International Journal of Pure and Applied Mathematics
Volume 81 No. 4 2012, 613-617
ISSN: 1311-8080 (printed version)
url: http://www.ijpam.eu

ijpam.eu

EMBEDDINGS OF GENERAL CURVES IN PROJECTIVE SPACES IN THE RANGE OF THE CUBICS

E. Ballico

Department of Mathematics
University of Trento
38 123 Povo (Trento) - Via Sommarive, 14, ITALY
e-mail: ballico@science.unitn.it

Abstract: Here we prove the existence of linearly normal smooth curves $C \subset \mathbb{P}^r$, $r \geq 5$, with maximal rank, $h^1(X, \mathcal{O}_C(1)) \leq r + \lfloor r/2 \rfloor - 2$, any genus at most of order $r^3/12$ and general moduli. In this range of degrees and genera we prove the surjectivity of the restriction map $H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(3)) \to H^0(C, \mathcal{O}_C(3))$.

AMS Subject Classification: 14H50, 14H51
Key Words: postulation, curve with general moduli, maximal rank conjecture, cubic hypersurfaces

1. Introduction

This paper is continuation of [3]. Notice that the thesis in Theorem 1 below is far stronger than the one in [3], Theorem 1. For all integers g, r, d let $\rho(g, r, d) := (r+1)d - rg - r(r+1)$ denote the Brill-Noether number of the triple (g, r, d). Let $W(d, g, r)$ be the irreducible component of the Hilbert scheme $\mathrm{Hilb}(\mathbb{P}^r)$ of $\mathbb{P}^r$ introduced in [5]. We just recall that if $d \geq r$ and $\rho(g, r, d) \geq 0$, then it is the only irreducible component of $\mathrm{Hilb}(\mathbb{P}^r)$ containing curves with general moduli.

Let $C \subset \mathbb{P}^r$ be any projective curve. The curve C is said to have *maximal rank* if for every integer $x > 0$ the restriction map $H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(x)) \to H^0(C, \mathcal{O}_C(x))$ has maximal rank, i.e. either it is injective or it is surjective.

Received: August 20, 2012

For any smooth curve X and any spanned line bundle L on X let $h_L : X \to \mathbb{P}^n$, $n := h^0(X, L) - 1$, be the morphism induced by the complete linear system $|L|$.

For any integer $r > 0$ set $\epsilon_r := 0$ if r is even and $\epsilon_r := \lfloor r/2 \rfloor$ if r odd. In this note we prove the following result.

Theorem 1. *Fix integers r, e, g such that $r \geq 5$ and $0 \leq e \leq r + \lfloor r/2 \rfloor - 2$. Set $d := g + r - e$. Assume $g \geq e(r+1)$ and $3d + 1 - g \leq \binom{r+3}{3} - \epsilon_r - 2$. Let X be a general smooth curve of genus g. Then $W^r_d(X) \neq \emptyset$, a general $L \in W^r_d(X)$ is very ample, $h^0(X, L) = r + 1$ and the curve $h_L(X) \subset \mathbb{P}^r$ has maximal rank with $h^1(\mathcal{I}_{h_L(X)}(t)) = 0$ for all $t \geq 3$.*

After the proof we outline an alternative way to prove results similar to Theorem 1. Sometimes it is more efficient and we think that to get much better results both methods must be combined.

2. The Proof

Lemma 1. *Fix integers $r \geq 3$ and $t \geq 0$, a hyperplane $H \subset \mathbb{P}^r$ and a general $S \subset H$ such that $\sharp(S) = r + t$. Fix integers $q \geq t(r+1)$ and $x \geq (q - t(r+1)) + r(t+1)$. Then there is a curve $C \in W(x, q, r)$ such that C intersects transversally H and $S \subset H$.*

Proof. Fix a rational normal curve $T \subset \mathbb{P}^r$ intersecting transversally H and containing r points of S. Set $\{P_1, \dots, P_t\} := S \setminus T \cap H$. For any $A \subset \mathbb{P}^r$ such that $\sharp(A) = r + 3$ and A is in linearly general position there is a unique rational normal curve $T_A \subset \mathbb{P}^r$ containing A. Fix general $B_i \subset T$, $1 \leq i \leq t$, such that $\sharp(B_i) = r + 2$. Set $A_i := B_i \cup \{P_i\}$. For general $B_1, \dots, B_t$ we have $\sharp(A_i) = r+3$ for all i, each A_i is in linearly general position, each T_{A_i} intersects quasi-transversally T, $T_{A_i} \cap T = B_i$ and $T_{A_i} \cap T_{A_j} = \emptyset$ for all $i \neq j$. Hence $Y := T \cup_{i=1}^{t} T_i \in H((r+1)t, t(r+1), r)$ ([7], [5], Lemma 2.3). Let C be the general union of Y, $q - (r+1)t$ secant lines of T and $x - (q - t(r+1)) + r(t+1)$ lines intersecting T ([7], [5], Lemma 2.2). Then $C \in W(x, q, r)$, C intersects transversally H and $S \subset H \cap C$. □

Let $W'(d, g, r)$ be the open subset of $W(d, g, r)$ formed by the reduced curves and $\mathcal{C} \to W'(d, g, r)$ the universal curve. Let $\mathcal{C}'$ be the open subset of $\mathcal{C}$ formed by the smooth points of the fibers of $\mathcal{C} \to W'(d, g, r)$.

Remark 1. Fix a hyperplane $H \subset \mathbb{P}^r$. For all integers x, q such that $W(x, q, r)$ is defined set $W'(x, q, r)_H := \{C \in W'(x, g, r) : C$ is transversal to $H\}$. Let $\mathcal{C}'_H$ denote the restriction of $\mathcal{C}'$ to $W'(x, q, r)_H$. For each $C \in W'(x, q, r)$

and any integer y such that $0 \le y \le x$, let $S^y(C,H)$ be the set of all subsets of $C \cap H$ with cardinality y. We see $S^y(C,H)$ as a finite subscheme of the Hilbert scheme of degree y subschemes of C supported by C_{reg}. In this way we get a closed subscheme $\mathcal{C}_H(y)$ of $\mathcal{C}'_H$ contained in the Hilbert scheme $\mathrm{Hilb}^y(H)_0$ of all subsets of H with cardinality y. Fix an integer $t \ge 0$ and assume $q \ge t(r+1)$ and $x \ge (q - t(r+1)) + r(t+1)$. Fix a general $S \subset H$ such that $\sharp(S) = r+t$. Lemma 1 gives that the natural map $\mathcal{C}_H(y) \to \mathrm{Hilb}^y(H)_0$ is dominant.

Lemma 2. *Fix integers r, x such that $r \ge 5$, $y \ge 0$, $x \ge y + r - 2$ and $r-2 \le x \le \lfloor (\binom{r+1}{3} - r + 1)/2 \rfloor$. Let Y be the general element of $W(x, x-r+2, r)$. Then $h^1(\mathbb{P}^{r-2}, \mathcal{I}_Y(3)) = 0$.*

Proof. Use that the Maximal Rank Conjecture is true for non-special curves in $\mathbb{P}^{r-2}$ and that our assumptions give the inequality $3x+1-(x-r+2) \le \binom{r+1}{3}$. □

Proof of Theorem 1. By [2] and semicontinuity it is sufficient to prove $h^1(\mathcal{I}_{h_L(X)}(t)) = 0$ for all $t \ge 3$ for a general $X \in \mathcal{M}_g$ and a general $L \in W^r_d(X)$. The case $e \le \lfloor r/2 \rfloor$ is easy and essentially contained in [3]. Hence from now on we assume $e > \lfloor r/2 \rfloor$. Set $q := \lfloor r/2 \rfloor \cdot r$ and $d' := q + r - \lfloor r/2 \rfloor$. Since $e(r+1) \le g$, we have $g - q \ge \lfloor r/2 \rfloor$. Since $\rho(q, r, d') \ge 0$ and $r \ge 3$, the general smooth curve D of genus q has $W^r_{d'}(C) \ne \emptyset$ ([1], Ch. V) and a general $M \in W^r_{d'}(D)$ is very ample (e.g., see the proof of [7], Theorem at pages 26-27), with $h^0(D,M) = r+1$ and $h^1(D,M) = \lfloor r/2 \rfloor$. Set $C := h_M(D) \subset \mathbb{P}^r$. By [6], Theorem 1, C is projectively normal and in particular $h^1(\mathcal{I}_C(2)) = 0$. Let $H \subset \mathbb{P}^r$ be a general hyperplane. Since H is general, $C \cap H$ is a set of d' distinct points. Look at the exact sequence

$$0 \to \mathcal{I}_C(1) \to \mathcal{I}_C(2) \to \mathcal{I}_{C\cap H,H}(2) \to 0 \qquad (1)$$

Since C is linearly normal, we have $h^2(\mathcal{I}_C(1)) = h^1(C, \mathcal{O}_C(1)) = \lfloor r/2 \rfloor$. Since C has general moduli, Gieseker-Petri package gives $h^2(\mathcal{I}_C(2)) = h^0(C, \mathcal{O}_C(2)) = 0$ ([1], page 131). Since $h^1(\mathcal{I}_C(2)) = 0$, (1) gives $h^1(H, \mathcal{I}_{C\cap H,H}(2)) = \lfloor r/2 \rfloor$. Fix a set $S \subset C \cap H$ such that $\sharp(S) = e + r - \lfloor r/2 \rfloor$. Since $e \le \rfloor r/2 \rfloor + r$, we have $\sharp(S) \le r + \lfloor r/2 \rfloor$. By Remark 1 we may see S as a general union of $\sharp(S)$ points of H. Set $g_1 := d - d' - (r-1)$. Hence Y has genus g'. Since $g \ge e(r+1)$ and $g_1 = g - q - (e - \lfloor r/2 \rfloor) \ge (e - \lfloor r/2 \rfloor)r$, we have $g' \ge (\sharp(S) - r - 2) \cdot \lfloor (r-1)/2 \rfloor$. Hence [4], Lemma 1.5, applied to the integer $n := r-1$, gives that $\sharp(S)$ general points of H are contained in a general subcurve of H with genus g_1 and degree $d - d'$. Hence we may fix $Y \subset H$ with $S \subset H$. Since $C \cup Y \in W(d, g, r)$, it is sufficient to prove $h^1(\mathcal{I}_{C\cup Y}(3)) = 0$. We have an exact sequence

$$0 \to \mathcal{I}_C(2) \to \mathcal{I}_{C\cup Y}(3) \to \mathcal{I}_{A\cup Y,H}(3) \to 0 \qquad (2)$$

Hence it is sufficient to prove $h^1(H, \mathcal{I}_{A\cup Y}(3)) = 0$. By semicontinuity it is sufficient to flatly degenerate $Y \cup A$ inside H to a curve $Y' \cup A$ with $Y' \cap A = \emptyset$ and $h^1(H, \mathcal{I}_{A\cup Y'}(3)) = 0$. Fix a hyperplane M of H such that $C \cap M = \emptyset$. A non-special smooth curve $E \subset M$ with $\deg(E) = p_a(E) + r - 1$ is a flat degeneration of a family of non-special curves in H. Hence we may take as Y' a general non-special curve of M. We have the following exact sequence of coherent sheaves on H:

$$0 \to \mathcal{I}_{A,H}(2) \to \mathcal{I}_{A\cup Y',H}(3) \to \mathcal{I}_{Y',M}(3) \to 0 \qquad (3)$$

Notice that $d' + \epsilon_r = \binom{r+1}{2}$. Hence $3(d - d') + 1 - (d - d' - r + 1) \leq \binom{r+2}{3}$. Hence Lemma 2 gives $h^1(M, \mathcal{I}_{Y',M}(3)) = 0$. Hence it is sufficient to prove $h^1(H, \mathcal{I}_{A,H}(2))$. Since $h^1(H, \mathcal{I}_{C\cap H,H}(2)) = \lfloor r/2 \rfloor$, for each integer $z \leq d' - \lfloor r/2 \rfloor$ there is $B \subset C \cap H$ such that $\sharp(B) = z$ and $h^1(H, \mathcal{I}_{B,H}(2)) = 0$. Since C is integral and H is general, the hyperplane section $H \cap C$ is in uniform position ([1], page 109). Hence $h^1(H, \mathcal{I}_{B,H}(2)) = 0$ for every $B \subset C \cap H$ such that $\sharp(C \cap H) - \sharp(B) \geq \lfloor r/2 \rfloor$. Hence $h^1(H, \mathcal{I}_{A,H}(2)) = 0$. □

Alternative Tool. As in [3] we take a hyperplane M of H spanned by r points of $C \cap H$. Instead of Y we add (as in [3]) a curve $Y \in W(d - d', g - q - r - 1, r - 2)$ in M containing $M \cap C$. In the set-up of Theorem 1 we have $d - d' < g - q - r - 1 + (r - 2)$ and hence we need to use some kind of induction from $\mathbb{P}^{r-2}$ to $\mathbb{P}^r$. In this case one need to prove that $C \cup Y \in W(d, g, r)$ and this is not easy if $d - d' \ll g - q - r - 1 + (r - 2)$ Applying only [3] to $\mathbb{P}^{r-2}$ we get a results similar to Theorem 1. To go further one need to go further in the inductive step applying the stronger statement in $\mathbb{P}^{r-2}$, and so on, starting with $\mathbb{P}^3$ for r odd and with $\mathbb{P}^4$ for r even.

Acknowledgements

The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

References

[1] E. Arbarello, M. Cornalba, P. Griffiths and J. Harris, *Geometry of Algebraic Curves, I*, Springer, Berlin (1985).

[2] E. Ballico, Embeddings of general curves in projective spaces: the range of the quadrics, *Lit. J. Math.*, **52**, No. 2 (2012), 134-137.

[3] E. Ballico, Embeddings of general curves in projective spaces: Parts of the range of the cubics, *Int. J. Pure and Appl. Math.*, **74**, No. 3 (2012), 353-355.

[4] E. Ballico, Ph. Ellia, The maximal rank conjecture for non-special curves in $\mathbb{P}^n$, *Math. Z.*, **196** (1987), 355-367.

[5] E. Ballico, Ph. Ellia, On the existence of curves with maximal rank in $\mathbb{P}^n$, *J. Reine Angew. Math.*, **397** (1989), 1-22.

[6] E. Ballico, C. Fontanari, Normally generated line bundles on general curves, II, *J. Pure Appl. Algebra*, **214**, No. 8 (2010), 1450-1455.

[7] E. Sernesi, On the existence of certain families of curves, *Invent. Math.*, **75**, No. 1 (1984), 25-57.