International Journal of Pure and Applied Mathematics
Volume 80 No. 5 2012, 727-737
ISSN: 1311-8080 (printed version)
url: http://www.ijpam.eu
ijpam.eu

A CLASS OF EIGENVALUE PROBLEMS FOR THE (p, q)-LAPLACIAN IN $\mathbb{R}^N$

Nawel Benouhiba[1 §], Zahia Belyacine[2]
[1,2]Department of Mathematics
LAM, Badji Mokhtar-Annaba University
P.O. Box 12, El Hadjar, 23000, Annaba, ALGERIA

Abstract: This paper concerns the study of a nonlinear eigenvalue problem for the $(p, q)-$Laplacian with a positive weight

$$-\Delta_p u - \Delta_q u = \lambda g(x)|u|^{p-2}u \text{ in } \mathbb{R}^N.$$

Using the Mountain-Pass Theorem, we show the existence of a continuous set of positive eigenvalues.

AMS Subject Classification: 35B33, 35B45, 35J60, 35J70
Key Words: (p, q)-Laplacian, eigenvalue, Weak solution, Palais-Smale condition

1. Introduction

In recent years much attention was given to the study of stationary solutions of the reaction-diffusion equation

$$u_t = div\left(\left(|\nabla u|^{p-2} + |\nabla u|^{q-2}\right)\nabla u\right) + c(x, u) \tag{1.1}$$

that appears in physics and related sciences such as biophysics, plasma physics and chemical reaction design.

Received: August 28, 2012

[§]Correspondence author

When $q = p$ and $c(x,u) = \lambda g(x)|u|^{p-2}u$, stationary problem associated to 1.1 becomes an eigenvalue p-Laplacian problem of the form

$$-\Delta_p u = \lambda g(x)|u|^{p-2}u \tag{1.2}$$

which has been widely studied both in bounded domains and $\mathbb{R}^N$. By means Ljusternik-Schnirelmann theory, it was established the existence of a non decreasing positive sequence of eigenvalues $0 < \lambda_1 < ... < \lambda_n < ...$ (see [4] and [8]). A characterization of the first eigenvalue was given by

$$\lambda_1 = \inf_{\substack{u \in W^{1,p} \\ u \neq 0}} \frac{\|u\|_{1,p}^p}{\int g|u|^p dx} \tag{1.3}$$

It was also shown (see [1]) that λ_1 is simple, principle and isolated.

For the case $q \neq p$, few studies appeared for special cases of $c(x,u)$. For example in [6], the author gives an existence result of a non trivial solution of the problem $-div\left(\left(|\nabla u|^{p-2} + |\nabla u|^{q-2}\right)\nabla u\right) = m|u|^{p-2}u + n|u|^{q-2}u + f(x,u)$ on $\mathbb{R}^N$, under suitable conditions on the coefficients and the exponents. In [9], a result of the existence of an infinitely many weak solutions of a similar problem with a concave-convex nonlinearity in bounded domain is given. In [5], the authors established a multiplicity existence result for the $(p,q)-$Laplacian problem with critical exponent on a bounded domain.

In this paper, we are interested in finding eigenvalues of the problem

$$-div((|\nabla u|^{p-2} + |\nabla u|^{q-2})\nabla u) = \lambda g(x)|u|^{p-2}u \text{ in } \mathbb{R}^N. \tag{1.4}$$

under the hypotheses:

$$1 < q < p < q^* \tag{1.5}$$

and

$$0 \leq g \in L^{(\frac{q^*}{\alpha})'}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \tag{1.6}$$

where $\alpha = \frac{p(1-t)}{1-\frac{pt}{p^*}}$ for some $t \in (0,1)$ (so $0 < \alpha < p$).

We will look for weak solutions of 1.4 in the framework of the reflexive Banach space $W = W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$, defined as the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm $\|u\| = \|u\|_{1,p} + \|u\|_{1,q}$.

As we will see, the energy functional associated to problem 1.4 is given by

$$I(u) = \frac{1}{p}\|u\|_{1,p}^p + \frac{1}{q}\|u\|_{1,q}^q - \frac{\lambda}{p}\int_{\mathbb{R}^N} g(x)|u|^p dx. \tag{1.7}$$

and

$$\langle I'(u), v\rangle = \int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u \nabla v dx + \int_{\mathbb{R}^N} |\nabla u|^{q-2}\nabla u \nabla v dx - \lambda \int_{\mathbb{R}^N} g(x)|u|^{p-2}uv dx,$$

for all $v \in W$.

Notice that critical points of the functional I are precisely weak solutions of 1.4. To get critical points of I we will apply the Mountain-Pass Theorem. Establishing the Palais-Smale condition is the most difficulty in studying such problems. First, there is a lack of compactness for the Sobolev embedding. So the boundedness of the Palais-Smale sequence is not evident. Another difficulty, as it was mentioned in [5, 9] becomes from the fact that the Banach space framework does not ensure that

$$|\nabla u_n|^{p-2}\nabla u_n \rightharpoonup |\nabla u|^{p-2}\nabla u \text{ in } L^{\frac{p}{p-1}}(\mathbb{R}^N)$$

for a Palais-Smale sequence $\{u_n\}$. That is why in both works the authors used the Concentration-Compactness principal.

In our case, the use of the weight g in an appropriate Lebesgue space overcomes these difficulties.

The principal result of this paper is the following theorem.

Theorem 1. *If p, q, and g fulfill 1.5 and 1.6, then there exits $\lambda^* > 0$ such that any $\lambda > \lambda^*$ is an eigenvalue of problem 1.4.*

In what follows, the letter C will be indiscriminately used to denote various constants when the exact values are irrelevant. The symbol $\int$ will denote $\int_{\mathbb{R}^N}$.

2. Proof of the Results

We say that $\lambda \in \mathbb{R}$ is an eigenvalue of Problem 1.4 if there exists $u \in W, u \not\equiv 0$ such that

$$\int |\nabla u|^{p-2}\nabla u.\nabla v dx + \int |\nabla u|^{q-2}\nabla u.\nabla v dx = \lambda \int g(x)|u|^{p-2}uv dx, \quad \forall v \in W. \quad (2.1)$$

This is equivalent to

$$I'(u) = 0 \text{ in } W'.$$

Standard argument shows that $I \in C^1(W, \mathbb{R})$. Since the functional I is not bounded from below, we will look for local minimizers by means of the Mountain-Pass Theorem [7].

We begin this section by establishing some results required in the proof of Theorem 1.

Lemma 2. *For all $u \in W$ we have*

$$\int g|u|^p dx \leq C\|g\|_\infty \|g\|^{p(1-t)}_{(\frac{q^*}{\alpha})'} \|u\|^{pt}_{1,p} \|u\|^{p(1-t)}_{1,q}. \tag{2.2}$$

Proof. For any $u \in W$, we have by the Hölder inequality

$$\int g|u|^p dx \leq A_1 A_2.$$

Where $A_1 = \left(\int g|u|^\alpha dx\right)^{\frac{p(1-t)}{\alpha}}$ and $A_2 = \left(\int g|u|^{p^*} dx\right)^{\frac{pt}{p^*}}$. By Sobolev injection it yields

$$A_2 \leq C\|g\|_\infty \|u\|^{pt}_{1,p}.$$

To estimate A_1 we apply the Hölder inequality to get

$$A_1^{\frac{\alpha}{p(1-t)}} \leq \left(\int g^{(\frac{q^*}{\alpha})'} dx\right)^{\frac{1}{(\frac{q^*}{\alpha})'}} \left(\int |u|^{q^*}\right)^{\frac{\alpha}{q^*}}.$$

By Sobolev embedding we can conclude that

$$A_1 \leq C\|g\|^{p(1-t)}_{(\frac{q^*}{\alpha})'} \|u\|^{p(1-t)}_{1,q}$$

and Lemma 2 follows. □

Lemma 3. *The functional I satisfies the Palais-Smale condition $(PS)_c$ for any $c \in \mathbb{R}$.*

Proof. Let $(u_n) \subset W$ be a Palais-Smale sequence at a level $c \in \mathbb{R}$. This means that

$$I(u_n) \to c \text{ and } I'(u_n) \to 0 \text{ in } W'.$$

We will show that u_n is a bounded sequence. Since $I(u_n)$ is a real convergent sequence then there exists $M > 0$ such that

$$I(u_n) = \frac{1}{p}\|u_n\|^p_{1,p} + \frac{1}{q}\|u_n\|^q_{1,q} - \frac{\lambda}{p}\int g|u_n|^p dx < M$$

$$\frac{1}{p}\left(\|u_n\|^p_{1,p} + \|u_n\|^q_{1,q}\right) \leq \frac{1}{p}\|u_n\|^p_{1,p} + \frac{1}{q}\|u_n\|^q_{1,q} \leq M + \frac{\lambda}{p}\int g|u_n|^p dx.$$

We entail that

$$\langle I'(u_n), u_n\rangle = \|u_n\|_{1,p}^p + \|u_n\|_{1,q}^q - \lambda \int g|u_n|^p dx \le pM.$$

This means that $\langle I'(u_n), u_n\rangle$ is bounded. We conclude that $(\frac{1}{q} - \frac{1}{p})\|u_n\|_{1,q}^q = I(u_n) - \frac{1}{p}\langle I'(u_n), u_n\rangle$ is a bounded real sequence i.e. (u_n) is bounded in $W^{1,q}(\mathbb{R}^N)$.

Put

$$J(u_n) = \int |\nabla u_n|^p dx - \lambda \int g|u_n|^p dx.$$

By the above considerations, $J(u_n)$ is a real convergent sequence. So, there exists $C > 0$ such that

$$\int |\nabla u_n|^p dx \le C + \lambda \int g|u_n|^p dx.$$

By Lemma 2 and the boundedness of u_n in $W^{1,q}(\mathbb{R}^N)$ we can find a positive constant, still denoted C, such that

$$\|u_n\|_{1,p}^p \le C(1 + \|u_n\|_{1,p}^{pt}). \tag{2.3}$$

If u_n is not bounded in $W^{1,p}(\mathbb{R}^N)$, we can suppose that $\|u_n\|_{1,p} \to \infty$. By relation 2.3, we have that

$$\|u_n\|_{1,p}^{p(1-t)} \le C(1 + \|u_n\|_{1,p}^{-pt}).$$

Since $0 < t < 1$ we conclude that $\|u_n\|_{1,p}$ is bounded which is a contradiction.

Consequently, u_n is bounded in W and then there exists $u \in W$ such that $u_n \rightharpoonup u$.

It is easy to show that

$$\lim_{n\to+\infty} \langle I'(u_n) - I'(u), u_n - u\rangle = 0. \tag{2.4}$$

Next, we will show that

$$\lim_{n\to+\infty} \int g(|u_n|^{p-2}u_n - |u|^{p-2}u)v dx = 0 \text{ for all } v \in W. \tag{2.5}$$

We have by Hölder inequality and assumption 1.6 that for any $R > 0$

$$|\int_{B_R} g(|u_n|^{p-2}u_n - |u|^{p-2}u)v dx|$$

$$\leq \|g\|_\infty \Big(\int_{B_R} ||u_n|^{p-2}u_n - |u|^{p-2}u|^{(p^*)'}dx\Big)^{\frac{1}{(p^*)'}} \Big(\int_{B_R} |v|^{p^*}dx\Big)^{\frac{1}{p^*}}.$$

Here B_R denotes the ball of radius R in $\mathbb{R}^N$ centred at the origin and we will denote $B'_R = \mathbb{R}^N \setminus B_R$.

So by Sobolev embedding it yields

$$\Big|\int_{B_R} g(|u_n|^{p-2}u_n - |u|^{p-2}u)v dx\Big|$$

$$\leq C\|g\|_\infty \|v\| \left(\int_{B_R} ||u_n|^{p-2}u_n - |u|^{p-2}u|^{(p^*)'}dx\right)^{\frac{1}{(p^*)'}}$$

Since u_n is weakly convergent to u in $W^{1,p}(\mathbb{R}^N)$ then $(\chi_{B_R}u_n)$ is also weakly convergent to $(\chi_{B_R}u)$ in $W^{1,p}(B_R)$. We can deduce that $(\chi_{B_R}u_n)$ converges strongly to $(\chi_{B_R}u)$ in $L^{(p^*)'(p-1)}(B_R)$ since $(p^*)'(p-1) < p^*$. Then there exists a subsequence, still denoted $(\chi_{B_R}u_n)$, and $h \in L^{(p^*)'(p-1)}(B_R)$ such that $\chi_{B_R}u_n \to \chi_{B_R}u$ a.e. in B_R as $n \to \infty$ and for all n, $|\chi_{B_R}u_n| \leq h$ a.e. in B_R. It follows that $\chi_{B_R}|u_n|^{p-2}u_n \to \chi_{B_R}|u|^{p-2}u$ a.e. in B_R and $\chi_{B_R}|u_n|^{p-1} \leq h^{p-1}$ a.e. in B_R. By the Lebesgue Theorem there exists another subsequence, still denoted $(\chi_{B_R}u_n)$, such that $\chi_{B_R}|u_n|^{p-2}u_u \to \chi_{B_R}|u|^{p-2}u$ strongly in $L^{(p^*)'}(B_R)$.

In an other hand we have by the Hölder inequality that for any $v \in W$

$$\Big|\int_{B'_R} g(|u_n|^{p-2}u_n - |u|^{p-2}u)v dx\Big| \leq \left(\int_{B'_R} g||u_n|^{p-2}u_n - |u|^{p-2}u|^{\frac{\alpha p^*}{(p-1)q^*}}dx\right)^{\frac{(p-1)q^*}{\alpha p^*}} \cdot$$

$$\cdot \left(\int_{B'_R} g|v|^{\left(\frac{\alpha p^*}{(p-1)q^*}\right)'}dx\right)^{\frac{1}{\left(\frac{\alpha p^*}{(p-1)q^*}\right)'}}.$$

Applying again the Hölder inequality we get

$$\int_{B'_R} g|v|^{(\frac{\alpha p^*}{(p-1)q^*})'}dx \leq \|g\|_{L^{(\frac{q^*}{\alpha})'}(B'_R)}^{(\frac{q^*}{\alpha})'/(\frac{\alpha}{\alpha p^*-(p-1)q^*})'} \left(\int_{B'_R} g^{\frac{pq^*-\alpha p^*}{q^*-\alpha}}|v|^{p^*}dx\right)^{\frac{\alpha}{\alpha p^*-(p-1)q^*}}$$

$$\leq C\|g\|_{L^{(\frac{q^*}{\alpha})'}(B'_R)}^{(\frac{q^*}{\alpha})'/(\frac{\alpha}{\alpha p^*-(p-1)q^*})'} \|g\|_\infty^{\frac{pq^*-\alpha p^*}{q^*-\alpha}} \|v\|_{1,p}^{\frac{\alpha p^*}{\alpha p^*-(p-1)q^*}}$$

Boundedness of (u_n) together with the above assertions yield

$$\Big|\int_{B'_R} g(|u_n|^{p-2}u_n - |u|^{p-2}u)v dx\Big| \leq C\|g\|_{L^{(\frac{q^*}{\alpha})'}(B'_R)}^{A}\|v\|$$

where $A = \frac{(\frac{q^*}{\alpha})'}{(\frac{\alpha}{\alpha p^* - (p-1)q^*})'(\frac{\alpha p^*}{(p-1)q^*})'}$. It follows that

$$\int_{B_R'} g(|u_n|^{p-2}u_n - |u|^{p-2}u)v dx \to 0$$

when $R \to \infty$, since $g \in L^{(\frac{q^*}{\alpha})'}(\mathbb{R}^N)$.

Consequently, $\lim\limits_{n\to+\infty} \int g(|u_n|^{p-2}u_n - |u|^{p-2}u)v dx = 0$ for all $v \in W$.

From relations 2.4 and 2.5 we get

$$\lim_{n\to+\infty} \int (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u)(\nabla u_n - \nabla u)dx +$$

$$+ \int (|\nabla u_n|^{q-2}\nabla u_n - |\nabla u|^{q-2}\nabla u)(\nabla u_n - \nabla u)dx = 0.$$

By the Hölder inequality we have

$$\int (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u)(\nabla u_n - \nabla u)dx \geq \int |\nabla u_n|^p dx + |\nabla u|^p dx -$$

$$-(\int |\nabla u_n|^p dx)^{\frac{p-1}{p}}(\int |\nabla u|^p dx)^{\frac{1}{p}} - (\int |\nabla u_n|^p dx)^{\frac{1}{p}}(\int |\nabla u|^p dx)^{\frac{p-1}{p}}$$

$$= (\|u_n\|_{1,p}^{p-1} - \|u\|_{1,p}^{p-1})(\|u_n\|_{1,p} - \|u\|_{1,p}) \geq 0.$$

By the same argument it yields

$$\int (|\nabla u_n|^{q-2}\nabla u_n - |\nabla u|^{q-2}\nabla u)(\nabla u_n - \nabla u)dx$$
$$\geq (\|u_n\|_{1,q}^{q-1} - \|u\|_{1,q}^{q-1})(\|u_n\|_{1,q} - \|u\|_{1,q}) \geq 0.$$

It follows that $\lim\limits_{n\to+\infty} \|u_n\|_{1,p} = \|u\|_{1,p}$ and $\lim\limits_{n\to+\infty} \|u_n\|_{1,q} = \|u\|_{1,q}$. This together with the weak convergence of u_n to u in W implies that u_n is strongly convergent to u in W and the proof is complete. □

Next, we will show that the functional I given by 1.7 satisfies the Mountain pass geometry.

Lemma 4. *1. There exist $\rho, \beta > 0$ such that $I(u) \geq \beta$ on $\|u\| = \rho$.*

2. There exists $u_0 \in W$ with $\|u\| > \rho$ and $I(u_0) < 0$.

Proof. (1) Let $u \in W$, we put $\rho = \|u\| = \rho_1 + \rho_2$ were $\|u\|_{1,p} = \rho_1$ and $\|u\|_{1,q} = \rho_2$. By relation 2.2 it yields

$$I(u) \geq \frac{1}{p}\rho_1^p - \frac{\lambda}{p}C\|g\|_\infty \|g\|^{p(1-t)}_{(\frac{q^*}{\alpha})'} \rho_1^{pt}\rho_2^{q(1-t)}$$

$$\geq \frac{1}{p}\rho_1^{pt}\left(\rho_1^{p(1-t)} - \lambda C\|g\|_\infty \|g\|^{p(1-t)}_{(\frac{q^*}{\alpha})'} \rho_2^{q(1-t)}\right)$$

We can choose $\rho_2 = \varepsilon$ and $\rho_1 = \left(1 + \lambda C\|g\|_\infty \|g\|^{p(1-t)}_{(\frac{q^*}{\alpha})'} \varepsilon^{q(1-t)}\right)^{\frac{1}{p(1-t)}}$ for a sufficiently small $\varepsilon > 0$. Consequently

$$I(u) \geq \frac{1}{p}\left(1 + \lambda C\|g\|_\infty \|g\|^{p(1-t)}_{(\frac{q^*}{\alpha})'} \varepsilon^{q(1-t)}\right)^{\frac{t}{(1-t)}} > 0$$

for any $u \in W$ such that $\|u\| = \left(1 + \lambda C\|g\|_\infty \|g\|^{p(1-t)}_{(\frac{q^*}{\alpha})'} \varepsilon^{q(1-t)}\right)^{\frac{1}{p(1-t)}} + \varepsilon.$

(2) We denote by φ the normalized eigenfunction associated to the first eigenvalue λ_1 of the p-Laplacian with weight g, namely

$$-div(\nabla|\varphi|^{p-2}\nabla\varphi) = \lambda_1 g|\varphi|^{p-2}\varphi \text{ in } \mathbb{R}^N$$

and

$$\int |\nabla\varphi|^p dx = 1.$$

Hence,

$$I(\tau\varphi) = \frac{\tau^p}{p} + \frac{\tau^q}{q}\int |\nabla\varphi|^q dx - \frac{\lambda\tau^p}{p}\int g|\varphi|^p dx, \tau > 0.$$

Since $\int g|\varphi|^p dx = \frac{1}{\lambda_1}$ we get

$$I(\tau\varphi) = \frac{\tau^p}{p}(1 - \frac{\lambda}{\lambda_1}) + \frac{\tau^q}{q}\int |\nabla\varphi|^q dx.$$

We claim that any eigenvalue of problem 1.4 satisfies $\lambda > \lambda_1$. So $I(\tau\varphi) \to -\infty$ when $\tau \to +\infty$. Consequently, there exists $\tau_0 > 0$ such that $I(\tau_0\varphi) < 0$ and we put $u_0 = \tau_0\varphi$. □

We return now to the claim that any eigenvalue λ of problem 1.4 satisfies $\lambda > \lambda_1$. For this, we introduce the quantity

$$\lambda^* = \inf_{\substack{u \in W \\ u \neq 0}} \frac{\int |\nabla u|^p dx + \int |\nabla u|^q dx}{\int g|u|^p dx}.$$

For any $u \in W$ we have

$$\frac{\int |\nabla u|^p dx + \int |\nabla u|^q dx}{\int g|u|^p dx} \geq \inf_{\substack{u \in W \\ u \neq 0}} \frac{\int |\nabla u|^p dx}{\int g|u|^p dx} \geq \inf_{\substack{u \in W^{1,p}(\mathbb{R}^N) \\ u \neq 0}} \frac{\int |\nabla u|^p dx}{\int g|u|^p dx} = \lambda_1.$$

So it follows that λ^* is a positive real number.

We suppose that there exists an eigenvalue λ of problem 1.4 such that $\lambda < \lambda^*$. So there exists $v \in W$, $v \neq 0$ that verifies

$$\int |\nabla v|^p dx + \int |\nabla v|^q dx = \lambda \int g|v|^p dx.$$

Then we get

$$\lambda^* > \lambda = \frac{\int |\nabla v|^p dx + \int |\nabla v|^q dx}{\int g|v|^p dx} \geq \inf_{\substack{u \in W \\ u \neq 0}} \frac{\int |\nabla u|^p dx + \int |\nabla u|^q dx}{\int g|u|^p dx} = \lambda^*$$

which is a contradiction. So, there is no eigenvalue less then λ^* and it is clair that $\lambda_1 < \lambda^*$. In addition, λ^* cannot be an eigenvalue of problem 1.4. Indeed, let $u_n \in W$ a minimizing sequence of λ^*. Similar arguments us used in [3] show that u_n converges strongly to a nontrivial function $u \in W$ that satisfies

$$p \int |\nabla u|^{p-2} \nabla u . \nabla w dx + q \int |\nabla u|^{q-2} \nabla u . \nabla w dx = \lambda^* p \int g|u|^{p-2} w dx$$

for all $w \in W$. This fact together with definition 2.1 implies that

$$\int |\nabla u|^{q-2} \nabla u . \nabla w dx = 0 \text{ for all } w \in W.$$

Hence, $u \equiv 0$ which is a contradiction.

Proof of Theorem 1. Define the minimax class

$$\mathbf{B} = \{\psi \in C([0,1], W), \psi(0) = 0, \psi(1) = u_0\}$$

and the corresponding minimax level

$$c = \inf_{\psi \in \mathbf{B}} \max_{\tau \in [0,1]} I(\psi(\tau)).$$

By the previous lemmas it follows that the assumptions of the Mountain-Pass Theorem are fullfiled. Therefore for any $\lambda > \lambda^*$, c is a critical value of I associated to a critical point $u_\lambda \in W$. Namely, $I'(u_\lambda) = 0$ and $I(u_\lambda) = c$. By Lemma 4(1) we have necessarily $c \geq \frac{1}{p}(1 + \lambda C\|g\|_\infty \|g\|^{p(1-t)}_{(\frac{q^*}{\alpha})'} \varepsilon^{q(1-t)})^{\frac{t}{(1-t)}} > 0$. Hence, u_λ cannot be trivial since $I(0) = 0$. Hence, Theorem 1 is proved. □

Acknowledgments

Project supported by the grants of PNR 8/u23/272 ANDRU of Algeria.

References

[1] A. Anane, Simplicity and isolation of first eigenvalue of the p-Laplacian with weight, *Comptes rendus de l'Académie des sciences, Série 1, Mathématique*, **305**, No. 16 (1987), 725-728.

[2] V. Benci, A. M. Micheletti, D. Visetti, An eigenvalue problem for a quasilinear elliptic field equation, *J. Differential Equations*, **184**, No.2 (2002), 299-320.

[3] N. Benouhiba, On the eigenvalues of weighted $p(x)$-Laplacian on $\mathbb{R}^N$, *Nonlinear Analysis*, **74**, No.1 (2011), 235-243.

[4] A. Lê, Eigenvalue problems for the p-Laplacian, *Nonlinear Analysis*, **64**, No. 5 (2006), 1057-1099.

[5] G. Li, G. Zhang, Multiple solutions for the p-q-Laplacian problem with critical exponent, *Acta Mathematica Scientia*, **29B**, No.4 (2009), 903-918.

[6] G.B. Li, The existence of a nontrivial solution to the p-q-Laplacian problem with nonlinearity asymptotic to u^{p-1} at infinity in $\mathbb{R}^N$, *Nonlinear Analysis*, **68**(2008), 1100-1119.

[7] L. Nirenberg, Variational and topological methods in nonlinear problems, *Bull. Amer. Math. Soc.*, **4**, No. 3 (1981), 267-302.

[8] A. Szulkin, M. Willem, Eigenvalue problems with indefnite weight, Studia Math. 135(1999), 191-201.

[9] H. Yin, Z. Yang, A class of p-q-Laplacian type eqation with concave-convex nonlinearities in bounded domain, *J. Math. Anal. Appl.*, **382** (2011), 843-855.