International Journal of Pure and Applied Mathematics
Volume 81 No. 3 2012, 517-529
ISSN: 1311-8080 (printed version)
url: http://www.ijpam.eu

ijpam.eu

A SCHUR-TYPE THEOREM FOR $\mathcal{I}$-CONVERGENCE AND MAXIMAL IDEALS

A. Boccuto[1 §], P. Das[2], X. Dimitriou[3]

[1]Department of Mathematics and Computer Sciences
University of Perugia
via Vanvitelli 1, I-06123 Perugia, ITALY

[2]Department of Mathematics
Jadavpur University
Kolkata, 700032, West Bengal, INDIA

[3]Department of Mathematics
University of Athens
Panepistimiopolis, Athens 15784, GREECE

Abstract: A Schur-type theorem in the setting of ideal convergence is proved. Some properties of ideal convergence in relation with usual convergence of suitable subsequences are investigated, and a characterization of maximal ideals is given. Furthermore some open problem is posed.

AMS Subject Classification: 28A20, 28B15, 54A20
Key Words: admissible ideal, maximal ideal, P-ideal, ideal convergence, Schur theorem

1. Introduction

Ideal convergence was introduced in [20] and independently in [22] with the name of "cofilter convergence". It was recently studied in the literature by severals authors, among which we quote for example [13, 14, 15, 16, 20, 21]. In

Received: September 15, 2012

§Correspondence author

particular it has been deeply investigated in problems concerning convergence of functions (see for instance [4, 5, 19, 20]) and convergence of measures and integrals (see e.g.[4, 6, 7, 10, 11]).

In this paper we prove a Schur-type theorem, which connects weak and norm convergence in ℓ_1 in the ideal context, investigate some properties of ideal convergence and study them in relation with classical convergence of suitable subsequences. We present a characterization of maximal ideals of $\mathbb{N}$ and pose some open problems. In this framework, some Schur-type theorems for ideal convergence are given in [11].

Recently, some slightly different versions of Schur-type theorems are proved with respect to the filter convergence. In particular, in [3] there are some necessary and/or sufficient conditions on filters of $\mathbb{N}$ to satisfy the Schur property. With similar techniques, these theorems have been extended to the context of absolutely summable (ℓ)-group-valued sequences in [11] and, as applications and consequences, some versions of limit theorems ([7, 11]), Dieudonné-type theorems ([7, 11]) and uniform boundedness principle ([9]) are given with respect to filters. Furthermore, some related basic matrix theorems for ideal convergence are proved in [10], while in [5] also some other versions of limit theorems and some main properties of (weak) compactness in the ideal convergence setting were investigated.

2. Preliminaries

Throughout the paper we denote by ℓ_1 the space of all real sequences whose associate series is absolutely convergent, ℓ_∞ the space of all bounded real sequences and c_{00} the subspace of ℓ_∞ of the eventually null sequences.

We recall the fundamental properties of ideals (see also [13, 20, 21]).

Definitions 2.1. (a) A family $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ is called an *ideal* of $\mathbb{N}$ iff $A \cup B \in \mathcal{I}$ whenever $A, B \in \mathcal{I}$ and for each $A \in \mathcal{I}$ and $B \subset A$ we get $B \in \mathcal{I}$.

(b) An ideal $\mathcal{I}$ of $\mathbb{N}$ is called *non-trivial* iff $\mathcal{I} \neq \emptyset$ and $\mathbb{N} \notin \mathcal{I}$. A non-trivial ideal $\mathcal{I}$ is called *admissible* iff it contains all singletons.

(c) An ideal $\mathcal{I}$ of $\mathbb{N}$ is said to be *maximal* iff for every $A \subset \mathbb{N}$ we get either $A \in \mathcal{I}$ or $\mathbb{N} \setminus A \in \mathcal{I}$.

(d) An admissible ideal $\mathcal{I}$ of $\mathbb{N}$ is said to be a *P-ideal* iff for any sequence $(A_j)_j$ in $\mathcal{I}$ there is a sequence $(B_j)_j$ of subsets of $\mathbb{N}$, such that the symmetric difference $A_j \Delta B_j$ is finite for all $j \in \mathbb{N}$ and $\bigcup_{j=1}^{\infty} B_j \in \mathcal{I}$.

We now give some notion of convergence in the ideal setting.

Definitions 2.2. (a) Let (X, d) be a metric space, $\mathcal{I}$ be an admissible ideal of $\mathbb{N}$ and $\mathcal{F} = \mathcal{F}(\mathcal{I}) = \{\mathbb{N} \setminus A : A \in \mathcal{I}\}$ be its dual filter. A sequence $(x_n)_n$ in X is called *$\mathcal{I}$-convergent* to $x \in X$ iff for all $\varepsilon > 0$, $\{n \in \mathbb{N}: d(x_n, x) > \varepsilon\} \in \mathcal{I}$. We then write $\mathcal{I} - \lim_n x_n = x$.

(b) A sequence $(x_n)_n$ in (X, d) is called *$\mathcal{I}$-Cauchy* iff for each $\varepsilon > 0$ there exists $q \in \mathbb{N}$ such that $\{n \in \mathbb{N}: d(x_n, x_q) > \varepsilon\} \in \mathcal{I}$.

If X is a Banach space, then the concepts of *norm $\mathcal{I}$-convergence* and *norm $\mathcal{I}$-Cauchy* can be formulated analogously as above.

(c) Let X be a Banach space. A sequence $(x_n)_n$ is called *weakly-$\mathcal{I}$-convergent* iff the sequence $(x^*(x_n))_n$ is $\mathcal{I}$-convergent for every $x^* \in X^*$ (the dual space of X). A sequence $(x_n)_n$ is said to be *weakly-$\mathcal{I}$-Cauchy* iff the sequence $(x^*(x_n))_n$ is $\mathcal{I}$-Cauchy for every $x^* \in X^*$.

(d) A sequence $(x_n)_n$ in (X, d) *$\mathcal{I}^*$-converges to* $x \in X$ iff there exists a set $A_0 \in \mathcal{F}(\mathcal{I})$ with $\lim_{n \in A_0} x_n = x$.

Examples 2.3. (see [20]) (i) If $\mathcal{I}_{fin} = \{A \subset \mathbb{N} : A \text{ finite}\}$, then $\mathcal{I}_{fin}$ is a P-ideal of $\mathbb{N}$ and $\mathcal{I}_{fin}$-convergence coincides with the ordinary convergence.

(ii) Let $A \subset \mathbb{N}$. The *asymptotic density* of A is defined as

$$\delta(A) := \lim_n \frac{\sharp(A \cap \{1, \ldots, n\})}{n},$$

where $\sharp$ denotes the cardinality of the set in brackets. If $\mathcal{I}_\delta = \{A \subset \mathbb{N} : \delta(A) = 0\}$, then $\mathcal{I}_\delta$ is a P-ideal of $\mathbb{N}$ (see [20]), but it is not maximal, because, if $\mathbb{E}$ denotes the set of all even subsets of $\mathbb{N}$, then neither $\mathbb{E}$ nor $\mathbb{N} \setminus \mathbb{E}$ belongs to $\mathcal{I}_\delta$.

(iii) Let $\mathbb{N} = \bigcup_{j=1}^{\infty} \Delta_j$ be a partition of $\mathbb{N}$ such that Δ_j is an infinite set for every $j \in \mathbb{N}$ and $\mathcal{I}_g = \{A \subset \mathbb{N} : A$ intersects only a finite number of Δ_j's $\}$. Then $\mathcal{I}_g$ is not a P-ideal.

We now give the following results.

Proposition 2.4. *Let $\mathcal{I}$ be an admissible ideal, $\mathcal{F}(\mathcal{I})$ be its dual filter, $(x_n)_n$ be a sequence in (X, d), such that $\mathcal{I} - \lim_n x_n = x \in X$. Then there exists a strictly increasing subsequence $(x_{n_q})_q$ of $(x_n)_n$, such that $\lim_q x_{n_q} = x$.*

Moreover, if $\mathcal{I}$ is a P-ideal, then the subsequence $(x_{n_q})_q$ can be chosen in such a way that $\{n_1 < n_2 < \ldots < n_q < \ldots\} \in \mathcal{F}(\mathcal{I})$.

Proof. Choose arbitrarily $\varepsilon > 0$. By definition of ideal convergence, in correspondence with ε there exists a positive integer n_1 with $d(x_{n_1}, x) \leq \varepsilon$. At the second step, in correspondence with

$$\varepsilon_2 := \frac{\min\{\varepsilon, d(x_1,x), d(x_2,x), \ldots, d(x_{n_1},x)\}}{2}, \tag{1}$$

there is $n_2 \in \mathbb{N}$ such that $d(x_{n_2}, x) \leq \varepsilon_2$. By virtue of (1), we get $n_2 > n_1$. Proceeding by induction, supposed that ε_{q-1} and n_{q-1} have been determined, let

$$\varepsilon_q := \frac{\min\{\varepsilon_{q-1}, d(x_1,x), d(x_2,x), \ldots, d(x_{n_{q-1}},x)\}}{2}. \tag{2}$$

By $\mathcal{I}$-convergence, there exists an integer $n_q \in \mathbb{N}$ with $d(x_{n_q}, x) \leq \varepsilon_q$. By (2), we have $n_q > n_{q-1}$. Since $\lim_q \varepsilon_q = 0$, then we get $\lim_q d(x_{n_q}, x) = 0$, and hence $\lim_q x_{n_q} = x$. This concludes the proof of the first part.

For the last part, see [20, Theorem 3.2]. □

Proposition 2.5. (see also [20, Theorem 3.2]) *The $\mathcal{I}^*$-convergence of sequences implies always the $\mathcal{I}$-convergence.*

Moreover, if $(x_n)_n$ is a sequence in (X,d), $\mathcal{I}$-convergent to $x \in X$, and $\mathcal{I}$ is a P-ideal, then $(x_n)_n$ $\mathcal{I}^*$-converges to x.

Proposition 2.6. *Let $(x_{i,j})_{i,j}$ be a double sequence in (X,d), $\mathcal{I}$ be any P-ideal, $\mathcal{F} = \mathcal{F}(\mathcal{I})$ be its dual filter, and let us suppose that $\mathcal{I} - \lim_i x_{i,j} = x_j$ for every $j \in \mathbb{N}$. Then there exists $B_0 \in \mathcal{F}$ such that $\lim_{h \in B_0} x_{h,j} = x_j$ for all $j \in \mathbb{N}$.*

Proof. Since $\mathcal{I}$ is a P-ideal, by virtue of Proposition 2.5 we get

$$\mathcal{I}^* - \lim_i x_{i,j} = x_j$$

for every $j \in \mathbb{N}$. Hence there is a sequence $(A_j)_j$ in $\mathcal{F}$ such that $\lim_{i \in A_j} x_{i,j} = x_j$ for all $j \in \mathbb{N}$. As $\mathcal{I}$ is a P-ideal, there is a sequence of sets $(B_j)_j$ in $\mathcal{F}$ such that $A_j \Delta B_j$ is finite for all $j \in \mathbb{N}$ and $B_0 := \bigcap_{j=1}^{\infty} B_j \in \mathcal{F}$. Since $\lim_{i \in A_j} x_{i,j} = x_j$ for all j, then we get also $\lim_{i \in B_j} x_{i,j} = x_j$ for all j. Let $B_0 = \{p_1 < \ldots < p_h < \ldots\}$ and choose arbitrarily $j \in \mathbb{N}$: then, since $B_0 \subset B_j$, in correspondence with ε an integer $\overline{h} = \overline{h}(\varepsilon, j)$ can be found, with the property that $|x_{p_h,j} - x_j| \leq \varepsilon$ whenever $h > \overline{h}$. This concludes the proof. □

The following lemma will be useful to prove our Schur-type theorem.

Lemma 2.7. *Let $(x_k)_k \in \ell_1$. For every p, $q \in \mathbb{N}$ with $p < q$, let $\mathcal{S}_{p,q}$ be the class of all subsets of $\{p, p+1, \dots, q\}$. Then we get*

$$\sum_{k=p}^{q} |x_k| \le 2 \max_{S \in \mathcal{S}_{p,q}} \left| \sum_{k \in S} x_k \right| .$$

Proof. Let us define $\mu : \mathcal{P}(\mathbb{N}) \to \mathbb{R}$ as follows:

$$\mu(A) := \sum_{k \in A} x_k, \quad A \subset \mathbb{N}.$$

By [8, Lemma 3.6], we get

$$\sum_{k=p}^{q} |x_k| = \sum_{k=p}^{q} |\mu(\{k\})| \le 2 \max_{S \in \mathcal{S}_{p,q}} |\mu(S)| = 2 \max_{S \in \mathcal{S}_{p,q}} \Big| \sum_{k \in S} x_k \Big| .$$

This ends the proof. □

3. The Main Results

We begin with a Schur-type theorem in the context of ideal convergence, which uses a sliding hump technique (for a related literature, see also [2, 3, 10, 12]).

Theorem 3.1. *Let $(y_{n,k})_{n,k}$ be a double sequence in $\mathbb{R}$ such that $(y_{n,k})_k \in \ell_1$ for each $n \in \mathbb{N}$, $\mathcal{I}$ be a fixed P-ideal of $\mathbb{N}$, and $\mathcal{F} = \mathcal{F}(\mathcal{I})$ be its dual filter. Suppose that there exist a sequence $(y_k)_k \in \ell_1$, with*

a) $\mathcal{I} - \lim_n y_{n,k} = y_k$ for every $k \in \mathbb{N}$

and an element $F \in \mathcal{F}$, such that

b) $\mathcal{I} - \lim_r \sum_{k \in E} y_{i_r,k} = \sum_{k \in E} y_k$ for each strictly increasing sequence $(i_r)_r$ in F and for every $E \subset \mathbb{N}$ such that both E and $\mathbb{N} \setminus E$ are infinite.

Then $\mathcal{I} - \lim_n \sum_{k=1}^{\infty} |y_{n,k} - y_k| = 0$.

Proof. For every p, $q \in \mathbb{N}$ with $p < q$, let $\mathcal{S}_{p,q}$ be as in Lemma 2.7. For each n, $k \in \mathbb{N}$ put $b_{n,k} := y_{n,k} - y_k$.

First of all note that, by a) and Proposition 2.6, a set $K \in \mathcal{F}$ can be found, with

$$\lim_{n \in K} b_{n,k} = 0 \tag{3}$$

for any $k \in \mathbb{N}$. So, in order to prove the result, it will be enough to show that

$$\lim_{n \in F \cap K} \sum_{k=1}^{\infty} |b_{n,k}| = 0, \tag{4}$$

since $F \cap K \in \mathcal{F}$ and by the first part of Proposition 2.5. If (4) is not true, then there is a positive real number C with the property that for every $q \in K$ there exists $l_q \in F \cap K$, $l_q > q$, with

$$\sum_{k=1}^{\infty} |b_{l_q,k}| > C. \tag{5}$$

From (5) and (3) it follows that there is a positive integer $i_1 \in F \cap K$, with

$$\sum_{k=1}^{\infty} |b_{i_1,k}| > C \text{ and } |b_{i_1,1}| \leq \frac{C}{8}. \tag{6}$$

At the first step, let $k_1 := 1$. Since $(y_k)_k$ and $(y_{i_1,k})_k$ belong to ℓ_1, then

$$\sum_{k=1}^{\infty} |b_{i_1,k}| < +\infty,$$

and so we can choose a natural number $k_2 > k_1$ with

$$\sum_{k=k_2}^{\infty} |b_{i_1,k}| \leq \frac{C}{8}.$$

From this and (6) we obtain

$$\sum_{k=2}^{k_2-1} |b_{i_1,k}| > \frac{3C}{4}. \tag{7}$$

From (7) and Lemma 2.7 we have

$$\max_{A \in \mathcal{S}_{2,k_2-1}} \Big| \sum_{k \in A} b_{i_1,k} \Big| > \frac{3C}{8}.$$

At the second step, taking $q = i_1$ in (5), we get the existence of an element $i_2 \in F \cap K$, $i_2 > i_1$, with

$$\sum_{k=1}^{\infty} |b_{i_2,k}| > C \text{ and } \sum_{k=1}^{k_2} |b_{i_2,k}| \leq \frac{C}{8} : \tag{8}$$

such a choice is possible, by virtue of (5) and (3) respectively. By proceeding analogously as above, we can find an integer $k_3 > k_2$ with

$$\sum_{k=k_3}^{\infty} |b_{i_2,k}| \le \frac{C}{8}.$$

From this and (8) we get

$$\sum_{k=k_2+1}^{k_3-1} |b_{i_2,k}| > \frac{3C}{4}. \tag{9}$$

From (9) and Lemma 2.7 it follows that

$$\max_{A\in\mathcal{S}_{k_2+1,k_3-1}} \Big|\sum_{k\in A} b_{i_2,k}\Big| > \frac{3C}{8}.$$

By induction, we can construct two strictly increasing sequences $(i_r)_r$ and $(k_r)_r$ in $F \cap K$ and $\mathbb{N}$ respectively, such that $k_1 = 1$ and

1) $\displaystyle\sum_{k=1}^{k_r} |b_{i_r,k}| \le \frac{C}{8}$;
2) $\displaystyle\sum_{k=k_{r+1}}^{\infty} |b_{i_r,k}| \le \frac{C}{8}$;
3) $\displaystyle\max_{A\in\mathcal{S}_{k_r+1,k_{r+1}-1}} \Big|\sum_{k\in A} b_{i_r,k}\Big| > \frac{3C}{8}$

for every $r \ge 2$. From 3) it follows that for such r's there exists a set $E_r \in \mathcal{S}_{k_r+1,k_{r+1}-1}$ with

4) $\displaystyle\Big|\sum_{k\in E_r} b_{i_r,k}\Big| > \frac{3C}{8}$.

Let now $E := \displaystyle\bigcup_{r=1}^{\infty} E_r$. Note that, by construction, E is infinite and $E \cap \{k_r : r \ge 2\} = \emptyset$, and hence $\mathbb{N} \setminus E$ is infinite too. By b) and Proposition 2.4, we can associate to the sequence $(i_r)_r$ and the set E a subsequence $(i_{r_s})_s$ of $(i_r)_r$, with

$$\lim_s \Big(\sum_{k\in E} b_{i_{r_s},k}\Big) = 0,$$

where the limit is intended in the usual sense. Thus, in correspondence with $\frac{C}{8}$, there is a positive integer $s_0 \in \mathbb{N}$ such that for every $s \geq s_0$ we get:

$$\begin{aligned}\Big|\sum_{k\in E_{r_s}} b_{i_{r_s},k}\Big| &= \Big|\sum_{k\in E} b_{i_{r_s},k} - \sum_{k\in E,k=1}^{k_{r_s}} b_{i_{r_s},k} - \sum_{k\in E,k=k_{r_s+1}}^{\infty} b_{i_{r_s},k}\Big| \\ &\leq \Big|\sum_{k\in E} b_{i_{r_s},k}\Big| + \sum_{k=1}^{k_{r_s}} |b_{i_{r_s},k}| + \sum_{k=k_{r_s+1}}^{\infty} |b_{i_{r_s},k}| \\ &\leq \frac{C}{8} + \frac{C}{8} + \frac{C}{8} = \frac{3C}{8}.\end{aligned}$$

This contradicts 4) and proves the theorem. □

In [12] the following Schur-type theorem was proved (see also [2]).

Theorem 3.2. *Let $(y_{n,k})_{n,k}$ be a double sequence in $\mathbb{R}$, and assume that:*

(i) $\lim_n \sum_{k\in A} y_{n,k}$ *exists in $\mathbb{R}$ for every subset $A \subset \mathbb{N}$;*

(ii) $\lim_n y_{n,k} = y_k$ *exists for all $k \in \mathbb{N}$.*

Then we get:

(a) $(y_k)_k \in \ell_1$;

(b) $\lim_n \sum_{k=1}^{\infty} |y_{n,k} - y_k| = 0$.

Remarks 3.3. (a) Theorem 3.2 has also the following interpretation.

Let $x_n = (y_{n,k})_k \in \ell_1$, for each $n \in \mathbb{N}$. Theorem 3.2 says that $(x_n)_n$ is a Cauchy sequence in the topology $\sigma(\ell_1, c_{00})$. The conclusion is that $(x_n)_n$ is norm convergent in ℓ_1. In particular, this implies that any weakly convergent sequence in ℓ_1 is norm convergent (that is the classical Schur theorem).

(b) Observe that Theorem 3.1 is an extension of Theorem 3.2 in the context of P-ideals.

We now give a characterization of maximal ideals, related with the $\mathcal{I}$-limit of the subsequences of real bounded sequences. Note that, if we assume the continuum hypothesis, then there exists a large class of maximal P-ideals (see [18]).

Proposition 3.4. *An admissible ideal $\mathcal{I}$ of $\mathbb{N}$ is not maximal if and only if, for any bounded sequence $(a_n)_n$ in $\mathbb{R}$ with the property that $\mathcal{I} - \lim_h a_{i_h}$ exists in $\mathbb{R}$ for each strictly increasing sequence $(i_h)_h$ in $\mathbb{N}$, we get that the limit $l := \lim_n a_n$ exists in $\mathbb{R}$ in the classical sense. In this case, $\mathcal{I} - \lim_h a_{i_h} = l$ is the same for every strictly increasing sequence $(i_h)_h$.*

Proof. We begin with the "if" part. Suppose by contradiction that $\mathcal{I}$ is maximal. Then there exists a bounded real sequence $(a_n)_n$, which does not admit limit in the classical sense. For every strictly increasing sequence $(i_h)_h$ in $\mathbb{N}$, the sequence $(a_{i_h})_h$ is obviously bounded too, and hence, since $\mathcal{I}$ is maximal, the limit $\mathcal{I} - \lim_h a_{i_h}$ exists in $\mathbb{R}$ (see also [4]). This leads to a contradiction, and concludes the proof of the "if" part.

We now turn to the "only if" part. First of all we claim that, if $\mathcal{I}$ is any not maximal ideal of $\mathbb{N}$, then there exist two disjoint elements $B_1 \notin \mathcal{I}$, $B_2 \notin \mathcal{I}$ whose union is $\mathbb{N}$. Otherwise, for each partition of $\mathbb{N}$ formed by two elements (B_1, B_2) of $\mathcal{I}$, then either B_1 or B_2 belongs to $\mathcal{I}$. If $B_1 \in \mathcal{I}$, then $B_2 \notin \mathcal{I}$, otherwise $\mathbb{N} \in \mathcal{I}$, and hence $\mathcal{I}$ should be trivial. Similarly, if $B_2 \in \mathcal{I}$, then $B_1 \notin \mathcal{I}$. Thus the ideal $\mathcal{I}$ should be maximal. This leads to a contradiction and proves the claim. Moreover note that, since $\mathcal{I}$ is admissible, then every finite subset of $\mathbb{N}$ belongs to $\mathcal{I}$, and hence the two involved sets B_1, B_2 turn out to be infinite. Thus we can represent them in the form

$$B_1 := \{t_1 < t_2 < \ldots < t_j < \ldots\}, \quad B_2 := \{r_1 < r_2 < \ldots < r_j < \ldots\}. \tag{10}$$

Now suppose by contradiction that $\lim_n a_n$ does not exist in $\mathbb{R}$. So, since $(a_n)_n$ is bounded, there are two sequences in $\mathbb{N}$, $(p'_h)_h$, $(q'_h)_h$ such that $\lim_h a_{p'_h} = l_1$, $\lim_h a_{q'_h} = l_2$, where

$$\liminf_n a_n := l_1 < l_2 := \limsup_n a_n.$$

Set $P := \{p'_j : j \in \mathbb{N}\}$, $Q := \{q'_l : l \in \mathbb{N}\}$. Let now $p_i := p'_1$, and choose $q_1 > p_1$, $q_1 \in Q$: such an element does exist, since Q is infinite. Pick now $p_2 \in P$ such that $p_2 > q_1$: such a choice is possible, because P is infinite. Keeping on by induction, it is possible to construct two sequences $(p_h)_h$, $(q_h)_h$, with the properties that: $\lim_j a_{p_j} = l_1$, $\lim_s a_{q_s} = l_2$, and

$$p_1 < q_1 < p_2 < \ldots < q_{h-1} < p_h < q_h < p_{h+1} < \ldots$$

For example, if we have just defined $p_1 < q_1 < \ldots < p_{h-1} < q_{h-1}$, let us choose $p_h \in P$ such that $p_h > q_{h-1}$ and $q_h \in Q$ with $q_h > p_h$: this is possible, since P and Q are infinite.

Let now B_1, B_2 be as in (10). For every $n \in \mathbb{N}$ there exists one natural number j that $n = t_j$, or there is one positive integer s such that $n = r_s$. In the first case put $b_n := a_{p_j}$, and in the second case set $b_n := a_{q_s}$.

The next step is to prove that for all $l \in \mathbb{R}$ there exists $\delta(l) > 0$, such that

$$\{n \in \mathbb{N} : |b_n - l| > \delta\} \notin \mathcal{I}. \tag{11}$$

First of all, let us consider the case $l \neq l_1$. Take $\delta := \dfrac{|l - l_1|}{2}$ and set

$$\varepsilon := \frac{|l - l_1|}{4} = \frac{\delta}{2}.$$

By the definition of limit, we get: $|a_{p_j} - l_1| \le \dfrac{|l - l_1|}{4}$ in the complement of a finite number of indexes j. So there exists a finite subset $N_1 \subset \mathbb{N}$ such that, if $n \in B_1 \setminus N_1$, then $|b_n - l_1| \le \dfrac{|l - l_1|}{4}$. This implies that for all $n \in B_1 \setminus N_1$ we get: $|b_n - l| > \dfrac{|l - l_1|}{2}$. Otherwise we should have:

$$|l - l_1| \le |l - b_n| + |b_n - l_1| \le \frac{|l - l_1|}{2} + \frac{|l - l_1|}{4} = \frac{3}{4}\,|l - l_1|.$$

This is possible if and only if $l = l_1$, but this is absurd, because it contradicts our assumption.

Thus the set $\{n \in \mathbb{N} : |b_n - l| > \delta\}$ contains $B_1 \setminus N_1$, and so it does not belong to $\mathcal{I}$, since $B_1 \notin \mathcal{I}$, N_1 is finite and $\mathcal{I}$ is admissible. Thus (11) is proved, at least when $l \neq l_1$.

We now turn to the case $l = l_1$. Take $\delta := \dfrac{l_2 - l_1}{2}$. Note that $\delta > 0$, since $l_1 < l_2$. Analogously as above, we get $|a_{r_s} - l_2| \le \dfrac{l_2 - 1_1}{4}$ in the complement of finitely many indexes s. Thus there is a finite subset $N_2 \subset \mathbb{N}$ such that $|b_n - l_2| \le \dfrac{l_2 - l_1}{4}$ whenever $n \in B_2 \setminus N_2$. This implies that for all $n \in B_2 \setminus N_2$ we have: $|b_n - l| > \dfrac{l_2 - l_1}{2}$. Otherwise, we get:

$$0 < l_2 - l_1 \le |l_1 - b_n| + |b_n - l_2| \le \frac{l_2 - l_1}{2} + \frac{l_2 - l_1}{4} = \frac{3}{4}\,(l_2 - l_1) < l_2 - l_1,$$

a contradiction. Thus the set $\{n \in \mathbb{N} : |b_n - l| > \delta\}$ contains $B_1 \setminus N_1$, and so it does not belong to $\mathcal{I}$, since $B_2 \notin \mathcal{I}$, N_2 is finite and $\mathcal{I}$ contains all the finite subsets of $\mathbb{N}$. This proves (11) in the case $l = l_1$.

From (11) it follows that the sequence $(b_n)_n$ does not have $\mathcal{I}$-limit. By construction, it follows easily that the sequence

$$(a_{p_1}, a_{q_1}, a_{p_2}, \ldots, a_{q_{h-1}}, a_{p_h}, a_{q_h}, a_{p_{h+1}}, \ldots,)$$

does not have $\mathcal{I}$-limit. Thus the assertion follows. □

Remark 3.5. Observe that Proposition 3.4 is a strengthening of [1, Theorem 2.1].

Open Problems

(a) Investigate the ideals $\mathcal{I}$ for which weak $\mathcal{I}$-convergence in ℓ_1 implies norm $\mathcal{I}$-convergence. A similar investigation considering filters was done in [3] and, in the (ℓ)-group setting, in [11].

(b) Study the ideals $\mathcal{I}$ for which weak* $\mathcal{I}$-convergence in ℓ_∞^* implies weak $\mathcal{I}$-convergence of sequences (see also [17, pp. 103-104]).

(c) Find results analogous to Proposition 3.4 when rearrangements of the initial sequence $(a_n)_n$ are considered.

Acknowledgments

Supported by Universities of Athens, Jadavpur and Perugia.

References

[1] H. Albayrak, S. Pehlivan, On the ideal convergence of subsequences and rearrangements of a real sequence, *Applied Math. Letters,* **23** (2010), 1203-1207.

[2] P. Antosik, C. Swartz, *Matrix Methods in Analysis*, Lecture Notes Math. **1113**, Springer - Verlag, Berlin-Heidelberg (1985).

[3] A. Aviles-Lopez, B. Cascales Salinas, V. Kadets, A. Leonov, The Schur ℓ_1 theorem for filters, *J. Math. Phys. Anal. Geom.,* **3**, No. 4 (2007), 383-398.

[4] A. Boccuto, D. Candeloro, Defining limits by means of integrals, *Operator Theory: Advances and Applications,* **201** (2009), 79-87.

[5] A. Boccuto, P. Das, X. Dimitriou, N. Papanastassiou, Ideal exhaustiveness, weak convergence and weak compactness in Banach spaces, *Real Anal. Exchange,* **37**, No. 2 (2012), 409-430.

[6] A. Boccuto, X. Dimitriou, N. Papanastassiou, Brooks-Jewett-type theorems for the pointwise ideal convergence of measures with values in (ℓ)-groups, *Tatra Mt. Math. Publ.,* **49** (2011), 17-26.

[7] A. Boccuto, X. Dimitriou, N. Papanastassiou, Some versions of limit and Dieudonné-type theorems with respect to filter convergence for (ℓ)-group-valued measures, *Cent. Eur. J. Math.,* **9**, No. 6 (2011), 1298-1311.

[8] A. Boccuto, X. Dimitriou, N. Papanastassiou, Limit theorems in (ℓ)-groups with respect to (D)-convergence, *Real Anal. Exchange,* **37**, No. 2 (2012), 269-298.

[9] A. Boccuto, X. Dimitriou, N. Papanastassiou, Uniform boundedness principle, Banach-Steinhaus and approximation theorems for filter convergence in Riesz spaces, *Proceedings of International Conference on Topology and its Applications ICTA 2011, Cambridge Sci. Publ.,* (2012), 45-58.

[10] A. Boccuto, X. Dimitriou, N. Papanastassiou, Basic matrix theorems for $\mathcal{I}$-convergence in (ℓ)-groups, *Math. Slovaca,* **62**, No. 5 (2012), 885-908.

[11] A. Boccuto, X. Dimitriou, N. Papanastassiou, Schur lemma and limit theorems in lattice groups with respect to filters, *Math. Slovaca,* (2012), to appear.

[12] J. K. Brooks, J. Mikusiński, On some theorems in functional analysis, *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.,* **18**, No. 3 (1970), 151-155.

[13] P. Das, S. K. Ghosal, Some further results on $\mathcal{I}$-Cauchy results and condition (AP), *Comput. Math. Appl.,* **59**, No. 8 (2010), 2597-2600.

[14] P. Das, P. Kostyrko, W. Wilczyński, P. Malík, $\mathcal{I}$ and $\mathcal{I}^*$-convergence of double sequences, *Math. Slovaca,* **58**, No. 5 (2008), 605-620.

[15] P. Das, P. Malík, On extremal $\mathcal{I}$-limit points of double sequences, *Tatra Mt. Math. Publ.,* **40** (2008), 91-102.

[16] P. Das, P. Malík, On the statistical and $\mathcal{I}$-variation of double sequences, *Real Anal. Exchange,* **33**, No. 2 (2008), 351-363.

[17] J. Diestel, *Sequences and series in Banach spaces,* Springer - Verlag, Berlin-Heidelberg (1984).

[18] M. Henriksen, Multiplicative summability methods and the Stone-Čech compactification, *Math. Zeitschr.,* **71** (1959), 427-435.

[19] A. Komisarski, Pointwise $\mathcal{I}$-convergence and $\mathcal{I}$-convergence in measure of sequences of functions, *J. Math. Anal. Appl.,* **340** (2008), 770-779.

[20] P. Kostyrko, T. Šalát, W. Wilczyński, $\mathcal{I}$-convergence, *Real Anal. Exchange,* **26**, No. 2 (2000/01), 669-685.

[21] B. K. Lahiri, P. Das, $\mathcal{I}$ and $\mathcal{I}^*$-convergence of nets, *Real. Anal. Exchange,* **33**, No. 2 (2007/08), 431-442.

[22] F. Nuray, W. H. Ruckle, Generalized statistical convergence and convergence free spaces, *J. Math. Anal. Appl.,* **245** (2000), 513-527.