International Journal of Pure and Applied Mathematics
Volume 80 No. 3 2012, 403-409
ISSN: 1311-8080 (printed version)
url: http://www.ijpam.eu
ijpam.eu

LINEARLY INDEPENDENT SUBSETS OF EMBEDDED VARIETIES

E. Ballico
Department of Mathematics
University of Trento
38 123 Povo (Trento) - Via Sommarive, 14, ITALY
e-mail: ballico@science.unitn.it

Abstract: Let $X \subset \mathbb{P}^n$ be an integral and non-degenerate variety. Assume $m < n \le 2m+1$. We prove that here is no zero-dimensional scheme $Z \subset X$ witht $\deg(Z) \le 4$ and $\dim(\langle Z\rangle) \le \deg(Z) - 2$ and and only if $m = 1$, $n = 3$ and X is a rational normal curve.

AMS Subject Classification: 14N05
Key Words: zero-dimensiona scheme, linear dependent zero-dimensional scheme

1. Introduction

Let $X \subseteq \mathbb{P}^n$ be an integral and non-degenerate variety defined over an algebraically closed field $\mathbb{K}$. We recall that a zero-dimensional scheme $Z \subset \mathbb{P}^n$ is said to be *curvilinear* if for each $P \in Z_{red}$ the Zariski tangent space of Z at P has dimension ≤ 1. It is easy to check that Z is curvilinear if and only if it is contained in a smooth curve.

For any zero-dimensional scheme $Z \subset \mathbb{P}^n$ let $\langle Z\rangle$ denote the linear span of Z, i.e. the intersection of all hyperplanes of $\mathbb{P}^n$ containing Z, with the convention $\langle Z\rangle = \mathbb{P}^n$ if there is no such a hyperplane. We prove the following result.

Received: July 14, 2012

Theorem 1. *Let $X \subset \mathbb{P}^n$ be an integral and non-degenerate variety. Assume $m < n \le 2m+1$. The following conditions are equivalent:*

(i) there is no zero-dimensional scheme $Z \subset X$ such that $\deg(Z) \le 4$ and $\dim(\langle Z\rangle) \le \deg(Z) - 2$.

(ii) there is no zero-dimensional scheme $Z \subset X$ such that $\deg(Z) = 4$ and $\dim(\langle Z\rangle) = 2$.

(iii) There is no curvilinear zero-dimensional subscheme $Z \subset X$ such that $\deg(Z) \le 4$ and $\dim(\langle Z\rangle) \le \deg(Z) - 2$.

(iv) $m = 1$, $n = 3$ and X is a rational normal curve.

See

The inequality " $\dim(\langle Z\rangle) \le \deg(Z) - 2$ means that Z is not linearly independent. We give a class of examples (for any $m \ge 2$) in which X satisfies the following assertion ♣:

♣ There is no set $E \subset X$ such that $\sharp(E) = 4$, $\dim(\langle E\rangle) \le 2$, and $\langle E\rangle \cap X$ is zero-dimensional.

See Example 1) (each X is a cone over a curve). Of course, any cone contains many sets E with $\sharp(E) = 4$ and E linearly dependent. The crucial part in ♣ is the condition that $\langle E\rangle \cap X$ contains no curve.

In this note we discuss the following definition (see [3], Lemma 2.1.5, and [2], Proposition 11, for the integer $\beta(X)$).

Notation 1. Let $X \subset \mathbb{P}^n$ be an integral and non-degenerate variety. Let $\beta(X)$ (resp. $\gamma(X)$, resp. $\eta(X)$) denote the maximal integer t such that any zero-dimensional scheme (resp. zero-dimensional and curvilinear, resp. finite set) $Z \subset X$ with $\deg(Z) \le t$ is linearly independent, i.e. $\dim(\langle Z\rangle) = \deg(Z) - 1$. Let $\beta'(X)$ be the maximal integer t such that if $Z \subset X$ is a zero-dimensional scheme, $\deg(Z) \le t$ and $\dim(\langle Z) \le \deg(Z) - 2$, then $\langle Z\rangle \cap X$ contains a positive dimensional subvariety, with the convention $\beta'(X) = +\infty$ if there is no such integer. Define in the same way the integers $\gamma'(X)$ and $\eta'(X)$ using curvilinear schemes and finite sets, respectively.

Of course $\beta'(X) \ge \beta(X)$, $\gamma'(X) \ge \gamma(X)$, $\eta'(X) \ge \eta(X)$, $\beta(X) \le \gamma(X) \le \eta(X)$ and $\beta'(X) \le \gamma'(X) \le \eta'(X)$. If X is a smooth curve, then each zero-dimensional subscheme of X is curvilinear and hence $\gamma(X) = \beta(X)$. If X is a curve, then $\alpha'(X) = \alpha(X)$ for all $\alpha \in \{\beta, \gamma, \eta\}$. We prove the following result.

Proposition 1. *Let $X \subset \mathbb{P}^n$ be an integral and non-degenerate subvariety. Set $m := \dim(X)$.*

(a) We have $\beta(X) \le \gamma(X) \le \eta(X) \le n+2-m$.

(b) We have $\beta(X) = n+2-m$ *if and only if* $\eta(X) = n-m+2$ *if and only if* $m = 1$ *and* X *is a rational normal curve.*

Proposition 2. *Let* $X \subset \mathbb{P}^n$ *be an integral and non-degenerate subvariety. Set* $m := \dim(X)$.

We have $\beta'(X) = +\infty \Leftrightarrow \gamma'(X) = +\infty \Leftrightarrow \alpha'(X) = +\infty \Leftrightarrow \deg(X) = n-m+1$.

2. The Proofs

Remark 1. Let $X \subset \mathbb{P}^n$ be a non-degenerate subvariety. If X is set-theoretically cut out by quadrics, then $\beta'(X) \ge 3$.

Proof of Proposition 1. Part (a) is obvious if $m = 1$. Now assume $m \ge 2$. Fix a a general codimension $m-1$ linear subspace W of $\mathbb{P}^n$. By a characteristic free version of Bertini's theorem for quasi-projective schemes (see [6], pp. 66–67) the scheme $W \cap X$ is an integral curve spanning V. Hence $X \cap V$ contains at least $n-m+3$ points. Since $V \ne \mathbb{P}^n$, we have $\eta(X) \le n+2-m$.

Now we prove part (b). Obviously $\beta(X) \le \gamma(X) \le \eta(X)$. For every zero-dimensional scheme $A \subset \mathbb{P}^1$ we have $h^0(\mathbb{P}^1, \mathcal{I}_A(n)) = \max\{0, n+1-\deg(A)\}$ and $h^1(\mathbb{P}^1, \mathcal{I}_A(n)) = \max\{0, \deg(A)-n-1\}$. Hence $\beta(X) = \eta(X) = n+1$ if X is a rational normal curve. Hence the " if " part of Proposition 1 is true. Now we check the " only if " part.

First assume $m = 1$. Set $d := \deg(X)$. Assume $d \ge n+1$. Let $H \subset \mathbb{P}^n$ be a general hyperplane. Since $H \cap X$ contains d points and $d \ge \dim(H)+2$, we have $\beta(X) \le \dim(H) = n-1$.

Now assume $m \ge 2$. Let $V \subset \mathbb{P}^n$ be a general linear subspace of codimension $m-1$. By the characteristic free part of Bertini's theorem for quasi-projective schemes (see [6], pp. 66–67) the scheme $V \cap X$ is an integral curve spanning V. Since $\dim(V) = n-m+1$, we have $b\eta(X) \le \beta(X \cap V) \le n-m+2$. Now assume $\eta(X) = n-m+2$. The case $m = 1$ gives that $X \cap V$ is a rational normal curve. Hence $\deg(X) = n+1-m$, i.e. X is a minimal degree m-dimensional variety. All these varieties are described in [5]. First assume $m = 2$. Either $n = 5$ and X is a Veronese surface or X is a cone over a rational normal curve of $\mathbb{P}^{n-1}$ or X is a ruled surface. Since X contains no line, then $n = 5$ and X is a Veronese surface. Since $\beta(X) = 4$ if X is a Veronese surface, even this case is excluded. Now assume $m \ge 3$. Let $M \subset \mathbb{P}^n$ be a general linear subspace. Since $\beta(M \cap X) = n-m+2$, $n-m+2 = 5$ and $M \cap X$ is a Veronese surface.

Hence X is a cone over a Veronese surface (see [5]). Hence X contains lines, a contradiction. □

Lemma 1. *Assume* $\dim(X) = 1$.

(a) We have $\beta'(X) = +\infty \Leftrightarrow \gamma'(X) = +\infty \Leftrightarrow \alpha'(X) = +\infty \Leftrightarrow X$ *is a rational normal curve.*

(b). If X is not a rational normal curve, then $\beta'(X) = \beta(X)$, $\gamma'(X) = \gamma(X)$ *and* $\eta'(X) = \eta(X)$.

Proof. If X is a rational normal curve, then $\beta(X) = \gamma(X) = \eta(X) = n+1$ and hence $\beta'(X) = +\infty$, $\gamma'(X) = +\infty$ and $\eta'(X) = +\infty$. Now assume that X is not a rational normal curve. Proposition 1 gives $\beta(X) \leq n$, $\gamma(X) \leq n$ and $\eta(X) \leq n$. Hence to test $\beta(X)$, $\gamma(X)$ and $\alpha(X)$ we only need to check zero-dimensional schemes Z such that $\deg(Z) \leq n$ and hence $\langle Z \rangle \neq \mathbb{P}^n$. Hence $\langle Z \rangle \cap X$ is zero-dimensional. □

Proof of Proposition 2. If $m = 1$, then use Lemma 1. Now assume $m \geq 2$. Set $d := \deg(X)$. Let $V \subset \mathbb{P}^n$ be a general codimension $m-1$ linear subspace. By a characteristic free version of Bertini's theorem for quasi-projective schemes (see [6], pp. 66–67, the scheme $X \cap V$ is an integral curve of degree d spanning V. If $d \neq n - m + 1$, i.e. if $X \cap V$ is not a rational normal curve, then $\eta'(X \cap V) \neq +\infty$. Since $\beta'(X) \leq \gamma'(X) \leq \eta'(X) \leq \eta'(X \cap V)$, we get $\beta'(X) \neq +\infty$, $\gamma'(X) \neq +\infty$ and $\eta'(X) \neq +\infty$, if $d \neq n-m+1$. Now assume $d = n-m+1$, i.e. assume that X is a minimal degree subvariety. Apply [4], Theorem 2.2. □

Proposition 3. *Let* $X \subset \mathbb{P}^n$ *be an integral and non-degenerate subvariety. Set* $m := \dim(X)$. *If* $2m + \gamma(X) \geq n + 3$ *and* $\gamma(X) \geq 3$, *then X is smooth.*

Proof. Assume $2m \geq n + \gamma(X) - 3$, $\gamma(X) \geq 3$ and the existence of $P \in \mathrm{Sing}(X)$. Let T_PX denote the Zariski tangent space of X at P. Since X is singular at P, T_PX is a linear subspace of dimension $\geq m + 1$. Set $\rho := \dim(T_PX)$. Fix a general $S \subset X$ such that $\sharp(S) = \gamma(X) - 3$ and call V the linear span of T_PX and S. Since X is non-degenerate and S is general, we have $\dim(V) \geq \min\{n, \rho + \gamma(X) - 3\} \geq n + 1 - m$. Hence $X \cap V$ contains a curve. Hence there is $Q \in X \cap V$ such $Q \notin S \cup \{P\}$. For each line $L \subset T_PX$ with $P \in L$ either $L \subset T_PX \cap X$ or the zero-dimensional scheme $L \cap X$ contains P with multiplicity at least 2. Hence the linear space $\langle \{P, Q\} \cup S\} \rangle$ contains a curvilinear subscheme of X with degree at least $\gamma(X)$. Since $\dim(\langle \{P, Q, \} \cup S \rangle \leq \sharp(S) + 1 = \gamma(X) - 2$, we get a contradiction. □

Proposition 4. *Let $X \subset \mathbb{P}^n$ be an integral and non-degenerate variety. Set $m := \dim(X)$. We have $\eta(X) \le 2 \cdot \lceil (n+2)/(m+1) \rceil - 1$.*

Proof. For each integer $z \ge 1$ let $\sigma_z(X)$ denote the closure in $\mathbb{P}^n$ of the union of all $(z-1)$-dimensional linear subspaces spanned by z points of X. Set $z := \lceil (n+2)/(m+1) \rceil$. First assume $\sigma_z(X) = \mathbb{P}^n$. A dimensional count gives that for a general $P \in \mathbb{P}^n$ there are infinitely many $(z-1)$-dimensional linear subspaces spanned by z points of X. Fix two of them, say $\langle A \rangle$ and $\langle B \rangle$ with $A \subset X$, $B \subset X$, $\sharp(A) = \sharp(B) = z$ and $\langle A \rangle \ne \langle$. For general P we may also find these linear spaces with $A \cap B = \emptyset$ (again, a dimensional count). Hence $\sharp(A \cup B) = 2z$. Since $P \in \langle A \rangle \cap \langle B \rangle$, we have $\dim(\langle A \rangle \cap \langle B \rangle) \le 2z - 2$. Hence $\eta(X) \le 2 \cdot \lceil (n+2)/(m+1) \rceil - 1$. Now assume $\sigma_z(X) \ne \mathbb{P}^n$. Hence $\sigma_z(X)$ is an integral variety, but not with the expected dimension. For a general $P \in \sigma_z(X)$ there are infinitely many $(z-1)$-dimensional linear subspaces spanned by z points of X. Any two of them give $\eta(X) \le 2 \cdot \lceil (n+2)/(m+1) \rceil - 1$. □

Proof of Theorem 1. Of course, (i) $\Rightarrow$ (ii), but since X is non-degenerate also the implication (ii) $\Rightarrow$ (i) is obvious. Obviously (i) $\Rightarrow$ (iii). Let $\mathrm{Sec}(X) \subseteq \mathbb{P}^n$ denote the secant variety of X.

(a) If $m = 1$, then use Lemma 1. From now on we assume $m \ge 2$.

(b) In this step and in step (d) we assume $n = 2m+1$ and $\mathrm{Sec}(X) = \mathbb{P}^{2m+1}$. Fix a general $Q \in \mathbb{P}^{2m+1}$. A dimensional count gives the existence of finitely many, say k, lines $L \subset \mathbb{P}^{2m+1}$ such that $Q \in L$ and $\sharp(X \cap L) \ge 2$; moreover $\deg(L \cap X) = 2$ for all such L and $L \cap \mathrm{Sing}(X) = \emptyset$. First assume $k \ge 2$ and call L, L' any two such lines and V the plane spanned by $L \cup L'$. Since V is spanned by 4 points of X, (iii) is not satisfied.

(c) In this step we assume $\dim(\mathrm{Sec}(X)) \le 2m$ (this is always the case if $n \le 2m+1$). Hence $\mathrm{Sec}(X)$ is an irreducible variety of dimension $\rho \le 2m$. Fix a general $P \in \mathrm{Sec}(X)$. Since $\rho > m$, then $P \notin X$. Fix a hyperplane $H \subset \mathbb{P}^{2m+1}$ such that $P \notin H$. The set of all lines $L \subset \mathbb{P}^{2m+1}$ A dimensional count gives the existence of an $(2m+1-\rho)$-dimensional quasi-projective variety $T \subset H$ such that each line L_t, $t \in T$, we have $P \in L_t$ and $\sharp(L_t \cap X) \ge 2$. Fix a general $(t,s) \in T \times T$. Since $L_t \ne L_s$ and $P \in L_s \cap L_t$, $L_s \cup L_t$ spans a plane, V. By construction V contains at least 4 non-collinear points. Hence (iii) is not satisfied.

(d) In this step we assume $n = 2m+1$, $\mathrm{Sec}(X) = \mathbb{P}^{2m+1}$ and that a general point of $\mathbb{P}^{2m+1}$ is contained in a unique secant line of X. Proposition 3 gives that X is smooth. Fix $O \in X$ and assume the existence of $Q \in X \cap T_O X$ such that $Q \neq O$. The line $\langle\{O, Q\}\rangle$ contains O with multiplicity at least 2. Hence either $L \subseteq X$ or $\deg(L \cap X) \geq 3$. Hence (iii) is not satisfied. Fix $O \in X$. Let $\ell : \mathbb{P}^{2m+1} \setminus T_O X \to \mathbb{P}^m$ denote the linear projection from $T_P X$. Since $T_O X \cap (X \setminus \{O\}) = \emptyset$, ℓ induces a morphism $f : X \setminus \{O\} \to \mathbb{P}^m$. For any $Q \in (X \setminus \{O\})$ set $V_Q := \langle T_O X \cup \{Q\}\rangle$. Each V_Q has dimension $m+1$. Assume that $V_Q \cap X$ is a scheme containing a zero-dimensional scheme $Z \subset V_Q \setminus \{O\}$ with $\deg(Z) \geq 2$. Fix $W \subseteq Z$ with $\deg(W) = 2$. Set $L := \langle W\rangle$. Since L is a line contained in V_Q and $T_O X$ is a hyperplane of V_Q, the set $L \cap T_O X$ contains at least one point, P. First assume $P = O$. In this case $\deg(L \cap X) \geq 4$, because $L \cap X$ contains O with multiplicity at least 2 and the degree 2 scheme W, (iii) is not satisfied. Now assume $P \neq O$. Set $M := \langle\{P, O\}\rangle$. M is a line and $M \cap X$ contains O with multiplicity at least two. Hence the plane $\langle L \cup M\rangle$ contains a degree 4 zero-dimensional subscheme of X. Hence (i) is not satisfied, a contradiction. We just proved that for each $Q \in X \setminus \{O\}$ the scheme $V_Q \cap (X \setminus \{O\})$ is the set $\{Q\}$ with its reduced structure. Hence $f : X \setminus \{O\} \to \mathbb{P}^m$ is injective and unramified, i.e. an open embedding. Since $m \geq 2$ and O is a smooth point of X, f extends to a morphism $\phi : X \to \mathbb{P}^m$. Both X and $\mathbb{P}^n$ are smooth. Since $\phi|(X \setminus \{O\})$ has invertible differential, ϕ has invertible differential. Since $X \setminus \{O\}$ is injective, ϕ is finite. We get that ϕ is an isomorphism. Since ϕ is induced by a linear projection, we have $\phi^*(\mathcal{O}_{\mathbb{P}^m}(1))|X \cong \mathcal{O}_X(1)|(X \setminus \{O\})$. Since X is smooth and $m \geq 2$, we get $\mathcal{O}_X(1) \cong \phi^*(\mathcal{O}_{\mathbb{P}^m}(1))$. Hence $h^0(X, \mathcal{O}_X(1)) = m+1$. Hence X is degenerate, a contradiction. □

Example 1. Fix an integer $m \geq 2$. Let $Y \subset \mathbb{P}^{m+2}$ be an integral and non degenerate curve such that $\dim(\langle E\rangle) = \sharp(E) - 1$ for each finite set $E \subset Y$ with $\sharp(E) \leq 4$ and that Y is scheme-theoretically cut-out by quadrics. For instance, we may take as Y any linearly normal curve of arithmetic genus q and degree $q + m + 2$ if $q + m + 2 - 4 \geq 2q - 1$, i.e. if $q \leq m - 1$. See $\mathbb{P}^{m+2}$ as a linear subspace M of $\mathbb{P}^{2m+1}$. Take an $(m-2)$-dimensional linear subspace $W \subset \mathbb{P}^{2m+1}$ such that $W \cap M = \emptyset$. Let X be the cone with vertex W and Y as a basis. We will check that ♣ is not satisfied. Since X is scheme-theoretically cut out by quadrics, every line $L \subset \mathbb{P}^{2m+1}$ containing a degree 3 zero-dimensional subscheme of X is contained in X. Let $E \subset X$ be a finite set such that $\sharp(E) = 4$ and assume that $\langle E\rangle$ is a plane and that $\langle E\rangle \cap X$ is zero-dimensional. Hence $\langle E\rangle \cap W$ is either empty or a point. First assume $\langle E\rangle \cap W = \emptyset$. The linear projection from W induces an isomorphism of E onto a set $E' \subset Y$ such that $\sharp(E') = 4$ and $\dim(\langle E'\rangle) = 2$, contradicting our choice of Y. Now assume that the linear space $\langle E\rangle \cap W$ is a point, O. Let $R \subset M$ be the image of $\langle E\rangle \setminus \{O\}$ by the linear projection from W. R is a line containing the image E'' of $E \setminus \{O\}$.

Our assumption on Y gives $\sharp(E'') \le 2$. We easily see that $\langle E\rangle$ contains a line spanned by O and a point of E''. Hence $\langle E\rangle \cap X$ contains a line.

Acknowledgements

The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

References

[1] B. Ådlandsvik, Joins and higher secant varieties, *Math. Scand.*, **61** (1987), 213-222.

[2] A. Bernardi, A. Gimigliano, M. Idà, On the stratification of secant varieties of Veronese varieties via symmetric rank, *J. Symbolic. Comput.*, **46**, No. 1 (2011), 34-53.

[3] J. Buczyński, A. Ginensky, J. M. Landsberg, Determinantal equations for secant varieties and the Eisenbud-Koh-Stillman conjecture, *ArXiv:* 1007.0192v3 [math.AG].

[4] D. Eisenbud, M. Green, K. Hulek, S. Popescu, Small schemes and varieties of minimal degree, *Amer. J. Math.*, **128**, No. 6 (2006), 1363-1389.

[5] D. Eisenbud, J. Harris, On varieties of minimal degree (a centennial account), *Algebraic Geometry*, Bowdoin (1985), Brunswick, Maine, 3-13; *Proc. Sympos. Pure Math.*, **46**, Part 1, Amer. Math. Soc., Providence, RI (1987).

[6] J.-P. Jouanolou, *Théorèmes de Bertini et Applications*, Progress in Mathematics, **42**, Birkhäuser Boston, Inc., Boston, MA (1983).

[7] J. Rathmann, The uniform position principle for curves in characteristic p, *Math. Ann.*, **276**, No. 4 (1987), 565-579.