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STRONG a-CONVERGENCE AND IDEAL STRONG EXHAUSTIVENESS OF SEQUENCES OF FUNCTIONS

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Abstract: We introduce and study the notions of strong a-convergence, a stronger form of the known a-convergence (or continuous convergence), and of I-strong exhaustiveness, where I is an ideal of subsets of $\mathbb{N}$, of a sequence of functions from a metric space (X, d) to another metric space (Y, ρ) and, among others, necessary and sufficient conditions for the continuity of the I-pointwise limit of a sequence of functions are derived.

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1. Introduction

The notion of $a-$convergence (also called “continuous convergence” or “stetige konvergenz”) has been known since the beginning of the 20th century. It was used already by C. Caratheodory in [2], by H. Hahn in [4] and by A. Zygmund in the study of trigonometric series in [5]. For a more detailed exposition see [3]. We recall that the sequence $\{f_n\}_{n\in\mathbb{N}}$ of functions from X to Y a-converges

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to a function f from X to Y at $x_0 \in X$ iff for each sequence $\{x_n\}_{x\in\mathbb{N}} \subseteq X$ convergent to x_0 it holds that the sequence $\{f_n(x_n)\}_{n\in\mathbb{N}}$ converges to $f(x_0)$. In [3] the notions of exhaustiveness and weak exhaustiveness have been defined. More precisely, we recall from [3] the following definitions which will be useful in the sequel:

Definition 1. A sequence $(f_n)_{n\in\mathbb{N}}$ is exhaustive at $x_0 \in X$ iff

$$\forall\varepsilon>0\;\exists\,\delta>0\;\exists\,n_0\in\mathbb{N}: n\geq n_0,\; d(x,x_0)<\delta\Longrightarrow\rho(f_n(x),f_n(x_0))<\varepsilon$$

Definition 2. A sequence $(f_n)_{n\in\mathbb{N}}$ is weakly exhaustive at $x_0 \in X$ iff

$$\forall\varepsilon>0\;\exists\,\delta>0: d(x,x_0)<\delta\Longrightarrow\exists\,n_x\in\mathbb{N}:\rho(f_n(x),f_n(x_0))<\varepsilon$$
$$\text{for all } n\geq n_x.$$

From the above notions is derived in [3] an answer to the fundamental question: "when the pointwise limit of a sequence of functions is continuous". More precisely it holds that:

Theorem 3. (see Theorem 4.2.3 in [3]) *If $\{f_n\}_{n\in\mathbb{N}}$ converges pointwise to f and $x_0 \in X$ then the following are equivalent:*

(i) f is continuous at x_0.

(ii) The sequence $\{f_n\}_{n\in\mathbb{N}}$ is weakly exhaustive at x_0.

Recently G. Beer and S. Levi in [1] defined the notion of strong uniform continuity of a function f and the notion of strong equicontinuity of a family $\{f_i : i \in I\}$ of functions as follows:

Definition 4. Let $f : X \longrightarrow Y$ and $B \subseteq X$. The function f is strongly uniformly continuous on B iff $\forall\,\varepsilon > 0\;\exists\,\delta > 0$ such that if $d(x, y) < \delta$ and $\{x, y\} \cap B \neq 0$ then $\rho(f(x), f(y)) < \varepsilon$.

Definition 5. A family $\{f_i : i \in I\}$ of functions from X to Y is called strongly equicontinuous on $B \subseteq X$, iff

$$\forall\,\varepsilon>0\;\exists\,\delta>0: i\in I, b\in B, d(x,b)<\delta\Longrightarrow\rho(f_i(x),f_i(b))<\varepsilon.$$

In Section 2 we introduce the notion of strong exhaustiveness on $B \subset X$. This is closely connected to the notion of strong equicontinuity introduced by Beer and Levi in [1]. This new notion enables us to investigate the convergence of a sequence of functions in terms of properties of the sequence and not of properties of functions as single members (Theorem 12). Also we define strong

a-convergence on $B \subseteq X$, which is a stronger notion than $a-$convergence at $x_0 \in B$ (Definition 9). In fact, it is a notion of convergence related to the boundary behaviour of a sequence of functions (see Remarks 10), and we prove that the pointwise convergence turns to strong a-convergence under the assumption of strong exhuastiveness of the sequence(Theorem 12).

In Section 3, using an arbitrary ideal $I \subseteq \mathcal{P}(\mathbb{N})$, we extend the notion of strong exhaustiveness to I-strong exhaustiveness and I-strongly weak exhaustiveness and we obtain a characterization (Proposition 17) of the strong uniform continuity of the I-pointwise limit f of a sequence of functions $\{f_n\}_{n\in\mathbb{N}}$. We point out again that we obtain this result considering a global property of the sequence of functions instead of properties of each single member of the sequence.

Finally in Section 4 we define the notion of strong exhaustiveness for families of functions and we study its relation with strong equicontinuity (Proposition 20 and Theorem 23).

Notations 6. Throughout the paper we shall assume that (X, d) and (Y, ρ) are arbitrary metric spaces, $\{f_n\}_{n\in\mathbb{N}}, f$ are elements of Y^X, $\mathbb{N}$ is the set of all positive integers, $\mathcal{P}(\mathbb{N})$ is the powerset of $\mathbb{N}$ and I is an ideal of $\mathbb{N}$, that is a family of subsets of $\mathbb{N}$ such that:

(i) $A \in I, B \subseteq \mathbb{N}$ with $B \subseteq A$ implies that $B \in I$

(ii) $A \in I, B \in I$ implies that $A \cup B \in I$.

An ideal I of $\mathbb{N}$ is called admissible iff $I \neq 0$, $\mathbb{N} \notin I$ and $\{\{n\}, n \in \mathbb{N}\} \subseteq I$.

We recall also the following:

Definitions 7. Let $x_0 \in X$. Then:

(i) $\{f_n\}_{n\in\mathbb{N}}$ is said to converge I-pointwise to f at x_0 (we write $f_n(x_0) \xrightarrow{I} f(x_0)$) iff $\{n \in \mathbb{N} : \rho(f_n(x_0), f(x_0)) \geq \varepsilon\} \in I$, $\forall \varepsilon > 0$.

(ii) $\{f_n\}_{n\in\mathbb{N}}$ is called I-pointwise convergent to f on X iff $\{f_n\}_{n\in\mathbb{N}}$ converges I-pointwise to f at x_0, $\forall x_0 \in X$ (we write $f_n(x) \xrightarrow{I} f(x)$, $\forall x \in X$).

2. Strong a-Convergence

Definition 8. We say that the sequence $\{f_n\}_{n\in\mathbb{N}}$ is strongly exhaustive on $B \subseteq X$, iff

$$\forall\, \varepsilon > 0\; \exists\, \delta = \delta(\varepsilon, B) > 0\; \exists\, n_0 = n_0(\varepsilon, B) \in \mathbb{N} :$$

$$\beta \in B \text{ and } d(x,\beta) < \delta \text{ and } n \geq n_0 \Longrightarrow \rho(f_n(x), f_n(\beta)) < \varepsilon.$$

Definition 9. We say that the sequence $\{f_n\}_{n\in\mathbb{N}}$ converges strongly-a to f on $B \subseteq X$ (we write $f_n \overset{str-a,B}{\longrightarrow} f$), iff

$$\forall \{x_n\}_{n\in\mathbb{N}} \subseteq X \ \forall\, x_0 \in X :$$
$$x_n \longrightarrow x_0 \text{ and } \{x_n, x_0\} \cap B \neq \emptyset \text{ , n=1,2,...} \Longrightarrow f_n(x_n) \longrightarrow f(x_0).$$

Remarks 10. (i) If $B = \{x_0\}$, $x_0 \in X$, then the strong $a-$convergence of $\{f_n\}_{n\in\mathbb{N}}$ to f on $\{x_0\}$ coincides with the well known $a-$convergence of $\{f_n\}_{n\in\mathbb{N}}$ to f at x_0.

(ii) If $x_0 \notin \overline{B}$, the implication of Definition 9 is trivially satisfied. On the other hand, if x_0 is an isolated point of B, the implication of Definition 9 means that $f_n(x_0) \longrightarrow f(x_0)$. So the interesting case is at points x_0 belonging to the limit set of B and especially the case when $x_0 \notin B$ but x_0 is a limit point of B. Indeed, it is known that the linear means $\{\sigma_n\}_{n\in\mathbb{N}}$ of the trigonometric series of an integrable function $f \in L^1[0, 2\pi]$, with respect to a positive summability kernel, a-converge to f at the points of continuity of f (see [5], Theorem 2.30). Also it is not hard to see that if a sequence $\{f_n\}_n$ converges uniformly at x_0 (that is uniformly in a neighborhood of x_0) to a function f, which is continuous at x_0, then we get $a-$convergence. Hence, in all these cases Definition 9 introduces a notion of convergence involving the boundary behaviour of these sequences with respect to the set B of points of continuity of f.

Obviously the strong a-convergence of a sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ on $B \subseteq X$ implies both the a-convergence and the pointwise convergence of $\{f_n\}_{n\in\mathbb{N}}$ to the same limit. But the inverse implications fail in general. Indeed, we have the following:

Example 11. *Let $X = [0, 1]$, $Y = \mathbb{R}$ and $d = \rho$ be the usual metric. Let also $f_n = x^n$, $x \in [0, 1]$, $n \in \mathbb{N}$, and $f(x) = 0$, $x \in [0, 1)$, $f(1) = 1$. It is not hard to see that $\{f_n\}_n$ a-converges and pointwise to f on $B = [0, 1)$. On the other hand $\{f_n\}_n$ does not converge strongly-a to f on B as it does not a-converge to f at $x_0 = 1$ (see also [3], Proposition 1.3).*

In the next theorem we examine when pointwise convergence implies strong a-convergence on a set $B \subseteq X$.

Theorem 12. *If $f_n(x) \longrightarrow f(x)$, $x \in X$ and $\{f_n\}_{n\in\mathbb{N}}$ is strongly exhaustive on a set $B \subseteq X$, then $f_n \overset{str-a,B}{\longrightarrow} f$.*

Proof. By Remarks 10 (ii) it is enough to consider the case when x_0 is a limit point of B. Let $x_n \in X, n = 1, 2, ..., x_n \longrightarrow x_0$ and $\{x_n, x_0\} \cap B \neq 0$. If $\varepsilon > 0$, we have to find $n_0 \in \mathbb{N}$ such that:

$$\rho(f_n(x_n), f(x_0)) < \varepsilon, \quad n \geq n_0 \tag{1}$$

Since $f_n(x_0) \longrightarrow f(x_0)$ we get that:

$$\exists\, n_1 \in \mathbb{N} : \rho(f_n(x_0), f(x_0)) < \frac{\varepsilon}{3}, \quad n \geq n_1 \tag{2}$$

Also by strong exhaustiveness it follows that:

$$\exists\, \delta > 0\, \exists\, n_2 \in \mathbb{N} : \beta \in B, d(x, \beta) < \delta, n \geq n_2 \Longrightarrow \rho(f_n(x), f_n(\beta)) < \frac{\varepsilon}{3}. \tag{3}$$

But x_0 is a limit point of B, hence there exists a sequence $\{y_n\}_{n\in\mathbb{N}} \subseteq B$ with $y_n \longrightarrow x_0$. Since also $x_n \longrightarrow x_0$ it follows that:

$$\exists\, n_3 \in \mathbb{N} : d(y_n, x_n) < \delta, \quad n \geq n_3 \tag{4}$$

Now, we set $n_0 = max(n_1, n_2, n_3)$. Then by (2), (3) and (4) we get for $n \geq n_0$ that:

$$\begin{aligned} \rho(f_n(x_n), f(x_0)) \leq & \rho(f_n(x_n), f_n(y_n)) + \rho(f_n(y_n), f_n(x_0)) \\ & + \rho(f_n(x_0), f(x_0)) \\ < & \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

This is (1) and the proof is complete. □

The above theorem gives rise to an interesting observation. Indeed, it is clear that a strongly exhaustive sequence of functions on $B \subseteq X$ is exhaustive on B. The opposite is not always true as the following shows:

Example 13. *Under the same assumptions and notations as in Example 11, by Theorem 12 we get that $\{f_n\}_n$ is not strongly exhaustive on B. But by [3, Theorem 2.6] we have that $\{f_n\}_n$ is exhaustive on B.*

3. *I*-Strong Exhaustiveness

Definition 14. Let I be an ideal of $\mathbb{N}$. The sequence $\{f_n\}_{n\in\mathbb{N}}$ is called I-strongly exhaustive on $B \subseteq X$, iff

$$\forall\, \varepsilon > 0\ \exists\, \delta > 0\ \exists\, A \in I : \beta \in B, d(x,\beta) < \delta, n \notin A \Longrightarrow \rho(f_n(x), f_n(\beta)) < \varepsilon.$$

Regarding the above extension of the Definition 8, the following proposition is valid.

Proposition 15. *Let $I \subseteq \mathcal{P}(\mathbb{N})$ be an admissible ideal. If $\{f_n\}_{n\in\mathbb{N}}$ converges I-pointwise to f and $\{f_n\}_{n\in\mathbb{N}}$ is I-strongly exhaustive on $B \subseteq X$, then f is strongly uniformly continuous on B.*

Proof. Let $\varepsilon > 0$. It is enough to find $\delta > 0$ such that:

$$\beta \in B \text{ and } d(x,\beta) < \delta \Longrightarrow \rho(f(x), f(\beta)) < \varepsilon \tag{5}$$

Since $\{f_n\}$ is I-strongly exhaustive on B it follows that:

$$\exists\, \delta_1 > 0\ \exists\, A_1 \in I : \quad \beta \in B, d(x,\beta) < \delta_1, n \notin A_1 \Longrightarrow \rho(f_n(x), f_n(\beta)) < \frac{\varepsilon}{3}. \tag{6}$$

Now, we fix $x \in X$ and $\beta \in B$ such that $d(x,\beta) < \delta_1$. By hypothesis $f_n(\beta) \xrightarrow{I} f(\beta)$ and $f_n(x) \xrightarrow{I} f(x)$. Hence, $\exists\, A_2, A_3 \in I$:

$$\rho(f_n(\beta), f(\beta)) < \frac{\varepsilon}{3}, \quad n \notin A_2 \tag{7}$$

and

$$\rho(f_n(x), f(x)) < \frac{\varepsilon}{3}, \quad n \notin A_3 \tag{8}$$

Since I is admissible, we have that $\mathbb{N} \setminus (A_1 \cup A_2 \cup A_3) \neq 0$. Hence if $n \notin A = A_1 \cup A_2 \cup A_3$ by (6), (7) and (8) we get:

$$\rho(f(x), f(\beta)) \leq \rho(f_n(x), f(x)) + \rho(f_n(x), f_n(\beta)) + \rho(f_n(\beta), f(\beta)) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Thus $\delta = \delta_1$ and the proof is complete. □

A refinement of ideal strong exhaustiveness on $B \subseteq X$ is the notion of ideal strongly-weak exhaustiveness on $B \subseteq X$. Using this new concept we obtain a necessary and sufficient condition for the strong uniform continuity, on B, of the ideal pointwise limit of a sequence of functions which are not necessarily continuous.

Definition 16. Let I be an ideal of $\mathbb{N}$. The sequence $\{f_n\}_{n\in\mathbb{N}}$ is called I-strongly weakly exhaustive on $B \subseteq X$, iff

$$\begin{aligned}&\forall\, \varepsilon > 0\, \exists\, \delta > 0 : \beta \in B, x \in S(\beta,\delta) \Longrightarrow \exists\, A = A(x,\beta) \in I :\\ &\rho(f_n(x), f_n(\beta)) < \varepsilon, \quad n \notin A,\end{aligned}$$

where $S(\beta,\delta) = \{x \in X : d(x,\beta) < \delta\}$.

Proposition 17. *Let $I \subseteq \mathcal{P}(\mathbb{N})$ be an admissible ideal and $B \subseteq X$. Assume that the sequence $\{f_n\}_{n\in\mathbb{N}}$ is I-pointwise convergent to f. Then, the following are equivalent:*

(i) f is strongly uniformly continuous on B.

(ii) $\{f_n\}_{n\in\mathbb{N}}$ is I-strongly weakly exhaustive on B.

Proof. (i)$\Longrightarrow$(ii). Let $\varepsilon > 0$. By hypothesis we get that:

$$\exists\, \delta > 0 : \beta \in B, d(\beta, x) < \delta \Longrightarrow \rho(f(x), f(\beta)) < \frac{\varepsilon}{3} \tag{9}$$

since $f_n(\beta) \xrightarrow{I} f(\beta)$ and $f_n(x) \xrightarrow{I} f(x)$, it follows that:

$$\exists\, A_\beta \in I : \rho(f_n(\beta), f(\beta)) < \frac{\varepsilon}{3}, \quad n \notin A_\beta \tag{10}$$

and

$$\exists\, A_x \in I : \rho(f_n(x), f(x)) < \frac{\varepsilon}{3}, \quad n \notin A_x \tag{11}$$

Now we set $A = A(x,\beta) = A_x \cup A_\beta$. From (9),(10),(11) we obtain that for $n \notin A$:

$$\begin{aligned}&\rho(f_n(x), f_n(\beta)) < \rho(f_n(x), f(x)) + \rho(f(x), f(\beta)) + \rho(f(\beta),\\ &f_n(\beta)) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,\end{aligned}$$

which means that $\{f_n\}_{n\in\mathbb{N}}$ is I-strongly weakly exhaustive on B.
$(ii) \Longrightarrow (i)$ The proof in this direction is similar to that of Proposition 15 with the additional assumption that the set $A_1 \in I$ depends on x, β. ☐

Remark 18. Obviously I-strongly weak exhaustiveness is weaker than I-strong exhaustiveness. In the next example we will see that I-strongly weak exhaustiveness is in general strictly weaker than I-strong exhaustiveness.

Example 19. *Let $X = Y = \mathbb{R}$ and $d = \rho$ be the usual metric. We set for $n \in \mathbb{N}$:*

$$\begin{aligned} f_n(x) &= 0, \text{ if } x \in \Big(-\infty, -\frac{1}{n}\Big] \cup \{0\} \cup \Big[\frac{1}{n}, +\infty\Big) \text{ and} \\ f_n(x) &= 1, \text{ otherwise.} \end{aligned}$$

Then, $\{f_n\}_{n\in\mathbb{N}}$ converges pointwise to $f = 0$. Since f is strongly uniformly continuous, say on $B = [-1, 1]$, we get by Proposition 17 that $\{f_n\}_{n\in\mathbb{N}}$ is I-strongly weakly exhaustive on B. But, since for $\beta = 0 \in B$ and for any $\delta > 0$, $\rho(f_n(x), f(x)) > \frac{1}{2}$ for $x \in (-\delta, \delta)$, except for a finite number of $n \in \mathbb{N}$, it follows that $\{f_n\}_{n\in\mathbb{N}}$ is not I-strongly exhaustive on B, for any I admissible ideal of $\mathbb{N}$ (see also Definition 14).

4. Strongly Exhaustive Families of Functions

It is not hard to see that the notion of strong exhaustiveness of a sequence $\{f_n\}_{n\in\mathbb{N}}$ (Definition 8) is strictly weaker than strongly equicontinuity of $\{f_n\}_{n\in\mathbb{N}}$. But, if f_n is strongly uniformly continuous, for each $n \in \mathbb{N}$, then these two notions coincide. More precisely we have the following proposition.

Proposition 20. *Suppose f_n is strongly uniformly continuous on $B \subseteq X$, for all $n \in \mathbb{N}$. Then the following are equivalent:*

(i) $\{f_n\}_{n\in\mathbb{N}}$ is strongly equicontinuous on B.

(ii) $\{f_n\}_{n\in\mathbb{N}}$ is strongly exhaustive on B.

Proof. The implication $(i) \Longrightarrow (ii)$ is obvious.

For the inverse implication, let $\{f_n\}_{n\in\mathbb{N}}$ be strongly exhaustive on B and $\varepsilon > 0$. Then

$$\begin{gathered} \exists\, \delta_0 > 0,\ \exists n_0 \in \mathbb{N}: \\ d(x,y) < \delta_0,\ \{x,y\} \cap B \neq 0,\ n \geq n_0 \Longrightarrow \rho(f_n(x), f_n(y)) < \varepsilon \end{gathered}$$

Also, for each $i = 1, 2, ..., n_0 - 1$, we get by Definition 4 and by hypothesis that:

$$\exists\, \delta_i > 0:\ d(x,y) < \delta_i,\ \{x,y\} \cap B \neq 0, \Longrightarrow \rho(f_i(x), f_i(y)) < \varepsilon$$

Hence the strong equicontinuity on B follows by taking $\delta = \min\{\delta_0, \delta_1, ..., \delta_{n_0-1}\}$ (see also Definition 5). $\square$

Remark 21. The notion of strong exhaustiveness can be naturally extended for arbitrary families of functions. If $S \neq \emptyset$ is any set by S_f we denote the ideal of all finite subsets of S.

Definition 22. Let $\mathcal{F}$ be an infinite family of functions from X to Y and $B \subseteq X$. We say that $\mathcal{F}$ is strongly exhaustive on B, iff

$$\forall\, \varepsilon > 0\, \exists\, \delta > 0\, \exists\, A \in \mathcal{F}_f\colon$$
$$\beta \in B, d(x,\beta) < \delta, g \in \mathcal{F} \setminus A \Longrightarrow \rho(g(x), g(\beta)) < \varepsilon.$$

In the next theorem we will see that for each family $\mathcal{F}$ strongly exhaustive on $B \subseteq X$, "suitable" limits of sequences from $\mathcal{F}$ give rise to a family, which is strongly equicontinuous on B.

Theorem 23. *Let $\Phi \subseteq Y^X$ be a family which is strongly exhaustive on $B \subseteq X$. If $I \subseteq \mathcal{P}(\mathbb{N})$ is an admissible ideal and σ is a symbol for a convergence stronger than I-pointwise, then the family $\Phi^\sigma = \{g \in Y^X \mid \exists \{f_n\}_{n\in\mathbb{N}} \subseteq \Phi : \{f_n\}_{n\in\mathbb{N}}$ is not eventually constant and $f_n \xrightarrow{\sigma} g\}$ is strongly equicontinuous on B.*

Proof. Let $\varepsilon > 0$. By definition of strong equicontinuity we have to find $\delta > 0$ such that

$$\beta \in B, d(x,\beta) < \delta \Longrightarrow \rho(g(x), g(\beta)) < \varepsilon, \text{ for all } g \in \Phi^\sigma \tag{12}$$

Since Φ is strongly exhaustive on B it follows that:

$$\exists\, \delta > 0\, \exists\, A \in \Phi_f : \beta \in B, d(x,\beta) < \delta \Longrightarrow \rho(g(x),$$
$$g(\beta)) < \frac{\varepsilon}{3}, \quad g \in \Phi \setminus A.$$

We claim that for the above δ (12) is true.
Indeed, let $g \in \Phi^\sigma$. Without loss of generality we can assume that there exists a sequence $\{f_n\}_n \subseteq \Phi$ such that

$$f_n \neq g \text{ for each } n \in \mathbb{N} \text{ and } f_n(x) \xrightarrow{I} g(x),\, x \in X. \tag{13}$$

by the definition of Φ^σ. Firstly, we observe that it is impossible infinite terms of $\{f_n\}_{n\in\mathbb{N}}$ to belong to the finite set A, hence by(13)

$$\exists n_0 \in \mathbb{N} : \rho(f_n(x), f_n(\beta)) < \frac{\varepsilon}{3}, \quad n \geq n_0. \tag{14}$$

Also, by I-pointwise convergence of f_n to g we get that:

$$\begin{aligned} \exists A_1, A_2 \in I : \rho(f_n(x), g(x)) &< \frac{\varepsilon}{3},\ n \notin A_1 \text{ and} \\ \rho(f_n(\beta), g(\beta)) &< \frac{\varepsilon}{3},\ n \notin A_2 \end{aligned} \tag{15}$$

Now, as I is admissible, it follows that there exists $n > n_0$ with $n \notin A_1 \cup A_2$. Hence by (15),(16) we get:

$$\rho(g(x), g(\beta)) \leq \rho(g(x), f_n(x)) + \rho(f_n(x), f_n(\beta)) + \rho(f_n(\beta), g(\beta)) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

So (12) holds and the proof is complete. □

Remark 24. We can easily construct examples of function families Φ, which are strongly exhaustive on $B \subseteq X$, each $f \in \Phi$ is not continuous on X and $\Phi^a \neq \emptyset$, where a denotes the $a-$convergence on X (see also [3], Proposition 1.3). Since each $f \in \Phi^a$ is continuous, it follows that $\Phi^a \cap \Phi = \emptyset$. Hence in Theorem 23 it can happen that $\Phi \cap \Phi^\sigma = \emptyset$ and $\Phi^\sigma \neq \emptyset$

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