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JACOBI ELLIPTIC NUMERICAL SOLUTIONS FOR THE TIME FRACTIONAL DISPERSIVE LONGWAVE EQUATION

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Abstract: The fractional derivatives in the sense of Caputo, and the homotopy analysis method (HAM) are used to construct the approximate solutions for nonlinear fractional dispersive long wave equation with reaspect to time fractional derivative. The HAM contains a certain auxiliary parameter which provides us with a simple way to adjust and control the convergence region and rate of convergence of the series solution.

AMS Subject Classification: homotopy analysis method, Caputo's fractional derivative, fractional dispersive long wave equation
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1. Introduction

Fractional differential equations FDEs have found applications in many problems in physics and engineering [1, 2, 3, 4]. Since most of the nonlinear FDEs cannot be solved exactly, approximate and numerical methods must be used. Some of the recent analytical methods for solving nonlinear problems include

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the Adomian decomposition method ADM, see [5, 6], variational iteration method VIM [7], Homotopy-perturbation method HPM, see [8, 9], and homotopy analysis method HAM, see [10]-[16]. The HAM, first proposed in 1992 by Liao [10], has been successfully applied to solve many problems in physics and science. This method is applied to solve linear and nonlinear fractional order systems.

In recent years, numerous studies and applications of fractional-order systems in many areas of science and engineering have been presented. This is a result of better understanding of the potential of fractional calculus revealed by problems such as viscoelasticity and damping, chaos, diffusion, wave propagation, percolation and irreversibility. Recently, many systems have been identified that display fractional-order dynamics, such as viscoelastic systems, quantum evolution of complex system, and the control of fractional-order dynamic systems, see [17, 18].

In this work, a new algorithm for solving the fractional order dispersive long wave equation (see [19]) is proposed based on HAM:

$$\begin{aligned} D_t^{\alpha} u + (uu)_x + v_x =&0,\\ D_t^{\alpha} v + (vu)_x + \frac{1}{3} u_{xxx} =&0, \end{aligned} \tag{1.1}$$

where $D_t^{\alpha} = \frac{\partial^{\alpha}}{\partial t^{\alpha}}$, α is the parameter standing for the order of the fractional time derivative, and satisfy $0 < \alpha \leq 1$. We use the Caputo fractional derivative on the half axis R^{+} (i.e. $t \in R^{+}$) ${}_t^C D_{0_+}^{\alpha}$ for time.

Subject to the initial conditions

$$\begin{aligned} u(x,0) = f(x) =&\lambda + \frac{2}{3}\sqrt{3}m\ sn(x,m),\\ v(x,0) = g(x) =&\frac{1}{3}(1 + m^2 - 2m^2\ sn^2(x,m)). \end{aligned} \tag{1.2}$$

where m is the modulus of the Jacobi elliptic sine functions ($0 < m < 1$) and the $sn(x,m)$, $cn[x,m], dn[x,m]$ are the Jacobi elliptic functions. These Jacobi elliptic function solutions degenerate to the soliton wave solutions and trigonometric function solutions at a certain limit condition.

In this paper, we present an alternative approach based on HAM to the system of FDEs (1.1) and (1.2), to obtain analytic approximations for fractional order dispersive long wave equation and demonstrate the effectiveness of the HAM algorithm.

2. Basic Definitions

In this section, we give some definitions and properties of the fractional calculus. Several definitions of fractional calculus have been proposed in the last two centuries. The fractional calculus is a name for the theory of integrals and derivatives of arbitrary order. There are many books [1, 2, 3, 4] that develop fractional calculus and various definitions of fractional integration and differentiation, such as Grunwald-Letnikov's definition, Riemann-Liouville definition, Caputo's definition and generalized function approach.

Definition 2.1. A real function $h(t)$, $t > 0$, is said to be in the space C_μ, $\mu \in \mathbb{R}$ if there exists a real number $p > \mu$, such that $h(t) = t^p h_1(t)$, where $h_1(t) \in C[0, \infty)$, and it is said to be in the space C_μ^n if and only if $h^{(n)} \in C_\mu$, $n \in \mathbb{N}$.

Definition 2.2. The Riemann-Liouville fractional integral operator (J^α) of order $\alpha \geq 0$, of a function $h \in C_\mu$, $\mu \geq -1$, is defined as

$$\begin{aligned} J^\alpha h(t) =& \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} h(\tau) d\tau \quad (\alpha > 0), \\ J^0 h(t) =& h(t), \end{aligned} \tag{2.1}$$

$\Gamma(\alpha)$ is the well- known Gamma function. Some of the properties of the operator J^α, which we will need here, are as follows:

(1) $J^\alpha J^\beta h(t) = J^{\alpha+\beta} h(t)$,

(2) $J^\alpha J^\beta h(t) = J^\beta J^\alpha h(t)$,

(3) $J^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma}$, where $\beta \geq 0$, and $\gamma \geq -1$.

Definition 2.3. The fractional derivative (D^α) of $h(t)$ in the Caputo's sense is defined as

$$\begin{aligned} D^\alpha h(t) =& \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} h^{(n)}(\tau) d\tau, \\ \text{for } n-1 <& \alpha \leq n, \quad n \in \mathbb{N}\ , t > 0, \quad h \in C_{-1}^n. \end{aligned} \tag{2.2}$$

The following are two basic properties of Caputo's fractional Derivative [4]:

(1) Let $h \in C_{-1}^n$, $n \in \mathbb{N}$. Then $D^\alpha h$, $0 \leq \alpha \leq n$ is well defined and $D^\alpha h \in C_{-1}$.

(2) Let $n-1<\alpha\leq n$, $n\in\mathbb{N}$ and $h\in C_\mu^n$, $\mu\geq -1$. Then

$$(J^\alpha D^\alpha)h(t)=h(t)-\sum_{k=0}^{n-1}h^{(k)}(0^+)\frac{t^k}{k!}. \tag{2.3}$$

3. Homotopy Analysis Method (HAM) for System of FDEs

Let us consider the following system of differential equation

$$N_i(u_1(t),\dots,u_n(t))=0,\quad i=1,2,\dots n, \tag{3.1}$$

subject to the following initial conditions at initial value:

$$u_k(t)=c_k,\qquad k=1,\dots,n,$$

where N_i are nonlinear operators, t denotes an independent operator and $u_i(t)$ are the unknown functions.

We can construct the following Zeroth-order deformation for $i=1,2,\dots n$,

$$(1-q)\mathscr{L}_i(\phi_i(t\ ;q)-u_{i\,0}(t))=qh_iH_i(t)N_i[\phi_1(t\ ;q),\dots,\phi_n(t\ ;q)], \tag{3.2}$$

where $q\in[0,1]$ is an embedding parameter, $h_i\neq 0$ are auxiliary parameters, $H_i(t)\neq 0$ are auxiliary functions, $\mathscr{L}_i=D_t^{\alpha_i}(n-1<\alpha_i\leq n)$ are auxiliary linear operators such that

$$\mathscr{L}_i[\phi_i(t)]=0 \text{ when } \phi_i(t)=0. \tag{3.3}$$

Generally, $u_{i\,0}(t)$ are initial guesses, which satisfy the initial conditions and $\phi_i(t$ $;q)$ are unknown functions where

$$\phi_i(t\ ;0)=u_{i0}(t),\quad \phi_i(t\ ;1)=u_i\ (t),\qquad i=1,2,\dots n. \tag{3.4}$$

Expanding $\phi_i(t\ ;q)$ in Taylor series we have

$$\phi_i(t\ ;q)=u_{i0}(t)+\sum_{m=1}^{\infty}u_{im}(t)q^m,\quad i=1,2,\dots n, \tag{3.5}$$

where

$$u_{im}(t)=\frac{1}{m!}\frac{\partial^m\phi_i(t\ ;q)}{\partial q^m}\,|_{q=0},\quad i=1,2,\dots n. \tag{3.6}$$

If the auxiliary parameters h_i, the auxiliary functions $H_i(t)$, the initial approximations $u_{i0}(t)$ and the auxiliary linear operators $\mathscr{L}_i$ are so properly chosen the series (3.5) converges at $q = 1$. Then, using (3.4) the series (3.5) becomes

$$u_i\,(t) = u_{i0}(t) + \sum_{m=1}^{\infty} u_{im}(t), \qquad i = 1, 2, \dots n. \tag{3.7}$$

Let us, we define the following vectors

$$\vec{u_i}(t) = \{u_{i0}(t), u_{i1}(t), u_{i2}(t), \dots, u_{in}(t)\}, \qquad i = 1, 2, \dots n. \tag{3.8}$$

then differentiating (3.2) m times with respect to q, setting $q = 0$ and dividing by $m!$, we have the mth-order deformation equation

$$\mathscr{L}_i(u_{im}(t) - \varkappa_m u_{im-1}(t)) = h_i H_i(t) \mathcal{R}_{im}(\vec{u_{1m-1}}, \vec{u_{2m-1}}, \dots, \vec{u_{nm-1}}), \tag{3.9}$$

where

$$\begin{aligned}\mathcal{R}_{im}&(\vec{u_{1m-1}}, \vec{u_{2m-1}}, \dots, \vec{u_{nm-1}})\\ &= \frac{1}{(m-1)!} \frac{\partial^{m-1} N_i[\phi_1(t\,;q), \dots, \phi_n(t\,;q)]}{\partial q^{m-1}} \Big|_{q=0},\end{aligned} \tag{3.10}$$

and

$$\varkappa_m = \begin{cases} 0 & m \le 1, \\ 1 & m > 1. \end{cases} \tag{3.11}$$

Applying the Riemann-Liouville integral operator J^{α_i} on both side of equation (3.9), we get

$$\begin{aligned} u_{im}(t) &= \varkappa_m u_{im-1}(t) \\ &- \varkappa_m \sum_{j=0}^{n-1} u_{im-1}^{(j)}(0^+) \frac{t^j}{j!} + h_i H_i(t) J^{\alpha_i}[\mathcal{R}_{im}(\vec{u_{1m-1}}, \vec{u_{2m-1}}, \dots, \vec{u_{nm-1}})], \end{aligned} \tag{3.12}$$

The mth-order deformation equations. (3.9) are linear and they can be easily solved, especially by means of symbolic computation software such as Mathematica, Maple, Matlab.

4. Application

To demonstrate the effectiveness of the method, we consider the system of nonlinear fractional initial-value problem (1.1) for $0 < \alpha < 1$ with the initial conditions (1.2) by choosing the linear operators

$$\begin{aligned} \mathscr{L}_1[\phi_1(t\ ;q)] &= D_t^\alpha[\phi_1(t\ ;q)],\\ \mathscr{L}_2[\phi_2(t\ ;q)] &= D_t^\alpha[\phi_2(t\ ;q)], \end{aligned} \tag{4.1}$$

with the property $\mathscr{L}_i[c_i] = 0, i = 1, 2$ where c_i are the integral constant and the nonlinear operators are defined as

$$\begin{aligned} N_1[\phi_1, \phi_2] &= \quad D_t^\alpha \phi_1 + (\phi_1\phi_1)_x + (\phi_2)_x,\\ N_2[\phi_1, \phi_2] &= \quad D_t^\alpha \phi_2 + (\phi_1\phi_2)_x + \tfrac{1}{3}(\phi_1)_{xxx}. \end{aligned}$$

Choosing $H_i(t) = 1$ for $i = 1, 2$ the zeroth-order deformation equations are

$$\begin{aligned} (1-q)\mathscr{L}_1[\phi_1(t\ ;q) - u_0(t)] =& qhN_1[\phi_1, \phi_2],\\ (1-q)\mathscr{L}_2[\phi_2(t\ ;q) - v_0(t)] =& qhN_2[\phi_1, \phi_2], \end{aligned} \tag{4.2}$$

where

$$\begin{aligned} \phi_1(t\ ;0) =& u_0(t), \quad \phi_1(t\ ;1) = u(t),\\ \phi_2(t\ ;0) =& v_0(t), \quad \phi_2(t\ ;1) = v(t). \end{aligned} \tag{4.3}$$

Then, the mth-order deformation equations become

$$\begin{aligned} \mathscr{L}_1[u_m(t) - \varkappa_m u_{m-1}(t)] =& h\mathcal{R}_{1m}(\overrightarrow{u_{m-1}}, \overrightarrow{v_{m-1}}),\\ \mathscr{L}_2[v_m(t) - \varkappa_m v_{m-1}(t)] =& h\mathcal{R}_{2m}(\overrightarrow{u_{m-1}}, \overrightarrow{v_{m-1}}), \end{aligned} \tag{4.4}$$

where

$$\begin{aligned} \mathcal{R}_{1m}(\overrightarrow{u_{m-1}}, \overrightarrow{v_{m-1}}) &= \quad D_t^\alpha u_{m-1} + \textstyle\sum_{j=0}^{m-1} u_j(u_{m-1-j})_x + (v_{m-1})_x,\\ \mathcal{R}_{2m}(\overrightarrow{u_{m-1}}, \overrightarrow{v_{m-1}}) &= \quad D_t^\alpha v_{m-1} + (\textstyle\sum_{j=0}^{m-1} u_j v_{m-1-j})_x + \tfrac{1}{3}(u_{m-1})_{xxx}. \end{aligned}$$

The system (4.4) have the following general solutions

$$u_m(t) = \varkappa_m u_{m-1}(t) + hJ^\alpha[D_t^\alpha u_{m-1} + \sum_{j=0}^{m-1} u_j(u_{m-1-j})_x + (v_{m-1})_x], \tag{4.5}$$

$$v_m(t) = \varkappa_m v_{m-1}(t) + hJ^\alpha[D_t^\alpha v_{m-1} + (\sum_{j=0}^{m-1} u_j v_{m-1-j})_x + \frac{1}{3}(u_{m-1})_{xxx}]. \tag{4.7}$$

In this case, where u_0 and v_0 are constant, the general solution (4.5), (4.6) and (4.7) are taking the following form

$$\begin{aligned}
u_m(t) =&(\varkappa_m + h)u_{m-1}(t)\\
&+ (\varkappa_m - 1)hu_{m-1}(t) + hJ^{\alpha}[\sum_{j=0}^{m-1} u_j(u_{m-1-j})_x + (v_{m-1})_x]\\
v_m(t) =&(\varkappa_m + h)v_{m-1}(t)\\
&+ (\varkappa_m - 1)hv_{m-1}(t) + hJ^{\alpha}[(\sum_{j=0}^{m-1} u_j v_{m-1-j})_x + \frac{1}{3}(u_{m-1})_{xxx}].
\end{aligned}$$

Finally, we have

$$\begin{aligned}
u(x,t) =&u_0 + \sum_{m=1}^{\infty} u_m(x,t),\\
v(x,t) =&v_0 + \sum_{m=1}^{\infty} v_m(x,t).
\end{aligned} \tag{4.8}$$

Substituting from (1.2) into (4.5), (4.6) and (4.7) we have

$$\begin{aligned}
u_1\,(x,t) =&h\ f_1(x)\frac{t^{\alpha}}{\Gamma(\alpha+1)},\\
v_1(x,t) =&h\ g_1(x)\frac{t^{\alpha}}{\Gamma(\alpha+1)},
\end{aligned} \tag{4.9}$$

$$\begin{aligned}
u_2(x,t) =&(1+h)u_1\,(x,t) + h^2 f_2(x)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},\\
v_2(x,t) =&(1+h)v_1\,(x,t) + h^2 g_2(x)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}
\end{aligned} \tag{4.10}$$

$$\begin{aligned}
u_3(x,t) =&(1+h)u_2\,(x,t) + h^3 f_3(x)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + h^2(1+h)f_2(x)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},\\
v_3(x,t) =&(1+h)v_2\,(x,t) + h^3 g_3(x)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + h^2(1+h)g_2(x)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}.\\
&\ldots
\end{aligned} \tag{4.11}$$

and so on. After some calculation, we get:

$$
\begin{aligned}
f(x) =& \lambda + \frac{2}{3}\sqrt{3}m\ sn(x,m),\\
f_1(x) =& g_x + f f_x,\\
f_2(x) =& (g_1)_x + f_1 f_x + f(f_1)_x,\\
f_3(x) =& (g_2)_x + f_2(f_x) + f_1(f_1)_x\frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)} + f(f_2)_x,\\
& \ldots
\end{aligned}
\tag{4.12}
$$

and

$$
\begin{aligned}
g(x) =& \frac{1}{3}(1+m^2-2m^2\ sn^2(x,m)),\\
g_1(x) =& \frac{1}{3}(f)_{xxx} + g f_x + f(g)_x,\\
g_2(x) =& \frac{1}{3}(f_1)_{xxx} + g_1 f_x + f_1(g)_x + f(g_1)_x + g(f_1)_x,\\
g_3(x) =& \frac{1}{3}(f_2)_{xxx} + g(f_2)_x + f_2(g)_x + g_1(f_1)_x\frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)} +\\
& f_1(g_1)_x\frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)} + g_2(f)_x + f(g_2)_x,\\
& \ldots
\end{aligned}
\tag{4.13}
$$

In this case the approximate solution of time-fractional equation (1.1) according to the HAM, we can conclude that

$$
\begin{aligned}
u_{app} =&\ u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + \ldots,\\
v_{app} =&\ v_0(x,t) + v_1(x,t) + v_2(x,t) + v_3(x,t) + \ldots,
\end{aligned}
$$

$$
\begin{aligned}
u_{app} =& f(x) + h\ f_1\frac{t^\alpha}{\Gamma(\alpha+1)} + h(1+h)f_1\frac{t^\alpha}{\Gamma(\alpha+1)} + h^2 f_2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\\
& + (1+h)(h(1+h)f_1\frac{t^\alpha}{\Gamma(\alpha+1)} + h^2 f_2(x)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)})\\
& + h^3 f_3\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + h^2(1+h)f_2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \ldots,
\end{aligned}
\tag{4.14}
$$

$$
v_{app} = g(x) + h\ g_1\frac{t^\alpha}{\Gamma(\alpha+1)} + h(1+h)g_1\frac{t^\alpha}{\Gamma(\alpha+1)} + h^2 g_2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}
$$

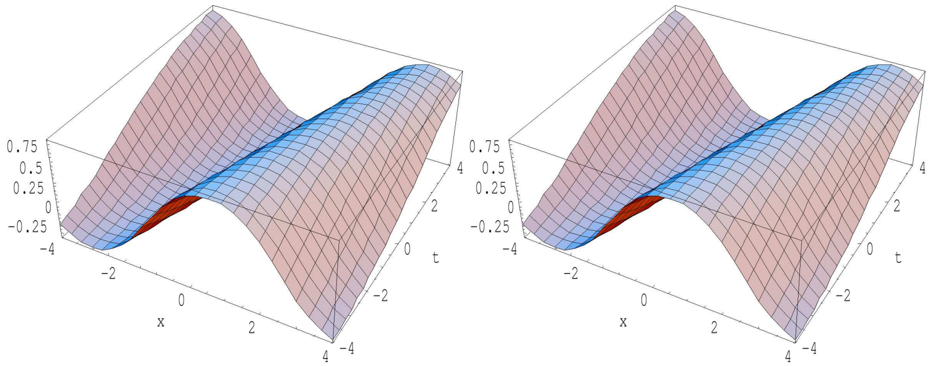

Figure 1: The numerical results for $\phi_5(x,t)$: (a) in comparison with the exact solution $u(x,t)$; (b) when $m = 0.5$ and $\lambda = 0.3$ for the Jacobi elliptic solution with the initial condition of equation (1.1).

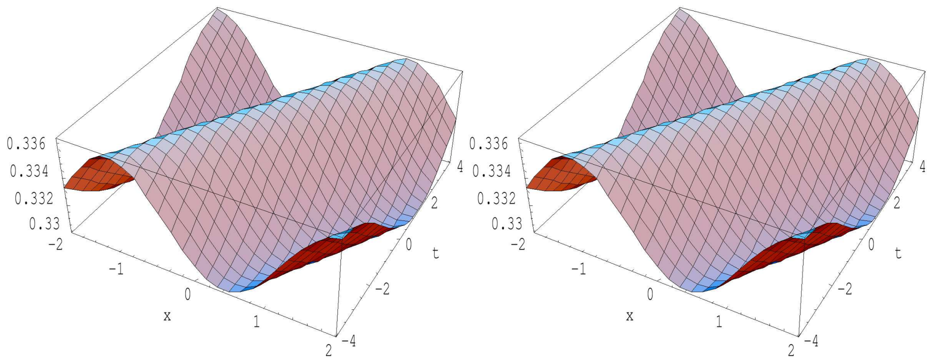

Figure 2: The numerical results for $\phi_4(x,t)$: (a) in comparison with the exact solution $v(x,t)$; (b) when $m = 0.1$ and $\lambda = 0.3$ for the Jacobi elliptic solution with the initial condition of equation (1.1).

$$
\begin{aligned}
&+ (1+h)(h(1+h)g_1 \frac{t^{\alpha}}{\Gamma(\alpha+1)} + h^2 g_2 \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}) \\
&+ h^3 g_3 \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + h^2(1+h) g_2 \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \ldots \qquad (4.15)
\end{aligned}
$$

And so on setting $h = -1$ and $\alpha = 1$,we get an accurate approximation solution in the following form:

$$u_{app}\,|_{\mathrm{ADM}} = u_{app}\,|_{\mathrm{HAM}}$$

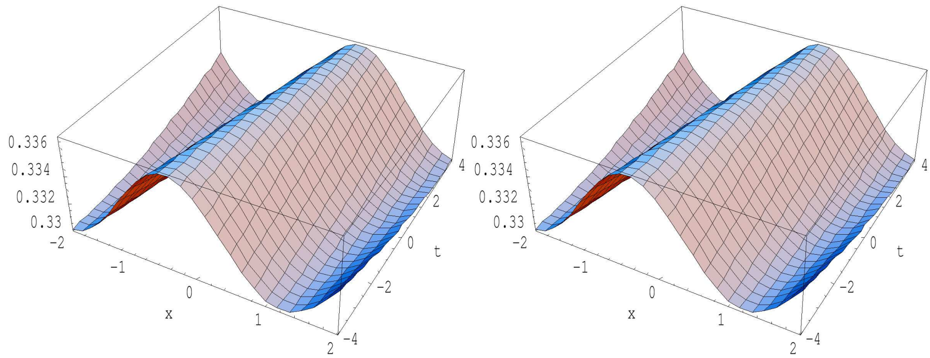

Figure 3: The numerical results for $\phi_5(x,t)$: (a) in comparison with the exact solution $v(x,t)$; (b) when $m = 0.1$ and $\lambda = 0.1$ for the Jacobi elliptic solution with the initial condition of equation (1.1).

$$
\begin{aligned}
&= f - f_1 \frac{t^{\alpha}}{\Gamma(\alpha+1)} + f_2 \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - f_3 \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \dots, \\
v_{app} &\mid_{\text{ADM}} = v_{app} \mid_{\text{HAM}} \\
&= g - g_1 \frac{t^{\alpha}}{\Gamma(\alpha+1)} + g_2 \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - g_3 \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \dots
\end{aligned} \tag{4.16}
$$

Using Taylor series expansion near $t = 0$,we get:

$$u(x,t) = \lambda + \frac{2}{3}\sqrt{3} msn[x - \lambda t, m], \tag{4.17}$$

and

$$v(x,t) = \frac{1}{3}(1 + m^2 - 2m^2 sn^2[x - \lambda t, m]). \tag{4.18}$$

These are the exact solutions of the dispersive long wave equations partial differential equation (1.1) obtained when replacing $\alpha = 1$ and $h = -1$ in (4.14) and (4.15). These solutions (4.17) and (4.18) are exactly the same solutions obtained in [19].

5. Conclusion

In this paper, the Homotopy analysis method (HAM) has been successfully applied to obtain the numerical solutions of the time fractional dispersive long wave equation. From Figures $1-3$, we deduce the behavior of the approximate solutions are the same behavior of the exact solutions at some different values α. Consequently, we deduce the approximate solutions are rapidly convergent series as the exact solutions. The HAM contains a certain auxiliary parameter which provides us with a simple way to adjust and control the convergence region and rate of convergence of the series solution.

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