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TOTALLY COFINITELY WEAK RAD-SUPPLEMENTED MODULES

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Abstract: Let R be a ring and M be a left $R-$module. M is called *cofinitely weak Rad-supplemented* if every cofinite submodule of M has a weak Rad-supplement in M. In this paper, we will define totally cofinitely weak Rad-supplemented modules. In general, the finite sum of totally cofinitely weak Rad-supplemented modules need not to be totally cofinitely weak Rad-supplemented. However a module totally cofinitely weak Rad-supplemented if and only if it is the direct sum of a semisimple module and a totally cofinitely weak Rad-supplemented module. We will prove a module M is totally cofinitely weak Rad-supplemented if and only if $\frac{M}{K}$ is totally cofinitely weak Rad-supplemented for a linearly compact submodule K of M. Similarly, a module M is totally cofinitely weak Rad-supplemented if and only if $\frac{M}{U}$ is totally cofinitely weak Rad-supplemented for a uniserial submodule U of M.

AMS Subject Classification: 16D10, 16L30, 16D99
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1. Introduction and Preliminiaries

Throughout the paper, R will be an associative ring with identity and all modules will be unital left $R-$modules unless otherwise specified. Let M be a

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module. The symbol $N \leq M$ means that N is a submodule of M. $Rad(M)$ will indicate Jacobson radical of M. K is a supplement of N in M if and only if $N + K = M$ and $N \cap K \ll K$, where K and N are submodules of M [9]. M is called *supplemented*, if every submodule N of M has a *supplement* in M, i.e. a submodule K is minimal with respect to $N + K = M$. If $N + K = M$ and $N \cap K \ll M$, then K is called a *weak supplement* of N in M, ([6], [11]), and clearly in this situation N is the weak supplement of K. M is a *weakly supplemented* module if every submodule of M has a weak supplement in M.

Let M be a module, N and K be any submodules of M with $N + K = M$. If $N \cap K \leq Rad(K)$ ($N \cap K \leq Rad(M)$) then K is called a *(weak) Rad-supplement* of N in M. For characterizations of Rad-supplemented and weak Rad-supplemented modules, we refer to [8] and [10].

A module M is called *locally artinian*, if every finitely generated submodule of M is artinian. A submodule N of M is said to be *cofinite* if $\frac{M}{N}$ is finitely generated. M is called a *cofinitely (weak) supplemented* module if every cofinite submodule of M has a (weak) supplement in M (see [1], [2]). Clearly supplemented modules are cofinitely supplemented and weakly supplemented modules are cofinitely weak supplemented.

M is called *cofinitely Rad-supplemented* if every cofinite submodule of M has a Rad-supplement [4].

In [7], an $R-$module M is called totally supplemented if every submodule of M is supplemented. M is called totally cofinitely supplemented if every submodule of M is cofinitely supplemented [3].

In this paper, we will say a module is totally cofinitely weak Rad-supplemented if every submodule of M is cofinitely weak Rad-supplemented. And we will investigate some properties of these modules.

2. Cofinitely Weak Rad-Supplemented Modules

Definition 1. *A module M is called a cofinitely weak Rad-supplemented if every cofinite submodule of M has a weak Rad-supplement.*

To prove that an arbitrary sum of cofinitely weak Rad-supplemented modules is a cofinitely weak Rad-supplemented module, we use the following standard lemma.

Lemma 2. *Let M be a module, N and U be submodules of M with cofinitely weak Rad-suplemented N and cofinite U. If $N + U$ has a weak Rad-supplement in M, then U also has a weak Rad-supplement in M.*

Proof. Let X be a weak Rad-supplement of $N+U$ in M. Then we have

$$\frac{N}{[N\cap(X+U)]}\cong\frac{N+(X+U)}{X+U}=\frac{M}{X+U}\cong\frac{\left(\frac{M}{U}\right)}{\left(\frac{(X+U)}{U}\right)}.$$

Since U is a cofinite submodule, $\frac{M}{U}$ is a finitely generated module. The last module in the right hand side of the preceding equation is a finitely generated module. Hence $N\cap(X+U)$ has a weak Rad-supplement Y in N, i.e.

$$\begin{aligned} Y+[N\cap(X+U)] &= N \\ Y\cap[N\cap(X+U)] &= Y\cap(X+U)\leq Rad(N)\leq Rad(M). \end{aligned}$$

Since

$$M=U+X+N=U+X+Y+[N\cap(X+U)]=X+U+Y,$$

Y is a weak Rad-supplement of $X+U$ in M. Therefore

$$U\cap(X+Y)\leq[X\cap(Y+U)]+[Y\cap(X+U)]\leq Rad(M).$$

This means that $X+Y$ is a weak Rad-supplement of U in M. □

Proposition 3. *Any arbitrary sum of cofinitely weak Rad-supplemented modules is a cofinitely weak Rad-supplemented module.*

Proof. Let $M=\sum_{i\in I}M_i$ where each module M_i is a cofinitely weak Rad-supplemented and N be a cofinite submodule of M. Then $\frac{M}{N}$ is generated by some finite set $\{x_1+N,x_2+N,....,x_n+N\}$ and therefore $M=Rx_1+Rx_2+...+Rx_n+N$. Since each x_i is contained in the sum $\sum_{j\in J}M_j$ for some finite subset $J=\left\{1_1,...,1_{s(1)},..,n_{s(n)}\right\}$ of I, one can see that $M=M_{1_1}+\sum_{j\in J-\{1_1\}}M_j+N$ has a trivial weak Rad-supplement 0 in M. Also, being M_{1_1} is a cofinitely weak Rad-supplemented module, implies that $N+\sum_{j\in J}M_j$ has a weak Rad-supplement by Lemma 2. Continuing in this way we will obtain (after we have used Lemma 2 $\sum_{i=1}^{n}s(i)$ times) N has a weak Rad-supplement in M as a result. □

Theorem 4. *Let M be a module and N be a submodule with $N\leq Rad(M)$. If $\frac{M}{N}$ is a cofinitely weak Rad-supplemented module, then M is a cofinitely weak Rad-supplemented module.*

Proof. Let U be any cofinite submodule of M. If we remember $\left(\frac{M}{(U+N)}\right) \cong \left(\frac{\left(\frac{M}{U}\right)}{\frac{(U+N)}{N}}\right)$, then we have $U+N$ is a cofinite submodule of M. Since $\frac{U+N}{N}$ is a cofinite submodule of $\frac{M}{N}$, there is a submodule $\frac{X}{N}$ of $\frac{M}{N}$ such that $\left(\frac{(U+N)}{N}\right) + \frac{X}{N} = \frac{M}{N}$ and $\left(\frac{(U+N)}{N}\right) \cap \left(\frac{X}{N}\right) = \left(\frac{(U\cap X+N)}{N}\right) \leq Rad\left(\frac{M}{N}\right)$. Therefore $N \leq Rad(M)$, $Rad\left(\frac{M}{N}\right) = \frac{(RadM)}{N}$ and $U \cap X \leq Rad(M)$. Lastly, $U+X=M$ implies that X is a weak Rad-supplement of U in M. □

Let M and N be are modules. An epimorphism $f : M \to N$ is called a *small cover* if $Ker(f) \ll M$. Recall that an epimorphism $f : M \to N$ is called a *generalized cover* if $Ker(f) \leq Rad(M)$ and M is called a generalized cover of N with an epimorphism $f : M \to N$.

Corollary 5. *A generalized cover of a cofinitely weak Rad-supplemented module is a cofinitely weak Rad-supplemented module.*

Proposition 6. *Any factor module of a cofinitely weak Rad-supplemented module is a cofinitely weak Rad-supplemented module.*

Proof. Let M be a cofinitely weak Rad-supplemented module and L be a submodule of M. Suppose that $\frac{U}{L}$ is a cofinite submodule of $\frac{M}{L}$. Note that $\frac{\left(\frac{M}{L}\right)}{\left(\frac{U}{L}\right)} \cong \frac{M}{U}$. Then U is a cofinite submodule of M. Since M is a cofinitely weak Rad-supplemented module, U has a weak Rad-supplement V in M, i.e. $U+V=M$ and $U \cap V \leq Rad(M)$. Thus $\frac{M}{L} = \frac{U}{L} + \frac{(V+L)}{L}$. Let $f : M \to \frac{M}{L}$ be a canonical epimorphism. Since $U \cap V \leq Rad(M)$ and $\left(\frac{U}{L}\right) \cap \left(\frac{(V+L)}{L}\right) = \left(\frac{U\cap(V+L)}{L}\right) = \frac{(L+(U\cap V))}{L} = f(U \cap V) \leq f(Rad(M)) \leq Rad\left(\frac{M}{L}\right)$, we get the result. □

Theorem 7. *Let $0 \to L \to M \to N \to 0$ be a short exact sequence. If L and N are cofinitely weak Rad-supplemented modules and L has a weak supplement in M, then M is a cofinitely weak Rad-supplemented module.*

Proof. Without restriction of generality, we will assume that $L \leq M$. Let S be the weak supplement of L in M, i.e. $L+S=M$ and $L\cap S \ll M$. Then we have $\frac{M}{L\cap S} \cong \frac{L}{L\cap S} \oplus \frac{S}{L\cap S}$. $\frac{L}{L\cap S}$ is cofinitely weak Rad-supplemented as a factor module of L which is cofinitely weak Rad-supplemented. On the other hand, $\frac{S}{L\cap S} \cong \frac{M}{L} \cong N$ is cofinitely weak Rad-supplemented. Then $\frac{M}{L\cap S}$ is cofinitely weak Rad-supplemented as a sum of cofinitely weak Rad- supplemented. Therefore M is a cofinitely weak Rad-supplemented module by Corollary 5. □

Theorem 8. *Let M be a module. Then M is cofinitely weak Rad-supplemented if and only if $\frac{M}{K}$ is cofinitely weak Rad-supplemented for a linearly compact submodule K of M.*

Proof. ($\Rightarrow$) Follows from by Proposition 6.

($\Leftarrow$) Let $0 \to K \to M \to \frac{M}{K} \to 0$ be a short exact sequence. Since K is linearly compact, K is cofinitely weak Rad-supplemented and K has a weak supplement in M. Therefore M is cofinitely weak Rad-supplemented by Theorem 7. □

Theorem 9. *Let M be a module. Then M is cofinitely weak Rad-supplemented if and only if $\frac{M}{U}$ is cofinitely weak Rad-supplemented for a uniserial submodule U of M.*

Proof. ($\Rightarrow$) It follows from by Proposition 6.

($\Leftarrow$) Consider the following exact sequence $0 \to U \to M \to \frac{M}{U} \to 0$. Since U is uniserial, it is hollow by [5,2.17]. So U is cofinitely weak Rad-supplemented.

Case 1: If $U \leq Rad(M)$, then M is cofinitely weak Rad-supplemented by Theorem 4.

Case 2: If $U \nleq Rad(M)$, then $U \not\ll M$ and there is a proper submodule N of M such that $U + N = M$. Since $U \cap N \leq U$ and U is a hollow, we have $U \cap N \ll M$ and so $U \cap N \leq Rad(M)$. Hence U has a weak Rad-supplement in M. Consequently, M is cofinitely weak Rad-supplemented by Theorem 7. □

3. Totally Cofinitely Weak Rad-Supplemented Modules

Definition 10. *A module is totally cofinitely weak Rad-supplemented if every submodule of M is cofinitely weak Rad-supplemented.*

This definition is not meaningless, that is every submodule of cofinitely weak Rad-supplemented module is cofinitely weak Rad-supplemented. Let us consider $\mathbb{Z}-$modules, $\mathbb{Z}$ and $\mathbb{Q}$, where $\mathbb{Z}$ is the set of integers and $\mathbb{Q}$ is the rational numbers. $\mathbb{Q}$ is cofinitely weak Rad-supplemented, since its unique cofinite submodule is itself. But $\mathbb{Z}$ is not cofinitely weak Rad-supplemented. Because, if N and K proper submodules of $\mathbb{Z}$, then, there is $n, m \in \mathbb{Z}$ such that $N = \langle n \rangle$ and $K = \langle m \rangle$ which are different from 0 and ± 1. Note that $0 \neq nm \in N \cap K \neq 0$. So, there is not a submodule K of M such that $N + K = \mathbb{Z}$ and $N \cap K \leq Rad(\mathbb{Z}) = 0$ for N. Hence $\mathbb{Z}$ is not cofinitely weak Rad-supplemented.

Theorem 11. *Every factor module of a totally cofinitely weak Rad-supplemented module is totally cofinitely weak Rad-supplemented.*

Proof. Let M be a totally cofinitely weak Rad-supplemented module and $\frac{U}{L}$ be a submodule of $\frac{M}{L}$ for some submodule U which contains L. Suppose that $\frac{K}{L}$ be a cofinite submodule of $\frac{U}{L}$. Since $(\frac{\frac{U}{L}}{\frac{K}{L}}) \cong \frac{U}{K}$, K is a cofinite submodule of U. Thus, there is a submodule V of U such that $K+V=U$ and $K \cap V \leq Rad(U)$. By Proposition 3.2 in [8], $\frac{(V+L)}{L}$ is a weak Rad-supplement of $\frac{K}{L}$ in $\frac{U}{L}$. Hence $\frac{M}{L}$ is a totally cofinitely weak Rad-supplemented module. □

Corollary 12. *Every homomorphic image of a totally cofinitely weak Rad-supplemented module is totally cofinitely weak Rad-supplemented.*

Theorem 13. *Let M be a module, U be a totally cofinitely weak Rad-supplemented submodule and each submodule of U have a weak supplement in any module containing this submodule. Then M is totally cofinitely weak Rad-supplemented module if and only if $\frac{M}{U}$ is totally cofinitely weak Rad-supplemented module.*

Proof. Necessity is clear by Corollary 12. Let K be a submodule of M. If K is a submodule of U, then K is a cofinitely weak Rad-supplemented module.

Suppose that K is not a submodule of U. Note that $\frac{(K+U)}{U} \cong \frac{K}{(U \cap K)}$ and take the following short exact sequence: $0 \to U \cap K \to K \to \frac{K}{U \cap K} \to 0$. Since $\frac{M}{U}$ is totally cofinitely weak Rad-supplemented, then $\frac{K}{U \cap K}$ is cofinitely weak Rad-supplemented. Also, $U \cap K$ has a weak supplement in K by hypothesis. By Theorem 7 , K is a cofinitely weak Rad-supplemented module. □

In Section 2, we proved that if U and V are cofinitely weak Rad–supplemented submodules of a module M, then the submodule $U+V$ is also cofinitely weak Rad–supplemented module. Clearly this implies that any finite direct sum of cofinitely weak Rad–supplemented modules is also cofinitely weak Rad–supplemented module. This raises an obvious question, namely if M_1 and M_2 are totally cofinitely weak Rad-supplemented modules, when is $M_1 \oplus M_2$ totally cofinitely weak Rad-supplemented ? We shall begin to address this question by considering the case when of M_1, M_2 are semisimple.

Theorem 14. *Let $M = M_1 \oplus M_2$ be a direct sum of submodules M_1, M_2 such that M_2 is semisimple. Then M is totally cofinitely weak Rad-supplemented if and only if M_1 is totally cofinitely weak Rad-supplemented.*

Proof. The necessity follows from by Corollary 12. Conversely, suppose that M_1 is totally cofinitely weak Rad-supplemented. Let N be a submodule of M. Since M_2 is semisimple, $M_2 = (N \cap M_2) \oplus L$ for some submodule L of M_2. It follows that $M = M_1 \oplus M_2 = M_1 \oplus [(N \cap M_2) \oplus L]$ and hence $N = (N \cap M_2) \oplus [N \cap (M_1 \oplus L)]$. Consider the submodule $H = N \cap (M_1 \oplus L)$ of $M_1 \oplus L$. Note that $H \cap L = N \cap L = 0$. So H embeds in M_1. By hypothesis, H is cofinitely weak Rad-supplemented. Being M_2 semisimple, $N \cap M_2$ is cofinitely weak Rad-supplemented. Therefore N is cofinitely weak Rad-supplemented by Proposition 3. Thus M is totally cofinitely weak Rad-supplemented. □

Proposition 15. *Let M be a module such that every (cyclic) finitely generated submodule is cofinitely weak Rad-supplemented. Then M is totally cofinitely weak Rad-supplemented.*

Proof. Let N be a submodule of M, then for any $n \in N$, Rn is cofinitely weak Rad-supplemented and by Proposition 3, $N = \sum_{n \in N} Rn$ is cofinitely weak Rad-supplemented. □

Theorem 16. *Any direct sum of locally artinian modules is totally cofinitely weak Rad-supplemented.*

Proof. Let $\{M_i\}_{i \in I}$ be a family of locally artinian modules for any index set I and $M = \bigoplus_{i \in I} M_i$ be direct sum of these modules. Let N be a submodule of M. We will take any nonzero $n \in N$ and consider Rn. Clearly $Rn \leq Rm_1 + Rm_2 + ... + Rm_k$ for some $m_1, m_2, ..., m_k$ from $\{M_i\}_{i \in I}$. $Rm_1 + Rm_2 + ... + Rm_k$ is locally artinian. So Rn is artinian and Rn is cofinitely weak Rad-supplemented. Therefore by Proposition 3, $N = \sum_{n \in N} Rn$ is cofinitely weak Rad-supplemented. □

Lemma 17. *Let R be a commutative ring and $M = M_1 \oplus M_2 \oplus .... \oplus M_n$ be a finite direct sum of totally cofinitely weak Rad-supplemented submodules M_i $(1 \leq i \leq n)$ for some $n \geq 2$. If $R = Ann\,(M_i) + Ann\,(M_j)$ for all $1 \leq i < j \leq n$, then M is totally cofinitely weak Rad-supplemented.*

Proof. Let U, V be submodules of M such that V is cofinite in U. By Lemma 4.1 in [7], $U = (U \cap M_1) \oplus .... (U \cap M_n)$ and $V = (V \cap M_1) \oplus .... (V \cap M_n)$. Since $\frac{U}{V} \cong \bigoplus_{i=1}^{n} \left(\frac{U \cap M_i}{V \cap M_i} \right)$ for each $1 \leq i \leq n$, $V \cap M_i$ is cofinite submodule of $U \cap M_i$. By hypothesis $U \cap M_i$ is cofinitely weak Rad-supplemented. Then there exists a weak Rad-supplement K_i of $V \cap M_i$ in $U \cap M_i$. Let $K = K_1 \oplus$

$K_2 \oplus ... \oplus K_n$. Then $U = (U \cap M_1) \oplus .... (U \cap M_n) = ((V \cap M_1) + K_1) \oplus .... \oplus ((V \cap M_n) + K_n) = ((V \cap M_1) \oplus .... \oplus (V \cap M_n)) + (K_1 \oplus .... \oplus K_n) = V + K$. Since $(V \cap M_i) \cap K_i = V \cap K_i \leq Rad\,(U \cap M_i)$ for every $1 \leq i \leq n$, we have $V \cap K = \bigoplus_{i=1}^{n} (V \cap K_i) \leq \bigoplus_{i=1}^{n} Rad\,(U \cap M_i) = Rad\left(\bigoplus_{i=1}^{n} (U \cap M_i)\right) = Rad(U)$. Thus U is cofinitely weak Rad-supplemented and it follows that M is totally cofinitely weak Rad-supplemented. □

Theorem 18. *Let R be a commutative ring and $\{M_i\}_{i \in I}$ be a family of totally cofinitely weak Rad-supplemented modules such that $R = Ann\,(M_i) + Ann\,(M_j)$ for $i \neq j \in I$. Then $\bigoplus_{i \in 1} M_i$ is totally cofinitely weak Rad-supplemented.*

Proof. Let $M = \bigoplus_{i \in I} M_i$ and $N \leq M$. If we take a nonzero $n \in N$ and consider Rn, then we can say that $Rn \leq Rm_1 + Rm_2 + ... + Rm_k$ for some $m_i \in M_i$ where $1 \leq i \leq k$. We know that M_i is totally cofinitely weak Rad-supplemented for every $i \in I$. So Rm_i is totally cofinitely weak Rad-supplemented and $\bigoplus_{i=1}^{k} Rm_i$ totally cofinitely weak Rad-supplemented and Rn is cofinitely weak Rad-supplemented by Lemma 17. Consequently by Proposition 3, $N = \sum_{n \in N} Rn$ is cofinitely weak Rad-supplemented. □

Theorem 19. *Let K be a linearly compact submodule of a module M.Then M is totally cofinitely weak Rad-supplemented if and only if $\frac{M}{K}$ is totally cofinitely weak Rad-supplemented.*

Proof. The necessity is clear by Corollary 12. For sufficiency, suppose that $\frac{M}{K}$ is a totally cofinitely weak Rad-supplemented where K is a linearly compact submodule of M. Take a submodule N of M.

If $N \leq K$, then N is a linearly compact by [9, 29.8 (2)]. Therefore, N is cofinitely weak Rad-supplemented.

If $N \nleq K$, then $N \cap K$ is linearly compact by [9, 29.8 (2)]. Since $\frac{M}{K}$ is totally cofinitely weak Rad-supplemented and $\frac{N}{N \cap K} \cong \frac{N+K}{K}$, $\frac{N}{N \cap K}$ is cofinitely weak Rad-supplemented. Hence N is cofinitely weak Rad-supplemented by Theorem 8. □

Theorem 20. *Let M be a module. Then M is totally cofinitely weak Rad-supplemented if and only if $\frac{M}{U}$ is totally cofinitely weak Rad-supplemented for a uniserial submodule U of M.*

Proof. (⇒) It follows from Corollary 12.

(⇐)Let N be a submodule of M.

If $N \leq U$, then N is cofinitely weak Rad-supplemented because submodules of uniserial modules are uniserial and uniserial modules are cofinitely weak Rad-supplemented.

If $N \nleq U$, then consider the following short exact sequence. $0 \to N \cap U \to N \to \frac{N}{N \cap U} \to 0$. Note that $\frac{N}{N \cap U} \cong \frac{N+U}{U}$. Since $N \cap U$ is uniserial, it is cofinitely weak Rad-supplemented. Also, $\frac{N}{N \cap U}$ is isomorphic to a submodule of $\frac{M}{U}$ and so $\frac{N}{N \cap U}$ is cofinitely weak Rad-supplemented. Therefore N is cofinitely weak Rad-supplemented by Theorem 9. □

Theorem 21. *For a ring R, the following statements are equivalent:*

i) R is semilocal.

ii) Every left $R-$module is cofinitely weak Rad-supplemented.

iii) Every left $R-$module is totally cofinitely weak Rad-supplemented.

Proof. i)⇒ii) Let R be a semilocal ring. Then every left $R-$module is weak Rad-supplemented by [5, 17.2]. Hence every left $R-$module is cofinitely weak Rad-supplemented.

ii)⇒iii)This proof is clear.

iii)⇒i)This proof is clear. □

Corollary 22. *Let R be a semiperfect ring. Then M is cofinitely weak Rad-supplemented if and only if M is totally cofinitely weak Rad-supplemented.*

Proof. Since R is semiperfect, R is semilocal by [9, 42.6]. So the result follows from preceding Theorem. □

Corollary 23. *Let R be a discrete valution ring. Then M is a cofinitely weak Rad-supplemented module if and only if M is a totally cofinitely weak Rad-supplemented.*

Proof. Since a discrete valution ring is semiperfect, it follows from Corollary 22. □

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