CHROMATIC EQUIVALENCE OF
$K_4$-HOMEOMORPHS WITH GIRTH 9

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Abstract: For a graph $G$, let $P(G, \lambda)$ denote the chromatic polynomial of $G$. Two graphs $G$ and $H$ are chromatically equivalent (or simply $\chi-$equivalent), denoted by $G \sim H$, if $P(G, \lambda) = P(H, \lambda)$. A graph $G$ is chromatically unique (or simply $\chi-$unique) if for any graph $H$ such as $H \sim G$, we have $H \cong G$, i.e, $H$ is isomorphic to $G$. A $K_4$-homeomorph is a subdivision of the complete graph $K_4$. In this paper, we discuss a pair of chromatically equivalent of $K_4$-homeomorphs with girth 9, that is, $K_4(1,3,5,d,e,f)$ and $K_4(1,3,5,d',e',f')$. As a result, we obtain two infinite chromatically equivalent non-isomorphic $K_4$-homeomorphs.

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1. Introduction

All graphs considered here are simple graphs. For such a graph $G$, let $P(G, \lambda)$ denote the chromatic polynomial of $G$. Two graphs $G$ and $H$ are chromatically

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equivalent (or simply $\chi$-equivalent), denoted by $G \sim H$, if $P(G, \lambda) = P(H, \lambda)$. A graph $G$ is chromatically unique (or simply $\chi$-unique) if for any graph $H$ such as $H \sim G$, we have $H \cong G$, i.e., $H$ is isomorphic to $G$.

![Figure 1: $K_4(a, b, c, d, e, f)$](image)

A $K_4$-homeomorph is a subdivision of the complete graph $K_4$. Such a homeomorph is denoted by $K_4(a, b, c, d, e, f)$ if the six edges of $K_4$ are replaced by the six paths of length $a, b, c, d, e, f$, respectively, as shown in Figure 1. So far, the chromaticity of $K_4$-homeomorphs with girth $g$, where $3 \leq g \leq 7$ has been studied by many authors (see [2,5,6,7]). The chromaticity of $K_4$-homeomorphs with girth 8 or 9 is still remains open. For some results on chromatic equivalence of $K_4$-homeomorphs with girth 8, the reader is referred to [3,8,9].

In this paper, we shall discuss a chromatically equivalence pair of $K_4$-homeomorphs, $K_4(1, 3, 5, d, e, f)$ (as in Figure 2) and $K_4(1, 3, 5, d', e', f')$. We obtain two infinite chromatically equivalent non-isomorphic $K_4$-homeomorphs. This result can be extended to the study of chromatic equivalence classes of $K_4(1, 3, 5, d, e, f)$ and chromatic uniqueness of $K_4$-homeomorphs with girth 9.

2. Preliminary Results

In this section, we give some known results used in the sequel.
**Lemma 2.1.** Assume that $G$ and $H$ are $\chi$—equivalent. Then

1. $|V(G)| = |V(H)|$, $|E(G)| = |E(H)|$ (see [4]);
2. $G$ and $H$ has the same girth and same number of cycles with length equal to their girth (see [11]);
3. If $G$ is a $K_4$-homeomorph, then $H$ must itself be a $K_4$-homeomorph (see [1]);
4. Let $G = K_4(a, b, c, d, e, f)$ and $H = K_4(a', b', c', d', e', f')$, then
   (i) $\min(a, b, c, d, e, f) = \min(a', b', c', d', e', f')$ and the number of times that this minimum occurs in the list $\{a, b, c, d, e, f\}$ is equal to the number of times that this minimum occurs in the list $\{a', b', c', d', e', f'\}$ (see [10]);
   (ii) if $\{a, b, c, d, e, f\} = \{a', b', c', d', e', f'\}$ as multisets, then $H \cong G$ (see [5]).

**Theorem 2.1.** (Yanling Peng [8]) Let $K_4(1, 2, 5, d, e, f)$ and $K_4(1, 2, 5, d', e', f')$ be chromatically equivalent, then

$$K_4(1, 2, 5, i, i + 6, i + 1) \sim K_4(1, 2, 5, i + 2, i, i + 5),$$
In this section, we present our main results.

**Theorem 2.2.** (Yanling Peng [9]) Let \( K_4(2, 3, 3, d, e, f) \) and \( K_4(2, 3, 3, d', e', f') \) be chromatically equivalent, then \( K_4(2, 3, 3, d, e, f) \) is isomorphic to \( K_4(2, 2, 3, d', e', f') \).

**Theorem 2.3.** (Roslan et al. [3]) Let \( K_4(1, 3, 4, d, e, f) \) and \( K_4(1, 3, 4, d', e', f') \) be chromatically equivalent, then

\[
K_4(1, 3, 4, i, i + 5, i + 1) \sim K_4(1, 3, 4, i + 2, i, i + 4),
K_4(1, 3, 4, i, i + 1, i + 4) \sim K_4(1, 3, 4, i + 2, i + 3, i).
\]

where \( i \geq 1 \).

### 3. Main Results

In this section, we present our main results.

**Lemma 3.1.** Assume that \( G \cong K_4(1, 3, 5, d, e, f) \) and \( H \cong K_4(1, 3, 5, d', e', f') \), then

(1) \( P(G) = (-1)^{x-1}[s/(s - 1)^2][s^x - s^6 - s^5 - s^4 - s^3 + 2s + 2 + R(G)], \)

where

\[
R(G) = -s^d - s^e - s^f - s^{e+1} - s^{f+1} + s^{e+4} + s^{e+5} + s^{f+6} + s^{d+8} + s^{d+e+f},
\]

\( s = 1 - \lambda, x \) is the number of the edges of \( G \).

(2) If \( P(G) = P(H) \), then \( R(G) = R(H) \).

**Proof.** (1) Let \( s = 1 - \lambda \). From [10], we have the chromatic polynomial of \( K_4 \)-homeomorphs \( K_4(a, b, c, d, e, f) \) is as follows:

\[
P(K_4(a, b, c, d, e, f) = (-1)^{x-1}[s/(s - 1)^2][(s^2 + 3s + 2) - (s + 1)(s^a + s^b + s^c + s^d + s^e + s^f) + (s^{a+d} + s^{b+f} + s^{c+e} + s^{a+b+e} + s^{b+d+c} + s^{a+c+f} + s^{d+e+f} - s^{x-1})]
\]

So, when \( a = 1, b = 3 \) and \( c = 5 \), we have

\[
P(K_4(1, 3, 5, d, e, f) = (-1)^{x-1}[s/(s - 1)^2][(s^2 + 3s + 2) - (s + 1)(s + s^3 + s^5 + s^d + s^e + s^f) + \]

isomorphic. From Lemma 3.1, we have
\[ R = \left( s^{1+d} + s^{3+f} + s^{5+e} + s^{4+e} + s^{8+d} + s^{6+f} + s^{d+e+f} - s^{x-1} \right) \]
\[ = (-1)^{x-1} [s/(s-1)^2] [s^{x-1} - s^6 - s^5 - s^4 - s^3 + 2s + 2 + R(G)] \]
where \( R(G) = -s^d - s^e - s^f - s^{e+1} - s^{f+1} - s^{f+3} - s^{e+4} - s^{e+4} - s^{f+6} + s^{d+8} + s^{d+e+f} \) as required.

(2) If \( P(G) = P(H) \), then we can easily see that \( R(G) = R(H) \). \( \square \)

**Theorem 3.1.** Let \( K_4 \)-homeomorphs \( K_4(1, 3, 5, d, e, f) \) and \( K_4(1, 3, 5, d', e', f') \) be chromatically equivalent, then we have

\[ K_4(1, 3, 5, i, i + 6, i + 1) \sim K_4(1, 3, 5, i + 2, i, i + 5), \]
\[ K_4(1, 3, 5, i, i + 1, i + 4) \sim K_4(1, 3, 5, i + 2, i + 3, i). \]

where \( i \geq 1 \).

**Proof.** Assume that \( G \cong K_4(1, 3, 5, d, e, f) \) and \( H \cong K_4(1, 3, 5, d', e', f') \). We now solve for the equation \( R(G) = R(H) \) to find \( G \) and \( H \) which are not isomorphic. From Lemma 3.1, we have

\[ R(G) = -s^d - s^e - s^f - s^{e+1} - s^{f+1} - s^{f+3} - s^{e+4} - s^{e+4} - s^{f+6} + s^{d+8} + s^{d+e+f}, \]
\[ R(H) = -s^{d'} - s^{e'} - s^{f'} - s^{e'+1} - s^{f'+1} - s^{f'+3} - s^{e'+4} - s^{e'+4} - s^{f'+6} + s^{d'+8} + s^{d'+e'+f'}. \]

Let the lowest remaining power and the highest remaining power be denoted by l.r.p. and h.r.p., respectively. From Lemma 2.1 (1), \( d + e + f = d' + e' + f' \). We obtain the following after simplification. Note that our assumption in the following steps of the proof is \( R_j(G) = R_j(H) \), where \( 1 \leq j \leq 19 \).

\[ R_1(G) = -s^d - s^e - s^f - s^{e+1} - s^{f+1} - s^{f+3} - s^{e+4} - s^{e+5} + s^{f+6} + s^{d+8}, \]
\[ R_1(H) = -s^{d'} - s^{e'} - s^{f'} - s^{e'+1} - s^{f'+1} - s^{f'+3} - s^{e'+4} + s^{e'+5} + s^{f'+6} + s^{d'+8}. \]

Let us consider the h.r.p. in \( R_1(G) \) and the l.r.p. in \( R_1(H) \). We have max \( \{e + 5, f + 6, d + 8\} = \max \{e' + 5, f' + 6, d' + 8\} \). Without loss of generality, we will consider only the following six cases.

Case 1. If max \( \{e + 5, f + 6, d + 8\} = e + 5 \) and max \( \{e' + 5, f' + 6, d' + 8\} = e' + 5 \), then \( e = e' \). Thus, we can cancel the following pairs of terms in the equations: \(-s^e\) with \(-s^{e'}, -s^{e+1}\) with \(-s^{e'+1}, s^{e+4}\) with \(s^{e'+4}\) and \(s^{e+5}\) with \(s^{e'+5}\). Therefore, the l.r.p. in \( R_1(G) \) is \( d \) or \( f \) and the l.r.p. in \( R_1(H) \) is \( d' \) or \( f' \). So, \( d = f' \) or \( d = d' \) or \( f = f' \) or \( f = d' \). We have \( e = e' \) and \( d + e + f = d' + e' + f' \). So, we know that \( \{d, e, f\} = \{d', e', f'\} \) as multisets. From Lemma 2.1 (4(ii)), \( G \cong H \).
Case 2. If \( \max \{e + 5, f + 6, d + 8\} = f + 5 \) and \( \max \{e' + 5, f' + 6, d' + 8\} = f' + 5 \), then \( f = f' \). We can deal with this case in the same way as Case 1, thus, \( G \cong H \).

Case 3. If \( \max \{e + 5, f + 6, d + 8\} = d + 7 \) and \( \max \{e' + 5, f' + 6, d' + 8\} = d' + 7 \), then we can deal with this case in the same way as Case 1. So, we have \( G \cong H \).

Case 4. If \( \max \{e + 5, f + 6, d + 8\} = e + 5 \) and \( \max \{e' + 5, f' + 6, d' + 8\} = f' + 6 \), then \( e + 5 = f' + 6 \), that is
\[
f' = e - 1
\]
from \( d + e + f = d' + e' + f' \), we have
\[
d + f = d' + e' - 1.
\]

Consider the l.r.p. in \( R_1(G) \) and the l.r.p. in \( R_1(H) \). From Lemma 2.1(4(i)), \( \min \{d, e, f\} = \min \{d', e', f'\} \). Without loss of generality, let \( \min \{d, e, f\} = d \).
The following subcases need to be considered.

Subcase 4.1. If \( \min \{d, e, f\} = d \) and \( \min \{d', e', f'\} = d' \), then \( d = d' \).
Thus, we can consider this case the same way as Case 1. So, \( G \cong H \).

Subcase 4.2. If \( \min \{d, e, f\} = d \) and \( \min \{d', e', f'\} = e' \), then \( d = e' \). From Equation (2), we have \( d' = f + 1 \). Note that \( f' = e - 1 \) by Equation (1). We can write \( R_1(G) \) and \( R_1(H) \) as follows:
\[
R_2(G) = -s^d - s^e - s^f - s^{e+1} - s^{f+1} + s^{f+3} + s^{e+4} + s^{e+5} + s^{f+6} + s^{d+8},
\]
\[
R_2(H) = -s^{f+1} - s^{d} - s^{e-1} - s^{d+1} - s^e + s^{e+2} + s^{d+4} + s^{d+5} + s^{e+5} + s^{f+9}.
\]

After simplifying \( R_2(G) \) and \( R_2(H) \), we have:
\[
R_3(G) = -s^f - s^{e+1} + s^{f+3} + s^{e+4} + s^{f+6} + s^{d+8},
\]
\[
R_3(H) = -s^{e-1} - s^{d+1} + s^{e+2} + s^{d+4} + s^{d+5} + s^{f+9}.
\]

Consider the term \( -s^{d+1} \) in \( R_3(H) \). Since the min \( \{d, e, f\} = d \), \( -s^{d+1} \) in \( R_3(H) \) cannot be cancelled by any of the positive terms in \( R_3(H) \). Thus, \( -s^{d+1} \) must be equal to \( -s^f \) or \( -s^{e+1} \) in \( R_3(G) \). Note that max \( \{e + 5, f + 6, d + 8\} = e + 5 \), so \( e + 5 \geq d + 8 \), that is, \( e + 1 \geq d + 4 \). Thus, \( -s^{e+1} \neq -s^{d+1} \).

If \( -s^{d+1} = -s^f \), then \( d + 1 = f \). Thus, \( R_3(G) \) and \( R_3(H) \) can be written as follows:
\[
R_4(G) = -s^{d+1} - s^{e+1} + s^{d+4} + s^{e+4} + s^{d+7} + s^{d+8},
\]
\[
R_4(H) = -s^{e-1} - s^{d+1} + s^{e+2} + s^{d+4} + s^{d+5} + s^{d+10}.
\]

After simplifying \( R_4(G) \) and \( R_4(H) \), we have
\[
-s^{e+1} + s^{e+4} + s^{d+7} + s^{d+8} = -s^{e-1} + s^{e+2} + s^{d+5} + s^{d+10}.
\]
Therefore, we have \( e = d + 6 \). At this point, we acquire the following equations: \( e = d + 6, f' = e - 1 = d + 5, f = d + 1, d' = f + 1 = d + 2 \) and \( e' = d \). Let \( d = i \). Therefore, we obtain the solution where \( G \) is isomorphic to \( K_4(1, 3, 5, i, i + 6, i + 1) \) and \( H \) is isomorphic to \( K_4(1, 3, 5, i + 2, i, i + 5) \).

**Subcase 4.3.** If \( \min \{d, e, f\} = d \) and \( \min \{d', e', f'\} = f' \), then \( d = f' \). Note that \( \max \{e' + 5, f' + 6, d' + 8\} = f' + 6 \). So, \( f' + 6 \geq d' + 8 \), that is, \( f' \geq d' + 2 > d' \). This contradicts with the fact that \( \min \{d', e', f'\} = f' \).

**Case 5.** If \( \max \{e + 5, f + 6, d + 8\} = f + 6 \) and \( \max \{e' + 5, f' + 6, d' + 8\} = d' + 8 \), then \( f + 6 = d' + 8 \), that is,

\[
d' = f - 2 \tag{3}
\]

from \( d + e + f = d' + e' + f' \), we have

\[
d + e = e' + f' - 2. \tag{4}
\]

Consider the l.r.p. in \( R_1(G) \) and the l.r.p. in \( R_1(H) \), where \( \min \{d, e, f\} = \min \{d', e', f'\} \) by Lemma 2.1(4(i)). Without loss of generality, let \( \min \{d, e, f\} = d \). The following subcases need to be considered.

**Subcase 5.1.** If \( \min \{d, e, f\} = d \) and \( \min \{d', e', f'\} = d' \), then we deal with this case the same way with Case 1. So, we get \( G \cong H \).

**Subcase 5.2.** If \( \min \{d, e, f\} = d \) and \( \min \{d', e', f'\} = e' \), then \( d = e' \). From Equation (4), we have \( f' = e + 2 \). Note that \( d' = f - 2 \) by Equation (3). Thus, we can write \( R_1(G) \) and \( R_1(H) \) as follows:

\[
R_5(G) = -s^d - s^e - s^f - s^{e+1} - s^{f+1} + s^{f+3} + s^{e+4} + s^{e+5} + s^{f+6} + s^{d+8},
\]

\[
R_5(H) = -s^{f-2} - s^d - s^{e+2} - s^{d+1} - s^{e+3} + s^{e+5} + s^{d+4} + s^{d+5} + s^{e+8} + s^{f+6}.
\]

Consider the term \(-s^{d+1}\) in \( R_6(H) \). Since \( \min \{d, e, f\} = d \), then \(-s^{d+1}\) cannot be cancelled by any positive terms in \( R_5(H) \). Note that \( \max \{e + 5, f + 6, d + 8\} = f + 6 \), so \( f + 6 \geq d + 8 \), that is \( f + 1 \geq d + 3 \), \( d + 1 \), thus \( f + 1 \neq d + 1 \), i.e., \(-s^{d+1} \neq -s^f \). Moreover \( f \geq d + 2 > d + 1 \), thus \( f \neq d + 1 \), i.e., \(-s^{d+1} \neq -s^f \). Therefore the term \(-s^{d+1}\) in \( R_5(H) \) must be cancelled by the term \(-s^e\) or \(-s^{e+1}\) in \( R_5(G) \).

If \(-s^{d+1} = -s^e\), then \( e = d + 1 \). Thus, \( R_5(G) \) and \( R_5(H) \) can be written as follows:

\[
R_6(G) = -s^{d+1} - s^f - s^{d+2} - s^{f+1} + s^{f+3} + s^{d+5} + s^{d+6} + s^{f+6} + s^{d+8},
\]

\[
R_6(H) = -s^{f-2} - s^{d+3} - s^{f+1} - s^{d+4} + s^{d+6} + s^{d+4} + s^{d+5} + s^{d+9} + s^{f+6}.
\]

After simplifying, we obtain

\[
-s^f - s^{d+2} - s^{f+1} + s^{f+3} + s^{d+8} = -s^f - s^{d+3} + s^{d+9} + s^{f+6}.
\]
The resulting equations contradict $R_6(G) = R_6(H)$.

If $-s^{d+1} = -s^{e+1}$, then $e = d$. Thus, $R_5(G)$ and $R_5(H)$ can be written as follows:

$$R_7(G) = -s^d - s^f - s^{d+1} - s^{f+1} + s^{f+3} + s^{d+4} + s^{d+5} + s^{f+6} + s^{d+8},$$
$$R_7(H) = -s^{f-2} - s^{d+2} - s^{d+1} - s^{d+3} + s^{d+5} + s^{d+4} + s^{d+5} + s^{d+8} + s^{f+6}.$$  

After simplifying, we obtain

$$-s^d - s^f - s^{f+1} + s^{f+3} = -s^{f-2} - s^{d+2} - s^{d+3} + s^{d+5}.$$  

Therefore, we have $f = 2 = d$. But $e = d$, so $e = f - 2$. From Eq. (3), $d' = d = e$. Since $d = e'$, we have $d' = d = e = e'$. From $d + e + f = d' + e' + f'$, we have $f = f'$. Therefore, $G \cong H$.

**Subcase 5.3.** If min $\{d, e, f\} = d$ and min $\{d', e', f'\} = f'$, then $d = f'$. From Equation (4), $e' = e + 2$. Note that from Equation (3), we have $d' = f - 2$. We can write $R_1(G)$ and $R_1(H)$ as follows:

$$R_8(G) = -s^d - s^e - s^f - s^{e+1} - s^{f+1} + s^{f+3} + s^{e+4} + s^{e+5} + s^{f+6} + s^{d+8},$$
$$R_8(H) = -s^{f-2} - s^{e+2} - s^{d} - s^{e+3} - s^{d+1} + s^{d+3} + s^{e+6} + s^{e+7} + s^{d+6} + s^{f+6}.$$  

After simplifying $R_8(G)$ and $R_8(H)$, we have

$$R_9(G) = -s^e - s^f - s^{e+1} - s^{f+1} + s^{f+3} + s^{e+4} + s^{e+5} + s^{d+8},$$
$$R_9(H) = -s^{f-2} - s^{e+2} - s^{e+3} - s^{d+1} + s^{d+3} + s^{e+6} + s^{e+7} + s^{d+6}.$$  

For the same reason stated by Subcase 5.2, $-s^{d+1}$ in $R_9(H)$ must be equal to $-s^{e}$ or $-s^{e+1}$ in $R_9(G)$. If $-s^{d+1} = -s^{e}$, then $e = d + 1$. We can write $R_9(G)$ and $R_9(H)$ as follows:

$$R_{10}(G) = -s^{d+1} - s^f - s^{d+2} - s^{f+1} + s^{f+3} + s^{d+5} + s^{d+6} + s^{d+8},$$
$$R_{10}(H) = -s^{f-2} - s^{d+3} - s^{d+4} - s^{d+1} + s^{d+3} + s^{d+7} + s^{d+8} + s^{d+6}.$$  

After simplifying, we have

$$-s^f - s^{d+2} - s^{f+1} + s^{f+3} + s^{d+5} = -s^{f-2} - s^{d+4} + s^{d+7}.$$  

So, we get $f = d + 4$. We also have $e = d + 1$, $d' = f - 2 = d + 2$, $f' = d$ and $e' = e + 2 = d + 3$. Let $d = i$. Therefore, we obtain the solution where $G \cong K_4(1, 3, 5, i, i + 1, i + 4)$ and $H \cong K_4(1, 3, 5, i + 2, i + 3, i)$.

If $-s^{d+1} = -s^{e+1}$, we have $e = d$. Thus, we have the following:

$$R_{11}(G) = -s^d - s^f - s^{d+1} - s^{f+1} + s^{f+3} + s^{d+4} + s^{d+5} + s^{d+8},$$
$$R_{11}(H) = -s^{f-2} - s^{d+2} - s^{d+3} - s^{d+1} + s^{d+3} + s^{d+6} + s^{d+7} + s^{d+6}.$$  

After simplifying, we have

$$-s^d - s^f - s^{f+1} + s^{f+3} + s^{d+4} + s^{d+5} + s^{d+8} = -s^{f-2} - s^{d+2} + s^{d+6} + s^{d+7} + s^{d+6}. $$
The resulting equation contradicts $R_{11}(G) = R_{11}(H)$.

**Case 6.** If $\max \{e + 5, f + 6, d + 8\} = e + 5$ and $\max \{e' + 5, f' + 6, d' + 8\} = d' + 8$, then $e + 5 = d' + 8$, that is,

$$d' = e - 3 \quad (5)$$

from $d + e + f = d' + e' + f'$, we have

$$d + f = e' + f' - 3. \quad (6)$$

Consider the l.r.p. in $R_1(G)$ and the l.r.p. in $R_1(H)$. We have $\min \{d, e, f\} = \min \{d', e', f'\}$ by Lemma 2.1(4(i)). Without loss of generality, let $\min \{d, e, f\} = d$. The following subcases need to be considered.

**Subcase 6.1.** If $\min \{d, e, f\} = d$ and $\min \{d', e', f'\} = d'$, then we deal with this case the same way with Case 1. So, we get $G \cong H$.

**Subcase 6.2.** If $\min \{d, e, f\} = d$ and $\min \{d', e', f'\} = e'$, then $d = e'$. From Equation (6), we have $f' = f + 3$. Note that $d' = e - 3$ by Equation (5). Thus, we can write $R_1(G)$ and $R_1(H)$ as follows:

$$R_{12}(G) = -s^d - s^e - s^f - s^{e+1} - s^{f+1} + s^{f+3} + s^{e+4} + s^{e+5} + s^{f+6} + s^{d+8},$$

$$R_{12}(H) = -s^{e-3} - s^d - s^{f+3} - s^{d+1} - s^{f+4} + s^{f+6} + s^{d+4} + s^{d+5} + s^{f+9} + s^{e+5}.$$  

After simplifying, we have

$$R_{13}(G) = -s^e - s^f - s^{e+1} - s^{f+1} + s^{f+3} + s^{e+4} + s^{d+8},$$

$$R_{13}(H) = -s^{e-3} - s^{f+3} - s^{d+1} - s^{f+4} + s^{d+4} + s^{d+5} + s^{f+9}.$$  

Consider the term $-s^{d+1}$ in $R_{13}(H)$. Since $\min \{d, e, f\} = d$, $-s^{d+1}$ cannot be cancelled by any positive term in $R_{13}(H)$. From $\max \{e + 5, f + 6, d + 8\} = e + 5$, we have $e + 5 \geq d + 8$, i.e., $e + 1 \geq d + 4 > d + 1$. So, $-s^{d+1} \neq -s^{e+1}$. Moreover, $e \geq d + 3 > d + 1$, thus, $e \neq d + 1$, i.e., $-s^{e} \neq -s^{d+1}$. So, $-s^{d+1}$ in $R_{13}(H)$ must be equal to $-s^{f}$ or $-s^{f+1}$ in $R_{13}(G)$.

If $-s^{d+1} = -s^{f+1}$, then $d = f$. So, we have

$$R_{14}(G) = -s^e - s^d - s^{e+1} - s^{d+1} + s^{d+3} + s^{e+4} + s^{d+8},$$

$$R_{14}(H) = -s^{e-3} - s^{d+3} - s^{d+1} - s^{d+4} + s^{d+4} + s^{d+5} + s^{f+9}.$$  

After simplifying, we have

$$-s^e - s^d - s^{e+1} + s^{d+3} + s^{e+4} + s^{d+8} = -s^{e-3} - s^{d+3} + s^{d+5} + s^{f+9}.$$  

The resulting equation contradicts $R_{14}(G) = R_{14}(H)$.

If $-s^{d+1} = -s^f$, then $d + 1 = f$. Thus, we have

$$R_{15}(G) = -s^e - s^{d+1} - s^{e+1} - s^{d+2} + s^{d+4} + s^{e+4} + s^{d+8},$$

$$R_{15}(H) = -s^{e-3} - s^{d+4} - s^{d+1} - s^{d+5} + s^{d+4} + s^{d+5} + s^{d+10}.$$
After simplifying, we have

\[ -s^e - s^{e+1} - s^{d+2} + s^{e+4} + s^{d+8} = -s^{e-3} - s^{d+4} + s^{d+10}. \]

The resulting equation contradicts \( R_{15}(G) = R_{15}(H) \).

**Subcase 6.3.** If \( \min \{d, e, f\} = d \) and \( \min \{d', e', f'\} = f' \), then \( d = f' \).

From Equation (6), we have \( e' = f + 3 \). Note that \( d' = e - 3 \) by Equation (5).

Thus, we have

\[
R_{16}(G) = -s^d - s^e - s^f - s^{e+1} - s^{f+1} + s^{e+4} + s^{e+5} + s^{f+6} + s^{d+8},
\]

\[
R_{16}(H) = -s^{e-3} - s^{f+3} - s^d - s^{f+4} - s^{d+1} + s^{d+3} + s^{f+7} + s^{f+8} + s^{d+6} + s^{e+5}.
\]

After simplifying, we have

\[
R_{17}(G) = -s^e - s^f - s^{e+1} - s^{f+1} + s^{e+4} + s^{f+6} + s^{d+8},
\]

\[
R_{17}(H) = -s^{e-3} - s^{f+3} - s^{f+4} - s^{d+1} + s^{d+3} + s^{f+7} + s^{f+8} + s^{d+6}.
\]

For the same reason stated in Subcase 6.2, \(-s^{d+1}\) in \( R_{17}(H) \) can only be equal to \(-s^f\) or \(-s^{f+1}\) in \( R_{16}(G) \).

If \(-s^{d+1} = -s^{f+1}\), then \( d = f \). So, we have

\[
R_{18}(G) = -s^e - s^d - s^{e+1} - s^{d+1} + s^{d+3} + s^{e+4} + s^{d+6} + s^{d+8},
\]

\[
R_{18}(H) = -s^{e-3} - s^{d+3} - s^{d+4} - s^{d+1} + s^{d+3} + s^{d+7} + s^{d+8} + s^{d+6}.
\]

After simplifying, we have

\[
-s^e - s^d - s^{e+1} + s^{e+4} = -s^{e-3} - s^{d+3} - s^{d+4} + s^{d+7}.
\]

Then, we know that the term \( s^{e-3} \) must be equal to \( s^d \). So, we have \( d = e - 3 \).

Also, we obtain \( d = f' = f \) and \( e' = f + 3 = f' + 3 = d + 3 = e \). From \( d + e + f = d' + e' + f' \), we have \( d = d' \). Therefore \( G \cong H \).

If \(-s^{d+1} = -s^f\), then \( d + 1 = f \). So, we have

\[
R_{19}(G) = -s^e - s^{d+1} - s^{e+1} - s^{d+2} + s^{d+4} + s^{e+4} + s^{d+7} + s^{d+8},
\]

\[
R_{19}(H) = -s^{e-3} - s^{d+4} - s^{d+5} - s^{d+1} + s^{d+3} + s^{d+8} + s^{d+9} + s^{d+6}.
\]

After simplifying, we obtain

\[
-s^e - s^{e+1} - s^{d+2} + s^{d+4} + s^{e+4} + s^{d+7} = -s^{e-3} - s^{d+4} - s^{d+5} + s^{d+3} + s^{d+9} + s^{d+6}.
\]

The resulting equation contradicts \( R_{19}(G) = R_{19}(H) \).

At this point, we have solved the equation \( R(G) = R(H) \) and the solution is as follows:

\[
K_4(1, 3, 5, i, i + 6, i + 1) \sim K_4(1, 3, 5, i + 2, i, i + 5),
\]

\[
K_4(1, 3, 5, i, i + 1, i + 4) \sim K_4(1, 3, 5, i + 2, i + 3, i),
\]

where \( i \geq 1 \). The proof is now completed.

We close the paper with the following problem.
**Problem.** Study the chromatic uniqueness of the graph $K_4(1, 3, 5, d, e, f)$, where $d + e \geq 6$, $d + f \geq 4$ and $e + f \geq 8$.

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**References**


