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HARMONIC ANALYSIS IN HYPERCOMPLEX SYSTEMS

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Abstract: This paper is devoted to give the necessary and sufficient conditions guarantees that the product of two positive definite functions defined in a hypercomplex system $L_1(Q, m)$ is also positive definite in $L_1(Q, m)$. Also, we prove that a continuous function with compact support ψ is negative definite if and only if $\exp(-\alpha\psi)$ is positive definite for each $\alpha > 0$. Moreover, we will give integral representations of some harmonic functions related to positive definite functions in hypercomplex system.

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1. Introduction

A central idea in harmonic analysis in various settings has been the existence of a product, usually called convolution, for functions and measures. In some

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cases, an investigation begins with a convolution algebra of measures as the primitive object upon which to build a theory; this is the case of the analysis of the objects called generalized hypergroups"hypercomplex system" which are the generalizations of the convolution algebra of Borel measures on a group. A hypercomplex system with a locally compact basis Q (see [2],[3] or [14]) is a set $L_1(Q,m)$ with generalized convolution, which can be defined in terms of a structure measure $c(A,B,r), \quad A,B \subset Q, r \in Q$. One important reason that explain why the harmonic analysts did not attracted to study Fourier algebra over hypercomplex system is that, the product of two continuous positive definite functions in a hypercomplex system is not necessarily positive definite in general. Moreover, one should observe that, a function ψ is negative definite if and only if $\exp(-\alpha\psi)$ is positive definite for each $\alpha > 0$. While this result holds for all semigroups it is not clear how to prove the 'only if' part for hypercomplex system since the usual technique does not apply(the 'if' part always holds provided that $\mathrm{Re}\psi$ is locally lower bounded). The problem is that except when x or y belong to the maximal subgroup of the hypercomplex system $\exp(-\alpha\psi(x*y))$ and $\exp(-\alpha\psi)(x*y)$ are usually not equal so that other methods have to be used to overcome this. Consider the space $L_1(Q,m) = L_1$ of functions on Q integrable with respect to the multiplicative measure m i.e., a regular Borel measure m positive on open sets such that

$$\int c(A,B,r)dm(r) = m(A)m(B) \quad (A,B \in B_0(Q))$$

It is some-times convenient to consider, together with the measure $c(A,B,r)$, its extension to the sets form $Q \times Q$. For this purpose, we fix r and, for any $(A,B \in (Q))$, put $m_r(A \times B) = c(A,B,r)$. The space $L_1(Q,m)$ with the convolution

$$(f*g)(r) = \int\int f(p)g(q)dm_r(p,q)$$

is called a hypercomplex system with basis Q. The rule played by the generalized translation R_p acting upon functions of a point $q \in Q$ and satisfying

$$(R_p\chi(.,\lambda))(q) = \chi(p,\lambda)\chi(q,\lambda)$$

A continuous bounded function $\phi(r) \quad (r \in Q)$ is called positive definite if

$$\sum_{i,j=1}^{n} \lambda_i\bar{\lambda_j}(R_{r_i} * \phi)(r_i) \geq 0$$

for all $r_1, r_2, ..., r_n \in Q, \lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{C}$ and $n \in \mathbb{N}$. As pointed in [2], every continuous positive definite function $\phi \in P(Q)$ admits a unique representation in the form of an integral

$$\phi(r) = \int_{X_h} \chi(r) d\mu(\chi) \quad (r \in Q)$$

where μ is a nonnegative finite regular measure on the space of continuous bounded characters X_h.

Harmonic analysis in hypercomplex system dates back to J. Delsartes and B. M. Levitans work during the 1930s and 1940s, but the substantial development had to wait till the 1990s when Berezansky and Kondratiev [2] put hypercomplex system in the right setting for harmonic analysis. The study of continuous conditionally exponential convex functions leads to the characterisation of convolution semigroups discussed in [12, section 2], which represents the essential datum for the potential theory. The main task in our previous works [8,10,11] was to give "the necessary and sufficient conditions guarantees that the product of two positive definite functions defined on hypergroup H is also positive definite on H". As a continuation of our work authors will give the necessary and sufficient conditions guarantees that the product of two positive definite functions defined in a hypercomplex system $L_1(Q,m)$ is also positive definite in $L_1(Q,m)$. Also, we prove that a continuous function with compact support ψ is negative definite if and only if $exp(-\alpha\psi)$ is positive definite for each $\alpha > 0$. Moreover, we will give integral representations of some harmonic functions related to positive definite functions in hypercomplex system.

This paper contains 4 sections. In §2, we give the necessary and sufficient conditions guarantee that the product of two positive definite functions defined in a hypercomplex system $L_1(Q,m)$ is also positive definite in $L_1(Q,m)$, then we resuming some properties of the set of τ-positive definite functions in hypercomplex system. §3, is devoted to give some properties of the set of negative definite functions in hypercomplex system which help us to prove that a continuous function with compact support ψ is negative definite if and only if $\exp(-\alpha\psi)$ is positive definite for each $\alpha > 0$. Finally, In §4, we give some relations between the set of τ-positive definite functions, the set of the completely monotone and completely alternating functions.

2. Product of Positive Definite Functions

A hypercomplex system $L_1(Q,m)$ is commutative if its structure measure is commutative. An essentially bounded measurable complex-valued function $\chi(r)$

$(r \in Q)$ not identically equal to zero on a set of positive measure is called a character of a hypercomplex system if

$$\int c(A,B,r)\chi(r)dx = \chi(A)\chi(B)$$

holds for any $A, B \in \mathbb{B}_0(Q)$. The following two lemmas are in fact, an adaption of whatever done for semigroups in [1, Berg et al]. We will not repeat the proof, wherever the proof for semigroups can be applied to the hypercomplex systems with necessary modification.

Lemma 2.1. *(i) The sum and the point-wise limit of positive definite functions in hypercomplex systems are also positive definite.*

(ii) Let ϕ be a continuous positive definite function on Q and define $L_1(Q,m) \to \mathbb{C}$ by $\Phi(s) := \int \phi(s)dm(s)$. Then Φ is positive definite in $L_1(Q,m)$.

Lemma 2.2. *A bounded measurable function $\phi \in C_c(Q)$ is positive definite if and only if there exists a ψ in $L_2(Q,m)$ such that $\phi = \psi \bullet \tilde{\psi}$, where*

$$f \bullet \tilde{g}(r) = \int_Q f(r * s)\overline{g(s)}dm(s).$$

for all $f, g \in C_c(Q)$.

Proof. The proof is as in Pederson[13, Lemma 7.2.4].

Theorem 2.3. *Let ϕ_1 and ϕ_2 belongs to $C_c(Q)$ then the product $\phi_1.\phi_2$ is positive definite on Q if and only if ϕ_1 and ϕ_2 are positive definite on Q.*

Proof. From the above lemma there exists $\psi_1, \psi_2 \in L_2(Q,m)$ such that $\phi_1 = \psi_1 \bullet \tilde{\psi_1}$ and $\phi_2 = \psi_2 \bullet \tilde{\psi_2}$, so

$$\begin{aligned}
\phi_1.\phi_2(r) =& (\psi_1 \bullet \tilde{\psi_1}(r)).(\psi_2 \bullet \tilde{\psi_2}(r)) \\
=& \int_Q \psi_1(r * s)\overline{\psi_1(s)}dm(s) \int_Q \psi_2(r * t)\overline{\psi_2(t)}dm(t) \\
=& \int_Q \int_Q \psi_1(r * s)\psi_2(r * t)\overline{\psi_1(s)\psi_2(t)}dm(s)dm(t) \\
=& \int_Q \int_Q \psi_1.\psi_2(r * s, r * t)\overline{\psi_1.\psi_2(r,t)}dm(s)dm(t) \\
=& \int_Q \int_Q \psi_1.\psi_2(r * (s,t))\overline{\psi_1.\psi_2(s,t)}dm(s)dm(t).
\end{aligned}$$

Applying Fubini's theorem to the right hand side we get

$$\phi_1.\phi_2(r) = \int\limits_{Q\times Q} \psi_1.\psi_2(r*(s,t))\overline{\psi_1.\psi_2(s,t)}dn(s,t)$$

This implies $\phi_1.\phi_2(r) = \psi_1.\psi_2 \bullet \widetilde{\psi_1.\psi_2}(r)$.

As a result of the above theorem, the reader can easily prove the following corollary: xms

Corollary 2.4. *Let $\phi \in C_c(Q)$ be positive definite such that $|\phi(x*x^*)| < \chi$ for all $x \in Q$. Then if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is holomorphic in $\{z \in \mathbb{C}; z < \chi\}$ and $a_n \geq 0$ for all $n \geq 0$, the composed kernel $f \circ \phi$ is again positive definite. In particular, if $\phi \in C_c(Q)$ is positive definite, then so is $\exp(\phi)$.*

Let $\mathbb{A}$ be a maximal algebra in $L_1(Q,m)$. A linear functional $L : \mathbb{A} \to \mathbb{C}$ is called τ-positive, where $\tau \subset \mathbb{A}$ is admissible, if

$$L(a) \geq 0 \quad \text{for all} \quad a \in algspan^+(\tau)$$

This holds if and only if

$$L(a_1 * ... * a_n) \geq 0 \quad \text{for all finite sets} \quad \{a_1, ..., a_n\} \subseteq \tau$$

Let $\alpha : L_1(Q,m) \to \mathbb{R}_+$ be an absolute value such that $\alpha(a) \geq 0$ for all $a \in L_1(Q,m)$. For $\sigma \in \mathbb{C}, a \in L_1(Q,m)$, we define

$$\Omega_{\sigma,a} = \frac{1}{2}(I + \frac{\sigma}{2\alpha(a)}E_a + \frac{\bar{\sigma}}{2\alpha(a^-)}E_{a^-}).$$

Theorem 2.5. *Every τ-positive function $\phi : L_1(Q,m) \to \mathbb{C}$, where $\tau = \{\Omega_{\sigma,a}; \sigma \in \{\pm 1, \pm i\}, a \in X_h\}$ is positive definite and has integral representation*

$$\phi(x) = \int\limits_{X_h} \chi(x)d\mu(\chi),$$

where $\mu \in M_+^b(X_h)$ is concentrated on the compact set of τ-positive characters.

Proof. Let Γ denote the set of τ-positive multiplicative linear functionals on $\mathbb{A}$, which are not identically zero. Clearly Γ is a compact subset of the set of τ-positive linear functionals in $L_1(Q,m)$. By [1,Theorem 4.5.4] the linear functional L corresponding to τ-positive function ϕ has a representation

$$L(T) = \int\limits_{\Gamma} \delta(T)d\tilde{\mu}(\delta), \quad T \in \mathbb{A},$$

where $\tilde{\mu} \in M_+(\Gamma)$. For $\delta \in \Gamma$ the function $x \to \delta(E_x)$ is a τ-positive character, and the mapping $j : \Gamma \to X_h$ given by $j(\delta)(x) = \delta(E_x)$ is a homeomorphism of Γ onto the compact set $j(\Gamma)$ of τ-positive characters. The image measure $\mu := \tilde{\mu}^j$ of $\tilde{\mu}$ under j is a Radon measure on X_h with compact support contained in $j(\Gamma)$, and replacing T by E_x we get

$$\phi(x) = \int_{X_h} \chi(x) d\mu(\chi), \quad x \in L_1(Q, m).$$

3. Negative Definite Functions

One should observe that, a function ψ is negative definite if and only if $\exp(-\alpha\psi)$ is positive definite for each $\alpha > 0$. While this result holds for all semigroups it is not clear how to prove the 'only if' part for hypercomplex systems since the usual technique do not apply(the 'if' part always holds provided that Reψ is locally lower bounded). The problem is that except when r or s belong to the maximal subgroup of the hypercomplex system $\exp(-\alpha\psi(r * s))$ and $\exp(-\alpha\psi)(r * s)$ are usually not equal so that other methods have to be used to overcome this. A locally bounded measurable function q is called a quadratic form if

$$q(r * s) + q(r * s^*) = 2q(r) + 2q(s)$$

for all $r, s \in Q$ and additive if $q(r * s) = 2q(r) + 2q(s)$ for all $r, s \in Q$. In the case Q is hermitian, that when Q carries the identity involution, then every quadratic form is an additive function and every negative definite function is real. A continuous bounded function $\phi(r)$ $(r \in Q)$ will be called negative definite if

$$\sum_{i,j=1}^{n} \lambda_i \bar{\lambda_j} (R_{r_i} * \phi)(r_i) \leq 0$$

for all $r_1, r_2, ..., r_n \in Q, n \in \mathbb{N}$ and $\lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{C}$ that satisfying $\sum_{i=1}^{n} \lambda_i = 0$.

A key result in the study of negative definite functions in hypercomplex systems is the following Levy-Khinchin representation(see,[10])

$$\psi(r) = \psi(e) + q(r) + \int_{X_h \setminus \{1\}} (1 - Re(\chi(r))) d\mu(\chi)$$

for all $r \in Q$ where q is a nonnegative quadratic form on Q and $\mu \in M_+(X_h \backslash \{1\})$. Both q and the integral part $\psi(r) - \psi(e) - q(r)$ belong to the set of negative definite functions on Q and the pair (q, μ) is uniquely determined by ψ with q being given by

$$q(r) = lim\{\frac{\psi(r^{*n})}{n^2} + \frac{\psi((r * r)^{*n})}{2n}\}.$$

Lemma 3.1. *If ϕ is positive definite then $\phi(e) - \phi$ is negative definite.*

For hypercomplex systems we have that if $\exp(-\alpha\psi)$ is positive definite for all $\alpha > 0$ then

$$\exp(-\alpha\psi(e)) - \exp(-\alpha\psi)$$

is negative definite and provided $Re\psi$ is locally lower bounded

$$lim \frac{\exp(-\alpha\psi(e)) - \exp(-\alpha\psi)}{\alpha} = \psi - \psi(e)$$

is also negative definite in which case so is ψ. In spit of the converse statement also holds for commutative semigroups, this result is not available for hypercomplex systems which hinges on deciding whether $\exp(-\psi)$ is positive definite.

Lemma 3.2. *Let $\psi : Q \times Q \to \mathbb{C}$. Put*

$$\phi(r, s) := \psi(r, r_0) + \overline{\psi(s, r_0)} - \psi(r, s) - \psi(r_0, r_0)$$

for fixed $r_0 \in Q$. Then ϕ is positive definite if and only if ψ is negative definite.

As a consequence of above discussion and Theorem 2.3 we will prove the following corollary:

Corollary 3.3. *A function $\psi \in C_c(Q)$ is negative definite if and only if $\exp(-\alpha\psi)$ is positive definite for each $\alpha > 0$.*

Proof. Suppose that ψ is negative definite. For obvious reasons we need only show that $\exp(-\alpha\psi)$ is positive definite for $\alpha = 1$. We choose $r_0 \in Q$ and with ϕ as in the above Lemma we have

$$-\psi(r, s) := \phi(r, s) - \overline{\psi(s, r_0)} - \psi(r, r_0) + \psi(r_0, r_0)$$

where ϕ is positive definite. Hence

$$\exp(-\psi(r, s)) = \exp(\phi(r, s)).\overline{\exp(-\psi(s, r_0))}. \exp(-\psi(r, r_0)). \exp(\psi(r_0, r_0))$$

Since, $\overline{\exp(-\psi(s, r_0))}. \exp(-\psi(r, r_0))$, is positive definite. From Theorem 2.3 we conclude that $\exp(-\alpha\psi)$ is positive definite.

4. Completely Monotone and Completely Alternating Functions

Suppose that $L^{\infty}_{loc}(Q)$ denotes the set of locally bounded measurable functions on Q, and $L^{c}_{1}(Q)$ the space of integrable functions on Q with compact support. Similar to the pointed out result for hypergroups in [4] and semigroups in [9], we have

$$L^{c}_{1}(Q) * L^{loc}_{\infty}(Q) \subset C(Q)$$

For each $r \in Q$, we define the shift operator E_s by $E_s\phi(r) = \phi(r * s)$ for all $r, s \in Q$ and $\phi \in \mathbb{C}^Q$. The complex span $\mathbb{A}$ of all such operators is a commutative algebra with identity $E_1 = I$ and involution $(\sum \alpha_i E_{r_i})^- = \sum \overline{\alpha_i} E_{r_i^-}$. For real valued $\phi \in L^{loc}_{\infty}(Q)$ and $r \in Q$ we define $\nabla_r \phi : Q \to \mathbb{R}$ by

$$(\nabla_r \phi)(s) := (I - E_r)(\phi)(s)$$

We call ϕ completely monotone if $\phi \geq 0$ and

$$\nabla_{r_1} \nabla_{r_2} ... \nabla_{r_n} \phi \geq 0$$

for all $n \in \mathbb{N}$ and $r_1, r_2, ..., r_n \subset Q$. For some applications of the completely monotone functions see [5-6]. The function ϕ is said to be completely alternating if

$$\nabla_{r_1} \nabla_{r_2} ... \nabla_{r_n} \phi \leq 0$$

for all $n \in \mathbb{N}$ and $r_1, r_2, ..., r_n \subset Q$. With $\Delta_r \psi := -\nabla_r \psi$ we see from

$$\nabla_{r_1} \nabla_{r_2} ... \nabla_{r_n} (\Delta_x \psi) = -\nabla_{r_1} \nabla_{r_2} ... \nabla_{r_n} \nabla_x \psi$$

that $\psi \in L^{loc}_{\infty}(Q)$ is completely alternating if and only if $\Delta_r \psi$ is completely monotone for each $r \in Q$. The set of completely monotone (res. alternating) functions is denoted $\mathbb{M}(Q)$ (resp. $\mathbb{A}(Q)$). It is clear that $\mathbb{M}(Q)$ and $\mathbb{A}(Q)$ are closed convex cones in $\mathbb{R}^Q$.

Theorem 4.1. *For a continuous function $\phi : Q \longrightarrow \mathbb{R}$ the following conditions are equivalent:*

(i) ϕ is completely monotone.

(ii) ϕ is τ-positive.

(iii) There exists a measure $\mu \in M^b_+(X_{h+})$ such that for all $r \in Q$

$$\phi(r) = \int_{X_{h+}} \chi(r) d\mu(\chi).$$

Proof. For $u_1, ..., u_n, r_1, ..., r_m \in Q$ we have

$$(I - E_{u_1})...(I - E_{u_n})E_{r_1}...E_{r_m}\phi(e) = \nabla_{u_1}...\nabla_{u_n}\phi(r_1 * ... * r_m)$$

so (i)⇒(ii). The implication "(ii)⇒(iii)" follows from the above Theorem since $0 \leq \chi(r) \leq 1$ for all $r \in Q$. Finally, if (iii) holds, then $\phi(r) \geq 0$ and since

$$\nabla_{u_1}...\nabla_{u_n}\chi(r) = \chi(r)\prod_{i=1}^{n}[1 - \chi(u_i)]$$

so,

$$\nabla_{u_1}...\nabla_{u_n}\phi(r) = \int_{X_{h+}} \chi(r)\prod_{i=1}^{n}[1 - \chi(u_i)]d\mu(\chi) \geq 0,$$

hence (i).

Theorem 4.2. *A continuous function $\psi : Q \to \mathbb{R}$ is completely alternating if and only if there exists an additive continuous function $h : Q \to \mathbb{R}_+$ and a unique measure $\mu \in M_+(X_{h+}\backslash\{1\})$ such that for all $r \in Q$*

$$\psi(r) = \psi(e) + h(r) + \int_{X_{h+}\backslash\{1\}} (1 - \chi(r))d\mu(\chi).$$

Proof. It follows from the definition that ψ is lower bounded by $\psi(e)$, firstly, we assume that $\psi(e) = 0$. Let $S \supseteq Q$ be a minimal semigroup containing the hypercomplex basis Q. Introducing

$$\Delta_s\psi(r) := \frac{1}{2}[\psi(r * s) + \psi(r * s^*)] - \psi(r); \quad r, s \in Q,$$

and as pointed in [1, Proposition 4.3.11] $\Delta_s\psi$ is bounded and positive definite on Q. Therefore appealing to Bochner's theorem for hypercomplex systems([7], Theorem 12.3B)

$$\Delta_s\psi(r) = \int_{X_{h+}} \rho_\chi(r)d\sigma_s(\chi)$$

for some $\sigma_s \in M_+^b(X_{h+})$, where we denote the canonical extension of $\chi \in X_{h+}$ to a function on Q by ρ_χ. A simple calculation implies

$$-\Delta_t\Delta_s\psi(r) = \int_{X_{h+}} \rho_\chi(r)[1 - Re\rho_\chi(t)]d\sigma_s(\chi)$$

$$= \int\limits_{X_{h+}} \rho_\chi(r)[1 - Re\rho_\chi(s)]d\sigma_t(\chi)$$

for $r, s, t \in Q$, implying

$$[1 - Re\rho_\chi(t)]d\sigma_s(\chi) = [1 - Re\rho_\chi(s)]d\sigma_t(\chi)$$

by the uniqueness of the Fourier transform ([7], Theorem 12.2A). Noting that the $\{\chi \in X_{h+}; Re\chi(s) \leq 1\}$ are open sets in X_{h+} with union(over s)given by $X_{h+}\backslash\{1\}$, we can find a unique Radon measure μ on $X_{h+}\backslash\{1\}$ such that for every $s \in S$

$$[1 - Re\rho_\chi(s)]d\mu(\chi) = d\sigma_s(\chi), \quad \text{on} \quad X_{h+}$$

The set $\hat{S}$ of all bounded semigroup characters on the semigroup S is a compact Hausdorff space with respect to the topology of point wise convergence. The canonical mapping $\zeta : X_{h+} \to \hat{S}, \zeta(\chi) := \rho_\chi$ is continuous, and obviously $\Delta_s\psi$ is the Laplace transform of σ_s^ζ(the image measure of σ_s under ζ) for each $s \in S$ hence from [1, Lemma 4.3.12, Definition 4.3.13 and Theorem 4.6.7] there exists an additive continuous function $h : Q \to \mathbb{R}_+$ on S such that for all $r \in S$

$$\psi(r) = h(r) + \int\limits_{\hat{S}_+\backslash\{1\}} (1 - \rho(r))d\mu^\zeta(\rho) = h(r) + \int\limits_{X_{h+}\backslash\{1\}} (1 - \chi(r))d\mu(\chi)$$

where $\mu \in M_+(X_{h+}\backslash\{1\})$.

Theorem 4.3. *The cone $\mathbb{M}(Q)$ is an extreme subset of $\mathbb{P}_b(Q)$, the set of bounded positive definite functions on X, and $\mathbb{M}^1(Q)$ is a Bauer simplex with $ex(\mathbb{M}^1(Q)) = X_{h+}$. For $\phi_1, \phi_2 \in \mathbb{M}(Q)$ also $\phi_1.\phi_2 \in \mathbb{M}(Q)$. A function $\phi \in \mathbb{P}_b(Q)$ is completely monotone if and only if the representing measure μ is concentrated on X_{h+}.*

Proof. As pointed out of Theorem 4.1. $\mathbb{M}(Q) \subseteq \mathbb{P}_b(Q)$. Suppose that $\phi = \phi_1 + \phi_2 \in \mathbb{M}(Q)$ and $\phi_i \in \mathbb{P}_b(Q)$ with

$$\phi_i(x) = \int\limits_{X_{h+}} \chi(x)d\mu_i(\chi), \quad i = 1, 2.$$

then $\mu = \mu_1 + \mu_2$, where $\mu \in M(X_{h+})$ is the representing measure for ϕ. Hence, μ_1, μ_2 are concentrated on X_{h+} so $\phi_1, \phi_2 \in \mathbb{M}(Q)$, So $\mathbb{M}(Q)$ is an extreme subset of $\mathbb{P}_b(Q)$. By transitivity of extremality an extreme point of $\mathbb{M}^1(Q)$ is also an extreme point of $\mathbb{P}_b^1(Q)$, hence $ex(\mathbb{M}^1(Q)) \subseteq X_{h+}$, and in fact there is equality since $X_{h+} \subseteq \mathbb{M}^1(Q) \cap ex(\mathbb{P}_b^1(Q))$. Recalling Theorem 2.3., then for any $\mu, \nu \in M(X_+)$ we have $supp(\mu * \nu) \subseteq X_{h+}$, and it follows that $\mathbb{M}(Q)$ is a stable under multiplication. By unicity of the representing measure for $\phi \in \mathbb{P}_b(Q)$ it follows that $\mathbb{M}^1(X_{h+})$ is a simplex, and that the representing measure for ϕ is concentrated on X_{h+} if $\phi \in \mathbb{P}_b(Q)$ is completely monotone.

Corollary 4.4. *A continuous function with compact support* $\psi \in \mathbb{A}(Q)$ *if and only if* $\exp(-t\psi) \in \mathbb{M}(Q)$ *for each* $t > 0$.

Proof. Suppose that $\exp(-t\psi) \in \mathbb{M}(Q)$ for all $t > 0$ then $1 - \exp(-t\psi) \in \mathbb{A}(Q)$, so $\frac{1-\exp(-t\psi)}{t} \in \mathbb{A}(Q)$. For the converse it suffices to prove that $\exp(-\psi) \in \mathbb{M}(Q)$ for $\psi \in \mathbb{A}(Q)$ with the representation

$$\psi(x) = \psi(e) + h(x) + \int_{X_{h+}\setminus\{1\}} (1 - \chi(x))d\mu(\chi)$$

Since $\mathbb{M}(Q)$ is closed under multiplication and $\exp(-h) \in X_{h+}$, it suffices to prove that

$$x \to exp[-\int_{X_{h+}\setminus\{1\}} (1 - \chi(x))d\mu(\chi)]$$

belongs to $\mathbb{M}(Q)$. This in fact true because $\chi \in \mathbb{M}(Q)$ and

$$\exp(c\chi) = \sum_{n=0}^{\infty} \frac{1}{n!} c^n \chi^n, \quad c \geq 0.$$

Remark. In [8], depending on the results given by Pederson[13, lemma 7.2.4], we were proved the stability of the set of continuous positive definite functions with compact support on hypergroups. In the same direction, the second part of Theorem 4.3. ensure that, the product of two completely monotone functions not only positive definite, but also completely monotone on $L_1(Q, m)$.

References

[1] C. Berg, J.P.R. Christensen, P. Ressel, *Harmonic Analysis on Semigroups. Theory of Positive Definite and Related Functions*, Graduated Texts in Math.,**100**, Springer-Verlag, Berlin-Heidelberg-New-York (1984).

[2] Y.M. Berezansky, A.A. Kalyuzhnyi, *Harmonic Analysis in Hypercomplex Systems*, Kluwer Academic Publisher, Dordrecht-Boston-London (1998).

[3] Y.M. Berezansky, A.A. Kalyuzhnyi, Nuclear function spaces on the base of hypercomplex systems, *Inst. Mat.*, **35** (1982), 9-17.

[4] W.R. Bloom, H. Heyer, *Harmonic Analysis of Probability Measures on Hypergroups*, De Gruyter, Berlin (1995).

[5] H.A. Ghany, Basic completely monotone functions as coefficients and solutions of linear q-difference equations with some applications, *Physics Essays*, **25**, No. 1 (2012), 38-42.

[6] H.A. Ghany, Computations of some integrals involving logarithmic function with some thermodynamical applications, *Physics Essays*, **24**, No. 3 (2011), 456-459.

[7] R.I. Jewett, Spaces with an abstract convolution of measures, *Adv. Math.*, **18** (1975), 1-101.

[8] A.S. Okb El Bab, H.A. Ghany, Harmonic analysis on hypergroups, *International Journal of Pure and Applied Mathematics*, **64**, No. 1 (2010), 22-31.

[9] A.S. Okb El Bab, H.A. Ghany, A. Hady, Remarks on some quasi classes of the moment problems, *International Journal of Pure and Applied Mathematics*, **74**, No. 2 (2012), 221-227.

[10] A.S. Okb El Bab, H. A. Ghany, M. Saad, On positive definite functions and some related functions on Hypergroups, *International Journal of Mathematical Analysis*, **6**, No. 13 (2012), 599-607.

[11] A.S. Okb El Bab, H.A. Ghany, S. Ramadan, On strongly negative definite functions for the product of commutative hypergroups, *International Journal of Pure and Applied Mathematics*, **71**, No. 4 (2011), 581-594.

[12] A.S. Okb El Bab, M.S. Shazly, Characterisation of convolution semigroups, *Indian Journal of Theoretical Physics*, **4** (1987), 300-311.

[13] G.K. Pederson, *C^*-Algebra and their Automorphism Groups*, London Mathematical Society Monograph, No. 14, Academic Press, London-New York (1979).

[14] A.M. Zabel, B.A. Bin Dehaish, Levy khinchin formula on hypercomplex systems, *Kyungpook Math. J.*, **48** (2008), 559-575.