m]_{q^\alpha}^n . \tag{2.3}$$

From the above, we can easily obtain that

$$\begin{aligned}
F_{q,w}^{(\alpha,\beta)}(t) &= \sum_{n=0}^{\infty} E_{n,q,w}^{(\alpha,\beta)} \frac{t^n}{n!} \\
&= [2]_{q^\beta} \sum_{m=0}^{\infty} (-1)^m w^m q^{\beta m} e^{[m]_{q^\alpha} t}.
\end{aligned} \tag{2.4}$$

Thus twisted q-Euler numbers $E_{n,q,w}^{(\alpha,\beta)}$ with weight (α,β) are defined by means of the generating function

$$F_{q,w}^{(\alpha,\beta)}(t) = [2]_{q^\beta} \sum_{m=0}^{\infty} (-1)^m w^m q^{\beta m} e^{[m]_{q^\alpha} t}. \tag{2.5}$$

Using similar method as above, we introduce twisted q-Euler polynomials $E_{n,q,w}^{(\alpha,\beta)}(x)$ with weight (α,β). $E_{n,q,w}^{(\alpha,\beta)}(x)$ with weight (α,β) are defined by

$$E_{n,q,w}^{(\alpha,\beta)}(x) = \int_{\mathbb{Z}_p} \phi_w(x)[x+y]_{q^\alpha}^n d\mu_{-q^\beta}(y). \tag{2.6}$$

By using p-adic q-integral on $\mathbb{Z}_p$, we have

$$E_{n,q,w}^{(\alpha,\beta)}(x) = [2]_{q^\beta} \left(\frac{1}{1-q^\alpha}\right)^n \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{\alpha x l} \frac{1}{1+wq^{\beta+\alpha l}} . \tag{2.7}$$

By using (2.6) and (2.7), we obtain

$$\begin{aligned} F_{q,w}^{(\alpha,\beta)}(t,x) &= \sum_{n=0}^{\infty} E_{n,q,w}^{(\alpha,\beta)}(x)\frac{t^n}{n!} \\ &= [2]_{q^\beta} \sum_{m=0}^{\infty} (-1)^m w^m q^{\beta m} e^{[m+x]_{q^\alpha} t}. \end{aligned} \tag{2.8}$$

Since $[x+y]_{q^\alpha} = [x]_{q^\alpha} + q^{\alpha x}[y]_{q^\alpha}$, we easily obtain that

$$\begin{aligned} E_{n,q,w}^{(\alpha,\beta)}(x) &= \int_{\mathbb{Z}_p} \phi_w(x)[x+y]_{q^\alpha}^n d\mu_{-q^\beta}(y) \\ &= \sum_{k=0}^{n} \binom{n}{k} [x]_{q^\alpha}^{n-k} q^{\alpha x k} E_{k,q,w}^{(\alpha,\beta)} \\ &= \left([x]_q + q^{\alpha x} E_{q,w}^{(\alpha,\beta)}\right)^n \\ &= [2]_{q^\beta} \sum_{m=0}^{\infty} (-1)^m w^m q^{\beta m} [x+m]_{q^\alpha}^n. \end{aligned} \tag{2.9}$$

Observe that, if $q \to 1, w = 1$, then $E_{n,q,w}^{(\alpha)} \to E_n$ and $E_{n,q,w}^{(\alpha)}(x) \to E_n(x)$.

By (2.7), we have the following complement relation:

Theorem 1. *For $n \in \mathbb{Z}_+$, we have*

$$E_{n,q^{-1},w^{-1}}^{(\alpha,\beta)}(1-x) = (-1)^n w q^{\alpha n} E_{n,q,w}^{(\alpha,\beta)}(x).$$

By (2.7), we have the following distribution relation:

Theorem 2. *For any positive integer m(=odd), we have*

$$E_{n,q,w}^{(\alpha,\beta)}(x) = \frac{[2]_{q^\beta}}{[2]_{q^{\beta m}}} [m]_{q^\alpha}^n \sum_{i=0}^{m-1} (-1)^i w^i q^{\beta i} E_{n,q^m,w^m}^{(\alpha,\beta)}\left(\frac{i+x}{m}\right), \quad n \in \mathbb{Z}^+.$$

By (1.4), (2.1) , and (2.6), we easily see that

$$[2]_{q^\beta} \sum_{l=0}^{n-1} (-1)^{n-1-l} w^l q^{\beta l} [l]_{q^\alpha}^m = q^{\beta n} w^n E_{m,q,w}^{(\alpha,\beta)}(n) + (-1)^{n-1} E_{m,q,w}^{(\alpha,\beta)}.$$

Hence, we have the following theorem.

Theorem 3. *Let $m \in \mathbb{Z}^+$. If $n \equiv 0 \pmod 2$, then*

$$q^{\beta n} w^n E_{m,q,w}^{(\alpha,\beta)}(n) - E_{m,q,w}^{(\alpha,\beta)} = [2]_{q^\beta} \sum_{l=0}^{n-1} (-1)^{l+1} w^l q^{\beta l} [l]_{q^\alpha}^m.$$

If $n \equiv 1 \pmod 2$, then

$$q^{\beta n} w^n E_{m,q,w}^{(\alpha,\beta)}(n) + E_{m,q,w}^{(\alpha,\beta)} = [2]_{q^\beta} \sum_{l=0}^{n-1} (-1)^{l} w^l q^{\beta l} [l]_{q^\alpha}^m.$$

From (1.3), we note that

$$\begin{aligned}
[2]_{q^\beta} &= q^\beta w \int_{\mathbb{Z}_p} \phi_w(x) e^{[x+1]_{q^\alpha} t} d\mu_{-q^\beta}(x) + \int_{\mathbb{Z}_p} e^{[x]_{q^\alpha} t} d\mu_{-q^\beta}(x) \\
&= \sum_{n=0}^{\infty} \left(q^\beta w \int_{\mathbb{Z}_p} \phi_w(x) [x+1]_{q^\alpha}^n d\mu_{-q^\beta}(x) + \int_{\mathbb{Z}_p} \phi_w(x) [x]_{q^\alpha}^n d\mu_{-q^\beta}(x) \right) \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \left(q^\beta w E_{n,q,w}^{(\alpha,\beta)}(1) + E_{n,q,w}^{(\alpha,\beta)} \right) \frac{t^n}{n!}.
\end{aligned}$$

Therefore, we obtain the following theorem.

Theorem 4. *For $n \in \mathbb{Z}^+$, we have*

$$q^\beta w E_{n,q,w}^{(\alpha,\beta)}(1) + E_{n,q,w}^{(\alpha,\beta)} = \begin{cases} [2]_{q^\beta}, & \text{if } n = 0, \\ 0, & \text{if } n \neq 0. \end{cases}$$

By Theorem 2.4 and (2.9), we have the following corollary.

Corollary 5. *For $n \in \mathbb{Z}^+$, we have*

$$q^\beta w (q^\alpha E_{q,w}^{(\alpha,\beta)} + 1)^n + E_{n,q,w}^{(\alpha,\beta)} = \begin{cases} [2]_{q^\beta}, & \text{if } n = 0, \\ 0, & \text{if } n \neq 0, \end{cases}$$

with the usual convention of replacing $(E_{q,w}^{(\alpha,\beta)})^n$ by $E_{n,q,w}^{(\alpha,\beta)}$.

3. Analogue of the Euler Zeta Function

By using twisted q-Euler numbers and polynomials with weight (α, β), twisted q-Euler zeta function and Hurwitz q-Euler zeta functions are defined. These functions interpolate the twisted q-Euler numbers and twisted q-Euler polynomials with weight (α, β), respectively. In this section we assume that $q \in \mathbb{C}$ with $|q| < 1$. From (2.4), we note that

$$\begin{aligned}\frac{d^k}{dt^k} F_{q,w}^{(\alpha,\beta)}(t)\bigg|_{t=0} &= [2]_{q^\beta} \sum_{m=0}^{\infty} (-1)^m w^m q^{\beta m} [m]_{q^\alpha}^k \\ &= E_{k,q,w}^{(\alpha,\beta)}, (k \in \mathbb{N}).\end{aligned} \tag{3.1}$$

By using the above equation, we are now ready to define twisted q-Euler zeta functions.

Definition 6. Let $s \in \mathbb{C}$.

$$\zeta_{q,w}^{(\alpha,\beta)}(s) = [2]_{q^\beta} \sum_{n=1}^{\infty} \frac{(-1)^n w^n q^{\beta n}}{[n]_{q^\alpha}^s}. \tag{3.2}$$

Note that $\zeta_{q,w}^{(\alpha,\beta)}(s)$ is a meromorphic function on $\mathbb{C}$. Note that, if $q \to 1, w = 1$, then $\zeta_{q,w}^{(\alpha,\beta)}(s) = \zeta(s)$ which is the Euler zeta functions. Relation between $\zeta_{q,w}^{(\alpha,\beta)}(s)$ and $E_{k,q,w}^{(\alpha,\beta)}$ is given by the following theorem.

Theorem 7. *For $k \in \mathbb{N}$, we have*

$$\zeta_{q,w}^{(\alpha,\beta)}(-k) = E_{k,q,w}^{(\alpha,\beta)}. \tag{3.3}$$

Observe that $\zeta_{q,w}^{(\alpha,\beta)}(s)$ function interpolates $E_{k,q,w}^{(\alpha,\beta)}$ numbers at non-negative integers.
By using (2.5), we note that

$$\begin{aligned}\frac{d^k}{dt^k} F_{q,w}^{(\alpha,\beta)}(t,x)\bigg|_{t=0} &= [2]_{q^\beta} \sum_{m=0}^{\infty} (-1)^m w^m q^{\beta m} [x+m]_{q^\alpha}^k \\ &= E_{k,q,w}^{(\alpha,\beta)}(x), (k \in \mathbb{N})\end{aligned}$$

and

$$\left(\frac{d}{dt}\right)^k \left(\sum_{n=0}^{\infty} E_{n,q,w}^{(\alpha,\beta)}(x) \frac{t^n}{n!}\right)\bigg|_{t=0} = E_{k,q,w}^{(\alpha,\beta)}(x), \text{ for } k \in \mathbb{N}. \tag{3.4}$$

By (3.2) and (3.4), we are now ready to define the Hurwitz twisted q-Euler zeta functions.

Definition 8. Let $s \in \mathbb{C}$.

$$\zeta_{q,w}^{(\alpha,\beta)}(s,x) = [2]_{q^\beta} \sum_{n=0}^{\infty} \frac{(-1)^n w^n q^{\beta n}}{[n+x]_{q^\alpha}^s}. \tag{3.5}$$

Note that $\zeta_{q,w}^{(\alpha,\beta)}(s,x)$ is a meromorphic function on $\mathbb{C}$. Relation between $\zeta_{q,w}^{(\alpha)}(s,x)$ and $E_{k,q,w}^{(\alpha)}(x)$ is given by the following theorem.

Theorem 9. *For $k \in \mathbb{N}$, we have*

$$\zeta_{q,w}^{(\alpha,\beta)}(-k,x) = E_{k,q,w}^{(\alpha,\beta)}(x).$$

Observe that $\zeta_{q,w}^{(\alpha,\beta)}(-k,x)$ function interpolates $E_{k,q,w}^{(\alpha,\beta)}(x)$ numbers at non-negative integers.

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