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MORE ON THE DIOPHANTINE EQUATION $8^x + 19^y = z^2$

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Abstract: In this paper, we show that the Diophantine equation $8^x+19^y = z^2$ has a unique non-negative integer solution. The solution (x, y, z) is $(1, 0, 3)$.

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1. Introduction

In 2012, Peker and Cenberci [4] suggested that the Diophantine equation $8^x + 19^y = z^2$ has no non-negative integer solution. In fact, $8^1 + 19^0 = 9 = 3^2$. In this paper, we will show that $(1, 0, 3)$ is a unique solution (x, y, z) for the Diophantine equation $8^x + 19^y = z^2$ where x, y and z are non-negative integers. For related papers, we list them as follows.

In 2007, Acu [1] showed that $(3, 0, 3)$ and $(2, 1, 3)$ are only two solutions (x, y, z) for the Diophantine equation $2^x + 5^y = z^2$ where x, y and z are non-negative integers. In 2011, Suvarnamani, Singta and Chotchaisthit [6] showed that the two Diophantine equations $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$ have no non-negative integer solution. In the same year, Suvarnamani [5] found some non-negative integer solutions for the Diophantine equation of type $2^x+p^y = z^2$ where p is a positive prime number. In 2012, Chotchaisthit [2] gave all non-negative integer solutions for the Diophantine equation of type $4^x + p^y = z^2$

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where p is a positive prime number.

2. Preliminaries

The Catalan's conjecture is a well known conjecture. This conjecture states that $(3, 2, 2, 3)$ is a unique solution (a, b, x, y) for the Diophantine equation $a^x - b^y = 1$ where a, b, x and y are integers with $\min\{a, b, x, y\} > 1$. In 2004, this conjecture was proven in 2004 by Mihailescu [3].

Proposition 2.1. [3] *$(3, 2, 2, 3)$ is a unique solution (a, b, x, y) for the Diophantine equation $a^x - b^y = 1$ where a, b, x and y are integers with $\min\{a, b, x, y\} > 1$.*

Next, we will prove two Lemmas by Proposition 2.1.

Lemma 2.2. *$(1, 3)$ is a unique solution (x, z) for the Diophantine equation $8^x + 1 = z^2$ where x and z are non-negative integers.*

Proof. Let x, y and z be non-negative integers such that $8^x + 1 = z^2$. If $x = 0$, then $z^2 = 2$ which is impossible. Then $x \geq 1$. Thus, $z^2 = 8^x + 1 \geq 8^1 + 1 = 9$. Then $z \geq 3$. Now, we consider on the equation $z^2 - 8^x = 1$. By Proposition 2.1, we have $x = 1$. Then $z = 3$. Hence, $(1, 3)$ is a unique solution (x, z) for the equation $8^x + 1 = z^2$ where x and z are non-negative integers. □

Lemma 2.3. *The Diophantine equation $1 + 19^y = z^2$ has no non-negative integer solution.*

Proof. Suppose that there are non-negative integers y and z such that $1 + 19^y = z^2$. If $y = 0$, then $z^2 = 2$ which is impossible. Then $y \geq 1$. Thus, $z^2 = 1 + 19^y \geq 1 + 19^1 = 20$. Then $z \geq 5$. Now, we consider on the equation $z^2 - 19^y = 1$. By Proposition 2.1, we have $y = 1$. Then $z^2 = 20$. This is a contradiction. Hence, the equation $1 + 19^y = z^2$ has no non-negative integer solution. □

3. Results

In [4], the Diophantine equation $8^x + 19^y = z^2$ has no non-negative integer solution. But we will show in this section that $(1, 0, 3)$ is a unique solution (x, y, z) for the Diophantine equation $8^x + 19^y = z^2$ where x, y and z are non-negative integers.

Theorem 3.1. *$(1, 0, 3)$ is a unique solution (x, y, z) for the Diophantine equation $8^x + 19^y = z^2$ where x, y and z are non-negative integers.*

Proof. Let x, y and z be non-negative integers such that $8^x + 19^y = z^2$. By Lemma 2.3, we have $x \geq 1$. Thus, z is odd. Then there is a non-negative integer t such that $z = 2t + 1$. Thus, $8^x + 19^y = 4(t^2 + t) + 1$. This implies that $19^y \equiv 1 \pmod 4$. Then y is even. Now, we will divide the number y into two cases.

Case $y = 0$. By Lemma 2.2, we have $x = 1$ and $z = 3$.

Case $y \geq 2$. Let $y = 2k$ where k is a positive integer. Then $z^2 - 19^{2k} = 2^{3x}$. Then $(z - 19^k)(z + 19^k) = 2^{3x}$. Thus, $z - 19^k = 2^u$ where u is a non-negative integer. Then $z + 19^k = 2^{3x-u}$. Thus, $2(19^k) = 2^{3x-u} - 2^u = 2^u(2^{3x-2u} - 1)$. We have two subcases.

Subcase $u = 0$. Then $z - 19^k = 1$. Thus, z is even. This is a contradiction.

Subcase $u = 1$. Then $2^{3x-2} - 1 = 19^k$. Then $2^{3x-2} - 19^k = 1$. If $x = 1$, then $k = 0$ so $y = 0$. Thus, $x \geq 2$. By Proposition 2.1, we have $k = 1$. Then $2^{3x-2} = 20$. This is impossible.

Therefore, $(1, 0, 3)$ is a unique solution (x, y, z) for the equation $8^x + 19^y = z^2$ where x, y and z are non-negative integers. □

Corollary 3.2. *The Diophantine equation $8^x + 19^y = w^4$ has no non-negative integer solution.*

Proof. Suppose that there are non-negative integers x, y and w such that $8^x + 19^y = w^4$. Let $z = w^2$. Then $8^x + 19^y = z^2$. By Theorem 3.1, we have $(x, y, z) = (1, 0, 3)$. Then $w^2 = z = 3$. This is a contradiction. Hence, the equation $8^x + 19^y = w^4$ has no non-negative integer solution. □

4. Open Problem

In [4], $(2, 1, 9)$ is a unique solution (x, y, z) for the Diophantine equation $8^x + 17^y = z^2$ where x, y and z are non-negative integers. But $8^1 + 17^0 = 9 = 3^2$. Thus, we may pose a question that "What's the set of all solutions (x, y, z) for the Diophantine equation $8^x + 17^y = z^2$ where x, y and z are non-negative integers?".

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