International Journal of Pure and Applied Mathematics
Volume 80 No. 5 2012, 657-663
ISSN: 1311-8080 (printed version)
url: http://www.ijpam.eu

SUPRA $D-$SETS AND ASSOCIATED SEPARATION AXIOMS

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Abstract: In this paper, we introduce and investigate some weak separation axioms by using the notion of supra open sets. We study the relationships between these new separation axioms and their relationships with some other properties.

AMS Subject Classification: 54D10
Key Words: supra open sets, supra T_i spaces (i=0,1,2), supra D_i spaces (i=0,1,2)

1. Introduction and Preliminaries

In 1983, A. S. Mashhour et al. [1] introduced the supra topological spaces. In 2010, O. R. Sayed et al. [2] introduced and studied a class of sets and maps between topological spaces called supra b-open sets and supra b-continuous functions respectively. Now we study the notions of supra T_i spces $i = 0, 1, 2$. Also we introduce and study the concepts of supra D_i spaces for $i = 0, 1, 2$ and investigate several properties for these concepts.

Throughout this paper (X, τ), (Y, ρ) and (Z, σ) (or simply X, Y and Z) denote topological spaces on which no separation axioms are assumed unless explicitly stated. For a subset A of (X, τ), the closure and the interior of A in X are denoted by $Cl(A)$ and $Int(A)$, respectively. The complement of A is denoted by $X - A$. A subcollection $\mu \subseteq 2^X$ is called a supra topology [1] on X if X,$\phi \in \mu$ and μ is closed under arbitrary union. (X, μ) is called a supra

Received: June 7, 2012

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topological space. The elements of μ are said to be supra open in (X,μ) and the complement of a supra open set is called a supra closed set. The supra closure of a set A, denoted by $Cl^{\mu}(A)$, is the intersection of supra closed sets including A. The supra interior of a set A, denoted by $Int^{\mu}(A)$, is the union of supra open sets included in A. The supra topology μ on X is associated with the topology τ if $\tau \subseteq \mu$.

2. Supra Separation Axioms

Definition 2.1. [1] Let (X,μ) be a supra topological space, then:

1) X is $S-T_0$ if for every two distinct points x and y in X there exists a supra open set U that contains only one of the points x and y.

2) X is $S-T_1$ if for every two distinct points x and y in X there exists two supra open sets U and V such that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$.

3) X is $S-T_2$ if for every two distinct points x and y in X there exists two disjoint supra open sets U and V such that $x \in U$ and $y \in V$.

Remark 2.2. Every $S-T_i$ space is $S-T_{i-1}$ for each $i = 1,2$ but the converse need not be true.

Example 2.3. Let $X = \{a,b,c\}$, $\mu = \{X,\phi,\{a\},\{a,b\}\}$. Then (X,μ) is an $S-T_0$ space but not $S-T_1$.

Example 2.4. Let $X = \{a,b,c\}$, $\mu = \{X,\phi,\{a,b\},\{a,c\},\{b,c\}\}$. Then (X,μ) is an $S-T_1$ space but not $S-T_2$.

Theorem 2.5. *A supra topological space (X,μ) is $S-T_0$ if and only if for each pair of distinct points x, y in X, $Cl^{\mu}(\{x\}) \neq Cl^{\mu}(\{y\})$.*

Proof. $\Rightarrow$) Let (X,μ) be an $S-T_0$ space and x, y be any two distinct points in X. There exists a supra open set U containing x or y, say x but not y. Then $X-U$ is a supra closed set containing y but not x. Now, $Cl^{\mu}(\{y\}) \subseteq X-U$, and therefore $x \notin Cl^{\mu}(\{y\})$. Hence $Cl^{\mu}(\{x\}) \neq Cl^{\mu}(\{y\})$.

$\Leftarrow$) Suppose that $x,y \in X$, $x \neq y$ and $Cl^{\mu}(\{x\}) \neq Cl^{\mu}(\{y\})$. Then there exists a point $z \in X$ such that z belongs only to one of the sets $Cl^{\mu}(\{x\})$ and $Cl^{\mu}(\{y\})$, say $z \in Cl^{\mu}(\{x\})$ but $z \notin Cl^{\mu}(\{y\})$. We claim that $x \notin Cl^{\mu}(\{y\})$. For, if $x \in Cl^{\mu}(\{y\})$ then $Cl^{\mu}(\{x\}) \subseteq Cl^{\mu}(\{y\})$. This contradicts the fact that $z \notin Cl^{\mu}(\{y\})$. Consequently x belongs to the supra open set $X - Cl^{\mu}(\{y\})$ to which y does not belong. $\square$

Definition 2.6. A subset A of a supra topological space X is called a supra neighborhood of a point x of X if there exists a supra open set U contains x such that $U \subseteq A$.

Lemma 2.7. *A subset A of a supra topological space X is supra open if and only if it is a supra neighborhood of each of its points.*

Definition 2.8. A point x of X is called a supra interior point of $A \subseteq X$ if $x \in Int^{\mu}(A)$.

Lemma 2.9. *Let X be a supra topological space and $A \subseteq X$, and $x \in X$. Then x is a supra interior point of A if and only if A is a supra neighborhood of x.*

Theorem 2.10. *A supra topological space (X, μ) is $S - T_1$ if and only if the singletons are supra closed sets.*

Proof. $\Rightarrow$) Let (X, μ) be $S - T_1$ and $x \in X$. Let $y \in X - \{x\}$. Then $x \neq y$ and so there exists a supra open set U_y such that $y \in U_y$ but $x \notin U_y$. Then $y \in U_y \subseteq X - \{x\}$ i.e., $X - \{x\} = \cup\{U_y : y \in X - \{x\}\}$ which is supra open.

$\Leftarrow$) Suppose $\{z\}$ is supra closed for every $z \in X$. Let $x, y \in X$ with $x \neq y$. Since $x \neq y$, $y \in X - \{x\}$. Hence $X - \{x\}$ is a supra open set containing y but not x. Similarly $X - \{y\}$ is a supra open set containing x but not y. So X is $S - T_1$. $\square$

Definition 2.11. A supra topological space (X, μ) is called a supra symmetric space if for x and y in X, $x \in Cl^{\mu}(\{y\})$ implies $y \in Cl^{\mu}(\{x\})$.

Theorem 2.12. *Let (X, μ) be a supra symmetric space. Then the following are equivalent:*

(1) (X, μ) is $S - T_0$;
(2) (X, μ) is $S - T_1$.

Proof. It is enough to show that (1) $\Rightarrow$ (2). Let $x \neq y$. Since (X, μ) is $S - T_0$, we may assume that $x \in U \subseteq X - \{y\}$ for some supra open set U. Then $x \notin Cl^{\mu}(\{y\})$ and hence $y \notin Cl^{\mu}(\{x\})$. Therefore there exists a supra open set V such that $y \in V \subseteq X - \{x\}$ and (X, μ) is an $S - T_1$ space. $\square$

Definition 2.13. [1] Let (X, τ) and (Y, ρ) be two topological spaces and μ be an associated supra topology with τ. A function $f : (X, \tau) \longrightarrow (Y, \rho)$ is called a supra continuous function if the inverse image of each open set in Y is supra open in X.

Definition 2.14. A function $f : (X, \tau) \to (Y, \rho)$ is called a supra open function if the image of each open set in X is a supra open set in (Y, η).

Definition 2.15. Let (X, τ) and (Y, ρ) be two topological spaces and μ, η be associated supra topologies with τ and ρ respectively. A function $f : (X, \tau) \to (Y, \rho)$ is called a supra irresolute function if the inverse image of each supra open set in Y is a supra open set in X.

The following two theorems can be easily proved:

Theorem 2.16. *The supra open image of any T_1-space is $S - T_1$.*

Theorem 2.17. *Let $f : (X, \tau) \to (Y, \rho)$ be an injective supra irresolute function. If Y is $S - T_1$ then X is $S - T_1$.*

Theorem 2.18. *The following properties are equivalent:*

1) X is $S - T_2$.
2) Let $x \in X$. For each $y \neq x$, there exists a supra open set U such that $x \in U$ and $y \notin Cl^{\mu}(U)$.
3) For each $x \in X$, $\cap\{Cl^{\mu}(U) : U$ is a supra open set with $x \in U\} = \{x\}$.

Proof. (1) $\Rightarrow$ (2). Let $x \in X$ and $y \neq x$. Then there are disjoint supra open sets U and V such that $x \in U$ and $y \in V$. Now $X - V$ is supra closed with $Cl^{\mu}(U) \subseteq X - V$ and $y \notin X - V$ and therefore $y \notin Cl^{\mu}(U)$.

(2) $\Rightarrow$ (3). If $y \notin \{x\}$, then there exists a supra open set U such that $x \in U$ and $y \notin Cl^{\mu}(U)$. So $y \notin \cap\{Cl^{\mu}(U) : U$ is a supra open set with $x \in U\}$.

(3) $\Rightarrow$ (1). If $y \neq x$. By assumption we have $\cap\{Cl^{\mu}(U) : U$ is a supra open set with $x \in U\} = \{x\}$, then there exists a supra open set U such that $x \in U$, $y \notin Cl^{\mu}(U)$. Let $V = X - Cl^{\mu}(U)$, then V is a supra open set with $y \in V$ and $U \cap V = \phi$. $\square$

Theorem 2.19. *The supra open image of any T_2-space is $S - T_2$ (i.e. if $f : X \longrightarrow Y$ is a supra open function and X is a T_2-space then $f(X)$ is $S - T_2$).*

Theorem 2.20. *Let $f : X \longrightarrow Y$ be an injective supra irresolute function. If Y is $S - T_2$ then X is $S - T_2$.*

3. Supra $D-$sets and Associated Separation Axioms

Definition 3.1. A subset A of a supra topological space (X, μ) is called a supra $D-$set if there are two supra open sets U and V such that $U \neq X$ and

$A = U - V$.

Observe that every supra open set U different from X is a supra $D-$set if $A = U$ and $V = \phi$.

Definition 3.2. A supra topological space (X, μ) is called:

1) $S - D_0$ if for any distinct pair of points x and y of X there exists a supra $D-$set in X containing x but not y or a supra $D-$set in X containing y but not x.

2) $S - D_1$ if for any distinct pair of points x and y in X there exists a supra $D-$set in X containing x but not y and a supra $D-$set in X containing y but not x.

3) $S - D_2$ if for any distinct pair of points x and y in X there exist disjoint supra $D-$sets G and E in X containing x and y, respectively.

Remark 3.3. 1) If (X, μ) is $S - T_i$, then (X, μ) is $S - D_i$, $i = 0, 1, 2$.

2) If (X, μ) is $S - D_i$, then it is $S - D_{i-1}$, $i = 1, 2$.

The supra toplogical space (X, μ) in Example 2.3 is $S - D_2$ and so $S - D_1$ but not $S - T_1$ and so not $S - T_2$.

Theorem 3.4. *For a supra topological space (X, μ) the following statements hold:*

1) (X, μ) is $S - D_0$ if and only if it is $S - T_0$.

2) (X, μ) is $S - D_1$ if and only if it is $S - D_2$.

Proof. (1) $\Rightarrow$) Let (X, μ) be $S - D_0$. Then for each distinct pair x, y in X, at least one of x, y, say x, belongs to a supra $D-$set U where $y \notin U$. Let $U = U_1 - U_2$ such that $U_1 \neq X$ and $U_1$and U_2 are supra open sets in X. Then $x \in U_1$. For $y \notin U$ we have two cases:

(a) $y \notin U_1$; (b) $y \in U_1$ and $y \in U_2$.

In case (a), $x \in U_1$ but $y \notin U_1$;

In case (b), $y \in U_2$ but $x \notin U_2$. Hence X is $S - T_0$.

$\Leftarrow$) By Remark 3.3.

(2) $\Rightarrow$) Suppose that X is $S - D_1$. Then for each distinct pair x, $y \in X$, we have supra $D-$sets G_1, G_2 such that $x \in G_1$, $y \notin G_1$; $y \in G_2$, $x \notin G_2$. Let $G_1 = U_1 - U_2$, $G_2 = U_3 - U_4$ where U_1, U_2, U_3 and U_4 are supra open sets such that $U_1 \neq X$ and $U_3 \neq X$. Since $x \notin G_2$, it follows that either $x \notin U_3$ or $x \in U_3$ and $x \in U_4$. Now we consider two cases:

(a) $x \notin U_3$. Since $y \notin G_1$ we have two subcases:

(a1) $y \notin U_1$. Since $x \in U_1 - U_2$, it follows that $x \in U_1 - (U_2 \cup U_3)$ and $y \in U_3 - U_4$ we have $y \in U_3 - (U_1 \cup U_4)$. Hence $(U_1 - (U_2 \cup U_3)) \cap (U_3 - (U_1 \cup U_4)) = \phi$.

(a2) $y \in U_1$ and $y \in U_2$. We have $x \in U_1 - U_2$, $y \in U_2$ and $(U_1 - U_2) \cap U_2 = \phi$.
(b) $x \in U_3$ and $x \in U_4$. We have $y \in U_3 - U_4$, $x \in U_4$ and $(U_3 - U_4) \cap U_4 = \phi$.
Therefore X is $S - D_2$.
$\Leftarrow$) By Remark 3.3. □

Theorem 3.5. *Let (X, τ) and (Y, ρ) be two topological spaces and μ, η be associated supra topologies with τ and ρ respectively. Let $f : (X, \tau) \to (Y, \rho)$ be a supra irresolute surjective function and G be a supra $D-$set in Y, then $f^{-1}(G)$ is a supra $D-$set in X.*

Proof. Let G be a supra $D-$set in Y. Then there are supra open sets U_1 and U_2 in Y such that $G = U_1 - U_2$ and $U_1 \neq Y$. By the supra irresoluteness of f, $f^{-1}(U_1)$ and $f^{-1}(U_2)$ are supra open in X. Since $U_1 \neq Y$, we have $f^{-1}(U_1) \neq X$. Hence $f^{-1}(G) = f^{-1}(U_1) - f^{-1}(U_2)$ is a supra $D-$set. □

Theorem 3.6. *Let (X, τ) and (Y, ρ) be two topological spaces and μ, η be associated supra topologies with τ and ρ respectively. Let $f : (X, \tau) \to (Y, \rho)$ be a supra irresolute bijective function. If (Y, η) is $S - D_1$ then (X, μ) is also $S - D_1$.*

Proof. Suppose that Y is a $S - D_1$ space. Let x and y be any pair of distinct points in X. Since f is injective and Y is $S - D_1$, there exist supra $D-$sets G_x and G_y of Y containing $f(x)$ and $f(y)$ respectively, such that $f(y) \notin G_x$ and $f(x) \notin G_y$. By Theorem 3.5, $f^{-1}(G_x)$ and $f^{-1}(G_y)$ are supra $D-$sets in X containing x and y, respectively, such that $y \notin f^{-1}(G_x)$ and $x \notin f^{-1}(G_y)$. This implies that X is a $S - D_1$ space. □

Theorem 3.7. *Let (X, τ) and (Y, ρ) be two topological spaces and μ, η be associated supra topologies with τ and ρ respectively. Then (X, μ) is $S - D_1$ if and only if for each pair of distinct points x, $y \in X$, there exists a supra irresolute surjective function $f : (X, \tau) \to (Y, \rho)$, where (Y, η) is a $S - D_1$ space such that $f(x)$ and $f(y)$ are distinct.*

Proof. $\Rightarrow$) For every pair of distinct points of X, it suffices to take the identity function on X.

$\Leftarrow$) Let x and y be any pair of distinct points in X. By assumption there exists a supra irresolute, surjective function f of a space (X, μ) onto a $S - D_1$ space (Y, η) such that $f(x) \neq f(y)$. Therefore, there exist two supra $D-$sets G_x and G_y of Y such that $f(x) \in G_x$ and $f(y) \in G_y$ but $f(y) \notin G_x$ and $f(x) \notin G_y$. Since f is supra irresolute and surjective, by Theorem 3.5, $f^{-1}(G_x)$ and $f^{-1}(G_y)$ are distinct supra $D-$sets in X containing x and y, respectively, such that $y \notin f^{-1}(G_x)$ and $x \notin f^{-1}(G_y)$. This implies that X is a $S - D_1$ space. $\square$

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