International Journal of Pure and Applied Mathematics
Volume 81 No. 5 2012, 773-784
ISSN: 1311-8080 (printed version)
url: http://www.ijpam.eu

ijpam.eu

# ON THE GENERATORS OF THE 2-CLASS GROUP OF THE FIELD $\mathbb{Q}(\sqrt{d}, i)$

Abdelmalek Azizi[1], Abdelkader Zekhnini[2 §], Mohammed Taous[3]

Department of Mathematics
Faculty of Science
Mohammed First University
Oujda, MOROCCO

[3]Department of Mathematics
Faculty of Science and Technology
Moulay Ismail University
Errachidia, MOROCCO

**Abstract:** Let $d$ be a square free integer such that the 2-class group of the field $\mathbb{Q}(\sqrt{d}, i)$ is of type (2, 2, 2). In this paper we give the generators of the 2-class group of $\mathbb{Q}(\sqrt{d}, i)$.

**AMS Subject Classification:** 11R37, 11R27, 11R29
**Key Words:** class group of type (2,2,2), Hilbert class field

## 1. Introduction

Let $\mathbb{k} = \mathbb{Q}(\sqrt{d}, i)$, where $d$ is a square free integer, we denote by $\mathbf{C}_{\mathbb{k},2}$ the 2-class group of $\mathbb{k}$; let $p$, $p_1$ and $p_2$ (resp. $q$, $q_1$ and $q_2$) be prime integers congruent to 1 (resp. 3) $\pmod 4$. According to [3], $\mathbf{C}_{\mathbb{k},2}$ is of type (2, 2, 2) if and only if $d$ is one of the following forms:

Received: August 1, 2012



§Correspondence author

(1) $d = p_1p_2$, where $\left(\frac{p_1}{p_2}\right) = -1$, $p_1 \equiv p_2 \equiv 1 \pmod 8$ and $\left(\frac{2}{a+b}\right) = -1$ with $p_1p_2 = a^2 + b^2$.

(2) $d = 2p_1p_2$, where $p_1 \equiv p_2 \equiv 1 \pmod 4$ and at least two elements of $\{\left(\frac{2}{p_1}\right), \left(\frac{2}{p_2}\right), \left(\frac{p_1}{p_2}\right)\}$ are equal to -1.

(3) $d = 2pq$, where $p \equiv 1$, $q \equiv 3 \pmod 8$ and $\left(\frac{p}{q}\right) = -1$.

(4) $d = pq_1q_2$, where $p$, $q_1$, $q_2$ satisfy the conditions $A$ and $B$:

$A : p \equiv -q_1 \equiv -q_2 \equiv 1 \pmod 4$ and $\left(\frac{2}{p}\right) = \left(\frac{q_1}{q_2}\right) = -\left(\frac{q_2}{q_1}\right) = 1$.

$B$ : One of the following three conditions is satisfied:

(I) $\left(\frac{p}{q_1}\right)\left(\frac{p}{q_2}\right) = -1$ and $\left(\frac{2}{q_1}\right) = \left(\frac{2}{q_2}\right) = -1$.

(II) $\left(\frac{p}{q_1}\right)\left(\frac{p}{q_2}\right) = -1$, $\left(\frac{2}{q_1}\right) = 1$ and $\left(\frac{2}{q_2}\right) = -1$.

(III) $\left(\frac{p}{q_1}\right) = \left(\frac{p}{q_2}\right) = -1$ and $\left(\frac{2}{q_1}\right)\left(\frac{2}{q_2}\right) = -1$.

(5) $d = p_1p_2q$, where $p_1 \equiv p_2 \equiv 1 \pmod 4$, $p_1$ or $p_2 \equiv 5 \pmod 8$ and at least two elements of $\{\left(\frac{p_1}{p_2}\right), \left(\frac{p_1}{q}\right), \left(\frac{p_2}{q}\right)\}$ are equal to -1.

In the case where $d = p_1p_2q$, we adopt the following definitions:

(i) $p_1$, $p_2$ and $q$ are called of type I if one of the following conditions holds:
(a) $\left(\frac{2}{p_1}\right) = 1$ and $\left(\frac{2}{p_2}\right) = \left(\frac{p_1}{p_2}\right) = \left(\frac{p_1}{q}\right) = -1$.
(b) $\left(\frac{2}{p_1}\right) = \left(\frac{2}{p_2}\right) = \left(\frac{p_1}{p_2}\right) = \left(\frac{p_2}{q}\right) = -1$ and $\left(\frac{p_1}{q}\right) = 1$.
(c) $\left(\frac{2}{p_2}\right) = \left(\frac{p_1}{p_2}\right) = 1$ and $\left(\frac{2}{p_1}\right) = \left(\frac{p_1}{q}\right) = \left(\frac{p_2}{q}\right) = -1$.

(ii) $p_1$, $p_2$ and $q$ are called of type II if one of the following conditions is satisfied:
(a) $\left(\frac{2}{p_2}\right) = 1$ and $\left(\frac{2}{p_1}\right) = \left(\frac{p_1}{p_2}\right) = \left(\frac{p_2}{q}\right) = -1$.
(b) $\left(\frac{2}{p_1}\right) = \left(\frac{2}{p_2}\right) = \left(\frac{p_1}{p_2}\right) = \left(\frac{p_1}{q}\right) = -1$ and $\left(\frac{p_2}{q}\right) = 1$.
(c) $\left(\frac{2}{p_1}\right) = \left(\frac{p_1}{p_2}\right) = 1$ and $\left(\frac{2}{p_2}\right) = \left(\frac{p_1}{q}\right) = \left(\frac{p_2}{q}\right) = -1$.

(iii) $p_1$, $p_2$ and $q$ are called of type III if one of the following conditions is satisfied:
(a) $\left(\frac{2}{p_1}\right) = \left(\frac{2}{p_2}\right) = \left(\frac{p_1}{q}\right) = \left(\frac{p_2}{q}\right) = -1$.

(b) $\left(\frac{2}{p_1}\right)=\left(\frac{p_1}{q}\right)=1$ and $\left(\frac{2}{p_2}\right)=\left(\frac{p_1}{p_2}\right)=\left(\frac{p_2}{q}\right)=-1$.
(c) $\left(\frac{2}{p_2}\right)=\left(\frac{p_2}{q}\right)=1$ and $\left(\frac{2}{p_1}\right)=\left(\frac{p_1}{p_2}\right)=\left(\frac{p_1}{q}\right)=-1$.

In this paper we are interested to give the generators of $\mathbf{C}_{\mathbb{k},2}$ and our main result it is a deduction from theorems 1, 2, 3 and 4 :

**Theorem** (Main Result). *Let $\mathbb{k}=\mathbb{Q}(\sqrt{d},i)$, where $d$ is a square free integer, $\mathbf{C}_{\mathbb{k},2}$ the 2-class group of $\mathbb{k}$. Suppose $\mathbf{C}_{\mathbb{k},2}$ is of type $(2,2,2)$, then we have:*

(1) *If $d$ is of the form (1), then $\mathbf{C}_{\mathbb{k},2}=\langle[\mathcal{H}_0^{\frac{h(d)}{2}}],[\mathcal{H}_1],[\mathcal{H}_2]\rangle$, where $\mathcal{H}_0$, $\mathcal{H}_1$ and $\mathcal{H}_2$ are prime ideals in $\mathbb{k}$ above 2 and $p_1$ respectively and $h(d)$ the class number of $\mathbb{Q}(\sqrt{d})$.*

(2) *If $d$ is of the form (2) or (3), then $\mathbf{C}_{\mathbb{k},2}=\langle[\mathcal{H}_0],[\mathcal{H}_1],[\mathcal{H}_2]\rangle$, where $\mathcal{H}_0$ is the prime ideal in $\mathbb{k}$ above 2 and $\mathcal{H}_1$, $\mathcal{H}_2$ are prime ideals in $\mathbb{k}$ above $p_1$ (resp. $p$) if $d$ takes the form (2) (resp. (3)).*

(3) *Assume $d$ is of the form (4), let $\mathcal{H}_1$, $\mathcal{H}_2$ (resp. $\mathcal{Q}_1$, $\mathcal{Q}_2$) be prime ideals in $\mathbb{k}$ above $p$ (resp. $q_1$, $q_2$), then:*

   (i) *If $p$, $q_1$ and $q_2$ satisfy B (I) or B (II) and $\left(\frac{p}{q_1}\right)=-\left(\frac{p}{q_2}\right)=1$, then $\mathbf{C}_{\mathbb{k},2}=\langle[\mathcal{H}_1],[\mathcal{H}_2],[\mathcal{Q}_2]\rangle$.*
   (ii) *Else, $\mathbf{C}_{\mathbb{k},2}=\langle[\mathcal{H}_1],[\mathcal{H}_2],[\mathcal{Q}_1]\rangle$.*

(4) *Suppose $d$ is of the form (5), let $\mathcal{H}_1$, $\mathcal{H}_2$ (resp. $\mathcal{H}_3$, $\mathcal{H}_4$) be prime ideals in $\mathbb{k}$ above $p_1$ (resp. $p_2$), then:*

   (i) *If $p_1$, $p_2$ and $q$ are of type I, then $\mathbf{C}_{\mathbb{k},2}=\langle[\mathcal{H}_1],[\mathcal{H}_3],[\mathcal{H}_4]\rangle$.*
   (ii) *If $p_1$, $p_2$ and $q$ are of type II or of type III, then $\mathbf{C}_{\mathbb{k},2}=\langle[\mathcal{H}_1],[\mathcal{H}_2],[\mathcal{H}_3]\rangle$.*

## 2. Generators of $\mathbf{C}_{\mathbb{k},2}$

First we give some results that will be useful later.

**Proposition 1.** *Let $d$ be a square free integer, $k=\mathbb{Q}(\sqrt{d},i)$, $a+ib$ an element of $\mathbb{Z}(i)$ and $\mathcal{H}$ an ideal of $k$ such that $\mathcal{H}^2=(a+ib)$. Let $\varepsilon_d=x+y\sqrt{d}$ be the fundamental unit of $\mathbb{Q}(\sqrt{d})$. So:*

(1) *If $\sqrt{a^2+b^2}\notin\mathbb{Q}(\sqrt{d})$, then $\mathcal{H}$ is not principal in $k$.*

(2) *If $a^2+b^2=d$, then we have:*

(a) *If the norm of* $\varepsilon_d$ *is 1, then* $\mathcal{H}$ *is not principal in* $k$.

(b) *If the norm of* $\varepsilon_d$ *is -1, then:*

*(i) If* $(ax \pm yd) \pm b$ *or* $2(-xb \pm yd) \pm a$ *is a square in* $\mathbb{N}$, *then* $\mathcal{H}$ *is principal in* $k$.

*(ii) Else* $\mathcal{H}$ *is not principal in* $k$.

*Proof.* Let $a+ib$ be an element of $\mathbb{Z}[i]$ and $\mathcal{H}$ an ideal of $k$ such that $\mathcal{H}^2 = (a+ib)$. We suppose that $\mathcal{H}$ is principal, then there exist $\alpha \in \Bbbk$ and a unit $\varepsilon$ in $\Bbbk$ such that: $\alpha^2 = (a+ib)\varepsilon$ (1). Let $\varepsilon_d$ be the fundamental unit of $\mathbb{Q}(\sqrt{d})$, then a fundamental system of units (UFS) of $k$ is $\{\varepsilon_d\}$ or $\{\sqrt{i\varepsilon_d}\}$, in the later case $\varepsilon_d$ is of norm 1. It comes down to cases $\varepsilon \in \{\pm 1, \pm i, \varepsilon_d, i\varepsilon_d\}$ or $\varepsilon \in \{\pm 1, \pm i, \varepsilon_d, i\varepsilon_d, i\sqrt{i\varepsilon_d}, \sqrt{i\varepsilon_d}\}$.

(1) Suppose that $\sqrt{a^2+b^2} \notin \mathbb{Q}(\sqrt{d})$, then we have:

(i) If $\varepsilon \in \{\pm 1, \pm i, \varepsilon_d, i\varepsilon_d\}$, then by applying the norm $N_{\Bbbk/\mathbb{Q}(\sqrt{d})}$ to Equation (1), we find that $\sqrt{a^2+b^2} \in \mathbb{Q}(\sqrt{d})$, which is not the case.

(ii) If $\varepsilon = \sqrt{i\varepsilon_d}$ or $\varepsilon = i\sqrt{i\varepsilon_d}$, then the norm $N_{\Bbbk/\mathbb{Q}(i)}$ applied to Equation (1), imply that $\sqrt{i} \in \mathbb{Q}(i)$, which is absurd. So $\mathcal{H}$ is not principal in $k$.

(2) Suppose that $a^2+b^2 = d$, then we have:

(i) If $\varepsilon = 1$, so by putting $\alpha = \alpha_1 + i\alpha_2$, where $\alpha_1$, $\alpha_2$ are in $\mathbb{Q}(\sqrt{d})$, Equation (1) imply that

$$\begin{cases} \alpha_1^2 - \alpha_2^2 = a, \\ 2\alpha_1\alpha_2 = b; \end{cases} \Leftrightarrow \begin{cases} \alpha_1^4 - 4a\alpha_1^2 - b^2 = 0, \ (\star) \\ \alpha_2 = \frac{b}{2\alpha_1}; \end{cases}$$

and $\Delta' = 4d$, where $\Delta'$ is the discriminant of equation $(\star)$ for the unknown $\alpha_1^2$, thus $\alpha_1^2 = \frac{1}{2}(a \pm \sqrt{d})$, therefore equation $(\star)$ admits solution if and only if $2(a \pm \sqrt{d})$ is a square in $\mathbb{Q}(\sqrt{d})$, hence there exist s, t in $\mathbb{Q}$ such that $2(a \pm \sqrt{d}) = (s+t\sqrt{d})^2 = s^2 + t^2 d + 2st\sqrt{d}$, which is equivalent to: $\begin{cases} s^4 - 2as^2 + d = 0, \\ t = \frac{\pm 1}{t}; \end{cases}$ and $\Delta' = a^2 - d = -b^2$, as $\Delta' < 0$, then $2(a \pm \sqrt{d})$ is not a square in $\mathbb{Q}(\sqrt{d})$. Similar proof if $\varepsilon = -1$.

(ii) Let $\varepsilon = \pm i$. So equation (1) is solvable if and only if $2(\pm b \pm \sqrt{d})$ is a square in $\mathbb{Q}(\sqrt{d})$, but this gives us a negative discriminant $\Delta = -a^2$.

(iii) Let $\varepsilon = \varepsilon_d$, then from Equation (1), there exist $\alpha_1$, $\alpha_2$ in $\mathbb{Q}(\sqrt{d})$ such that:

$\left\{\begin{array}{l}\alpha_1^2-\alpha_2^2=a\varepsilon_d,\\ 2\alpha_1\alpha_2=b\varepsilon_d;\end{array}\right. \Leftrightarrow \left\{\begin{array}{l}4\alpha_1^4-4a\varepsilon_d\alpha_1^2-b\varepsilon_d^2=0, \quad (*)\\ \alpha_2=\frac{b\varepsilon_d}{2\alpha_1};\end{array}\right.$ and $\Delta'=4\varepsilon_d^2 d$, so $\alpha_1^2=\frac{\varepsilon_d}{2}(a\pm\sqrt{d})$, thus equation $(*)$ admits solution if and only if $2\varepsilon_d(a\pm\sqrt{d})$ is a square in $\mathbb{Q}(\sqrt{d})$ i.e. if and only if there exist s, t in $\mathbb{Q}$ such that: $2\varepsilon_d(a\pm\sqrt{d})=(s+t\sqrt{d})^2=s^2+t^2d+2st\sqrt{d}$, as $\varepsilon_d=x+y\sqrt{d}$, so: $\left\{\begin{array}{l}s^2+t^2d=2xa\pm 2yd,\\ st=ya\pm x;\end{array}\right.$

thus $\left\{\begin{array}{l}s^4-2(ax\pm yd)s^2+(ya\pm x)^2d=0, \quad (\bullet)\\ t=\frac{ya\pm x}{s};\end{array}\right.$ the discriminant of equation $(\bullet)$ is:

$\Delta'=(ax\pm yd)^2-(ya\pm x)^2d=(a^2-d)(x^2-y^2d)=-b^2(x^2-y^2d)$.

— If the norm of $\varepsilon_d$ is 1, then equation $(\bullet)$ has no solution.

— If the norm of $\varepsilon_d$ is -1, then $s^2=(ax\pm yd)\pm b$. Hence equation $(\bullet)$ admits solution if and only if $(ax\pm yd)\pm b$ is a square in $\mathbb{N}$.

(iv) Let $\varepsilon=\varepsilon_d$, then by the same way we find similar results:

— If the norm of $\varepsilon_d$ is 1, then there is no solutions.

— If the norm of $\varepsilon_d$ is -1, then there is a solution if and only if $2(-xb\pm yd)\pm a$ is a square in $\mathbb{N}$.

(v) If $\varepsilon=\sqrt{i\varepsilon_d}$ or $\varepsilon=i\sqrt{i\varepsilon_d}$, as in the case (1) we find that $i$ is a square in $\mathbb{Q}(i)$, which is absurd. □

We proceed in the same way to prove the following result:

**Proposition 2.** *Let $d$ be a composite integer, even, square free and product at least of three prime numbers, $k=\mathbb{Q}(\sqrt{d},i)$, $p$ a prime number and $\mathcal{H}$ an ideal of $k$ such that $\mathcal{H}^2=(p)$. Let $\varepsilon_d=x+y\sqrt{d}$ be the fundamental unit of $\mathbb{Q}(\sqrt{d})$. Then we have:*

(1) *If the norm of $\varepsilon_d$ is $-1$, so $\mathcal{H}$ is not principal in $k$.*

(2) *If the norm of $\varepsilon_d$ is 1, we have:*

   (i) *If $\{\varepsilon_d\}$ is UFS of $k$, then $\mathcal{H}$ is principal if and only if $2p(x\pm 1)$ or $p(x\pm 1)$ is a square in $\mathbb{N}$.*

   (ii) *If not $\mathcal{H}$ is not principal in $k$.*

**Remark 1.** Proposition 2 holds if $d$ is a composite integer, odd, square free and product at least of three prime numbers and $\mathcal{H}^2=(p)$ or $\mathcal{H}^2=(pq)$, where $p$ and $q$ are prime numbers.

### 2.1. Generators of $\mathbf{C}_{\Bbbk,2}$ when $d$ is Even

If $d$ is even, then it is of the form (2) or (3); so $p_1$ (resp. $p$) splits in $\mathbb{Q}(i)$ in product of two primes which we denote by $\pi_1$ and $\pi_2$.

**Theorem 1.** *Let $\Bbbk = \mathbb{Q}(\sqrt{d}, i)$, where $d$ is of the form* (2) *or* (3) *and $\mathbf{C}_{\Bbbk,2}$ be the 2-class group of $\Bbbk$. We denote by $\mathcal{H}_0$, $\mathcal{H}_1$ and $\mathcal{H}_2$ the prime ideals of $\Bbbk$ laying above $1+i$, $\pi_1$ and $\pi_2$ respectively, then $\mathbf{C}_{\Bbbk,2} = \langle[\mathcal{H}_0], [\mathcal{H}_1], [\mathcal{H}_2]\rangle$.*

*Proof.* For both forms (2) and (3), the numbers $\pi_1$ and $\pi_2$ are ramified primes in $\Bbbk/\mathbb{Q}(i)$, then there exist $\mathcal{H}_1$ and $\mathcal{H}_2$ prime ideals in $\Bbbk$ such that: $\pi_j\mathcal{O}_\Bbbk = (\pi_j) = \mathcal{H}_j^2$, ($j \in \{1, 2\}$), where $\mathcal{O}_\Bbbk$ is the ring of integers of $\Bbbk$; on the other hand, 2 is totally ramified in $\Bbbk$, hence there exists $\mathcal{H}_0$ an ideal prime of $\Bbbk$ such that $\mathcal{H}_0^2 = (1+i)\mathcal{O}_\Bbbk$.

According to [3], if $d$ is of the form (3) (resp. (2)), then the norm of the fundamental unit of $\mathbb{Q}(\sqrt{d})$ is 1 (resp. -1) and the unit index of $\Bbbk$ is 2 (resp. 1), so as $(\mathcal{H}_1\mathcal{H}_2)^2 = (p_1)$ or $(p)$, Proposition 2 claims that $\mathcal{H}_1\mathcal{H}_2$ is not principal in $\Bbbk$; moreover if we put $p$ or $p_1 = e^2+4f^2$, we find that $\mathcal{H}_0^2 = (1+i)$, $\mathcal{H}_1^2 = (e+2if)$ and $\mathcal{H}_2^2 = (e-2if)$, as $\sqrt{2} \notin \mathbb{Q}(\sqrt{d})$ and $\sqrt{e^2 + (\pm 2f)^2} = \sqrt{p_1} \notin \mathbb{Q}(\sqrt{d})$, then Proposition 1 states that $\mathcal{H}_0$, $\mathcal{H}_1$ and $\mathcal{H}_2$ are of order 2 in $\Bbbk$; similar with the same argument we proof that $\mathcal{H}_0\mathcal{H}_1$, $\mathcal{H}_0\mathcal{H}_2$ and $\mathcal{H}_0\mathcal{H}_1\mathcal{H}_2$ are of order 2 in $\Bbbk$. This completes the proof. $\square$

**Numerical Examples 1.** $d$ is of the form (2).

| $d = 2.p_1.p_2$ | $\left(\frac{2}{p_1}\right)$ | $\left(\frac{2}{p_2}\right)$ | $\left(\frac{p_1}{p_2}\right)$ | $\mathbf{H}_0$ | $\mathbf{H}_1$ | $\mathbf{H}_2$ |
|---|---|---|---|---|---|---|
| $130 = 2.13.5$ | $-1$ | $-1$ | $-1$ | $[1,0,1]$ | $[0,0,1]$ | $[0,1,1]$ |
| $754 = 2.29.13$ | $-1$ | $-1$ | $1$ | $[0,1,1]$ | $[5,0,1]$ | $[5,0,0]$ |
| $986 = 2.17.29$ | $1$ | $-1$ | $-1$ | $[0,0,1]$ | $[0,1,0]$ | $[11,0,0]$ |
| $1066 = 2.13.41$ | $-1$ | $1$ | $-1$ | $[5,1,0]$ | $[5,0,1]$ | $[5,1,1]$ |

$d$ take the form (3).

| $d = 2.p.q$ | $\left(\frac{p}{q}\right)$ | $\mathbf{H}_0$ | $\mathbf{H}_1$ | $\mathbf{H}_2$ |
|---|---|---|---|---|
| $246 = 2.41.3$ | $-1$ | $[3,1,0]$ | $[3,0,0]$ | $[3,1,1]$ |
| $374 = 2.17.11$ | $-1$ | $[7,0,0]$ | $[0,1,1]$ | $[7,1,0]$ |

### 2.2. Generators of $\mathbf{C}_{\Bbbk,2}$ when $d = p_1p_2$

Suppose $d$ is of the form (1), defined in the introduction. We adopt that $\mathfrak{p}_k$ denotes an ideal of a number field $k$ above a prime number $p$. We need some lemmas.

**Lemma 1.** *Let $\Bbbk = \mathbb{Q}(\sqrt{d}, i)$, where $d = p_1p_2 = a^2 + b^2$, $p_1 \equiv p_2 \equiv 1 \pmod 8$ and $\mathcal{H}_0$ be the prime ideal of $\Bbbk$ above $1 + i$. If $(\frac{2}{a+b}) = -1$, then for all odd integer $n$, $\mathcal{H}_0^n$ is not principal ideal of $\Bbbk$.*

*Proof.* Suppose $\mathcal{H}_0^n$ is principal in $\Bbbk$, for some odd integer $n$, so there exists $\alpha = \alpha_1 + \sqrt{d}\alpha_2 \in \Bbbk$ such that $\alpha_i$ are in $\mathbb{Q}(i)$ and $\mathcal{H}_0^n = \alpha\mathcal{O}_{\Bbbk}$. As $p_1 \equiv p_2 \equiv 1 \pmod 8$, so $1+i$ splits in $\Bbbk/\mathbb{Q}(i)$, hence there exists a prime ideal $\mathcal{H}'_0$ in $\Bbbk$ above $1+i$ such that $\mathcal{H}_0\mathcal{H}'_0 = (1+i)\mathcal{O}_{\Bbbk}$. This allows us to write: $(1+i)^n = \varepsilon(\alpha_1^2 - d\alpha_2^2)$, with $\varepsilon$ is a unit of $\mathbb{Q}(i)$. As $\alpha_i^2$ are squares in $\mathbb{Q}(i)$ and $n$ is odd, then according to [6, p. 154 ] and [7, p. 323 ] we find that $\left(\dfrac{1+i}{\mathfrak{p}_{\mathbb{Q}(i)}}\right) = \left(\dfrac{\varepsilon}{\mathfrak{p}_{\mathbb{Q}(i)}}\right) = \left(\dfrac{2}{p_1}\right)_4 \left(\dfrac{p_1}{2}\right)_4 = \left(\dfrac{2}{p_2}\right)_4 \left(\dfrac{p_2}{2}\right)_4 = 1$. Using this result and according to [7, p. 323 ] we have $(\frac{2}{a+b}) = 1$, which contradicts our hypothesis. $\square$

**Lemma 2.** *Let $p_1$ and $p_2$ be two prime numbers such that $p_1 \equiv p_2 \equiv 1 \pmod 8$, $(\frac{p_1}{p_2}) = -1$ and $h(d)$ be the class number of $\mathbb{Q}(\sqrt{p_1p_2})$, then:*

(i) *The 2-class group of $\mathbb{Q}(\sqrt{p_1p_2})$ is generated by the class of $\mathfrak{p}_{i\mathbb{Q}(\sqrt{p_1p_2})}$;*

(ii) *The ideal $(2_{\mathbb{Q}(\sqrt{p_1p_2})})^{\frac{h(d)}{2}}$ is principal.*

*Proof.* (i) Since $(\frac{p_1}{p_2}) = -1$, then according to [4] the norm of the fundamental unit of the $\mathbb{Q}(\sqrt{p_1p_2})$ is equal to $-1$ and it's 2-class number is 2, therefore the 2-class group of $\mathbb{Q}(\sqrt{p_1p_2})$ is cyclic of order 2. As $p_i$ is ramified in $\mathbb{Q}(\sqrt{p_1p_2})/\mathbb{Q}$, so $p_i\mathcal{O}_{\mathbb{Q}(\sqrt{p_1p_2})} = \mathfrak{p}_{i\mathbb{Q}(\sqrt{p_1p_2})}^2$ and $\mathfrak{p}_{i\mathbb{Q}(\sqrt{p_1p_2})}$ is a no-principal ideal, if not there exists $\varepsilon$, unit of $\mathbb{Q}(\sqrt{p_1p_2})$, such that $p_i\varepsilon$ is square in $\mathbb{Q}(\sqrt{p_1p_2})$, this yields that $p_i\varepsilon_{p_1p_2}$ or $p_i$ is square in $\mathbb{Q}(\sqrt{p_1p_2})$, this contradicts the fact that the norm of the fundamental unity, $\varepsilon_{p_1p_2}$, is $-1$ and $\sqrt{p_i} \notin \mathbb{Q}(\sqrt{p_1p_2})$. Hence the result.

(ii) We know that $(2_{\mathbb{Q}(\sqrt{p_1p_2})})^{h(d)}$ is principal in $\mathbb{Q}(\sqrt{p_1p_2})$. If we suppose that $(2_{\mathbb{Q}(\sqrt{p_1p_2})})^{\frac{h(d)}{2}}$ is not principal, it follows that the class of $(2_{\mathbb{Q}(\sqrt{p_1p_2})})^{\frac{h(d)}{2}}$ is also a generator of the 2-class group of $\mathbb{Q}(\sqrt{p_1p_2})$. We deduce from (i) that $(2_{\mathbb{Q}(\sqrt{p_1p_2})})^{\frac{h(d)}{2}}\mathfrak{p}_{i\mathbb{Q}(\sqrt{p_1p_2})}$ is principal in $\mathbb{Q}(\sqrt{p_1p_2})$. As $p_i$ is ramified in $\mathbb{Q}(\sqrt{p_1p_2})/\mathbb{Q}$ and 2 splits completely in $\mathbb{Q}(\sqrt{p_1p_2})/\mathbb{Q}$, since $p_1 \equiv p_2 \equiv 1 \pmod 8$. This allows us to write by applying the norm that: $2p_i = \alpha^2 - p_1p_2\beta^2$, where $\alpha^2$, $\beta^2$ are in $\mathbb{Q}$; hence $1 = (\frac{2p_1}{p_2}) = (\frac{2}{p_2})(\frac{p_1}{p_2}) = (\frac{p_1}{p_2})$. Which contradicts our hypotheses. Finally $(2_{\mathbb{Q}(\sqrt{p_1p_2})})^{\frac{h(d)}{2}}$ is principal. $\square$

**Theorem 2.** *Let* $\mathbb{k} = \mathbb{Q}(\sqrt{p_1p_2}, i)$, *with* $p_1$, $p_2$ *are prime numbers congruent to 1* $\pmod 8$, $(\frac{p_1}{p_2}) = -1$ *and* $(\frac{2}{a+b}) = -1$, *where* $d = p_1p_2 = a^2 + b^2$, $\mathbf{C}_{\mathbb{k},2}$ *be the 2-class group de of* $\mathbb{k}$. *Put* $p_1 = \pi_1\pi_2$, *where* $\pi_1$ *and* $\pi_2$ *are in* $\mathbb{Z}[i]$, *let* $\mathcal{H}_0$, $\mathcal{H}_1$ *and* $\mathcal{H}_2$ *be the ideals of* $\mathbb{k}$ *above* $1+i$, $\pi_1$ *and* $\pi_2$ *respectively. Then* $\mathbf{C}_{\mathbb{k},2} = \langle [\mathcal{H}_0^{\frac{h(d)}{2}}], [\mathcal{H}_1], [\mathcal{H}_2]\rangle$, *where* $h(d)$ *is the class number of* $\mathbb{Q}(\sqrt{p_1p_2})$.

*Proof.* Put $p_1 = \pi_1\pi_2$, we know that $\pi_j$ are ramified primes in $\mathbb{k}/\mathbb{Q}(i)$, so there exist prime ideals $\mathcal{H}_j$ in $\mathbb{k}$ such that $\mathcal{H}_j^2 = (\pi_j)$.

As the norm of the fundamental unit of $\mathbb{Q}(\sqrt{d})$ is -1 and $\mathcal{H}_1^2 = (e + 2if)$, $\mathcal{H}_2^2 = (e - 2if)$ $(\mathcal{H}_1\mathcal{H}_2)^2 = (p_1)$, where $p_1 = e^2 + 4f^2$; so Propositions 1 above and 8 in [1], state that $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_1\mathcal{H}_2$ are of order 2 in $\mathbb{k}$.

Let us proof that $\mathcal{H}_0^{\frac{h(d)}{2}}$ is of order 2 in $\mathbb{k}$. We know from [4] that $\frac{h(d)}{2}$ is an odd integer, then Lemma 1 implies that $\mathcal{H}_0^{\frac{h(d)}{2}}$ is not principal, as $\mathcal{H}_0^{h(d)} = (\mathcal{H}_0^2)^{\frac{h(d)}{2}} = (\mathfrak{z}_{\mathbb{Q}(\sqrt{d})})^{\frac{h(d)}{2}}\mathcal{O}_{\mathbb{k}}$, because $\mathfrak{z}_{\mathbb{Q}(\sqrt{d})}$ is an ramified ideal in $\mathbb{k}/\mathbb{Q}(\sqrt{d})$. Lemma (1) leads that $\mathcal{H}_0^{h(d)}$ is principal in $\mathbb{k}$, i.e. $\mathcal{H}_0^{\frac{h(d)}{2}}$ is an ideal of order 2. $\mathcal{H}_0^{\frac{h(d)}{2}}\mathcal{H}_i$ is also of order 2 in $\mathbb{k}$, in fact if for example $\mathcal{H}_0^{\frac{h(d)}{2}}\mathcal{H}_1$ is principal. So by applying the norm in $\mathbb{k}/\mathbb{Q}(\sqrt{d})$, we find that $(\mathfrak{z}_{\mathbb{Q}(\sqrt{p_1p_2})})^{\frac{h(d)}{2}}\mathfrak{p}_{1\mathbb{Q}(\sqrt{d})}$ is an ideal principal in $\mathbb{Q}(\sqrt{d})$, afterward $\mathfrak{p}_{1\mathbb{Q}(\sqrt{d})}$ is principal in $\mathbb{Q}(\sqrt{d})$. This contradicts Lemma 1. Let us show by absurd also that the ideal $\mathcal{H}_0^{\frac{h(d)}{2}}\mathcal{H}_1\mathcal{H}_2$ is of order 2 in $\mathbb{k}$. If not, there exists $\alpha \in \mathbb{k}$ such that $\mathcal{H}_0^{\frac{h(d)}{2}}\mathcal{H}_1\mathcal{H}_2 = (\alpha)$, as $\mathcal{H}_1\mathcal{H}_2\mathcal{H}_3\mathcal{H}_4 = (\sqrt{\pi_1\pi_2\pi_3\pi_4}) = (\sqrt{p_1p_2})$ is principal in $\mathbb{k}$, then there exists $\beta \in \mathbb{k}$ such that $\mathcal{H}_0^{\frac{h(d)}{2}}\mathcal{H}_3\mathcal{H}_4 = (\beta)$. By taking norm in $\mathbb{k}/\mathbb{Q}(i)$, we find that $(1+i)^{\frac{h(d)}{2}}p_1 = \varepsilon(\alpha_1^2 - \alpha_2^2 d)$ and $(1+i)^{\frac{h(d)}{2}}p_2 = \varepsilon'(\beta_1^2 - \beta_2^2 d)$, with $\varepsilon$, $\varepsilon'$ are two units in $\mathbb{Q}(i)$, $\alpha_1$, $\alpha_2$, $\beta_1$ and $\beta_2$ are elements in $\mathbb{Q}(i)$. This imply that: $\left(\dfrac{1+i}{\mathfrak{p}_{2\mathbb{Q}(i)}}\right)\left(\dfrac{p_1}{\mathfrak{p}_{2\mathbb{Q}(i)}}\right) = 1$ and $\left(\dfrac{1+i}{\mathfrak{p}_{1\mathbb{Q}(i)}}\right)\left(\dfrac{p_2}{\mathfrak{p}_{1\mathbb{Q}(i)}}\right) = 1$. As $\left(\dfrac{p_1}{\mathfrak{p}_{2\mathbb{Q}(i)}}\right) = \left(\dfrac{p_1}{p_2}\right)$, $\left(\dfrac{p_2}{\mathfrak{p}_{1\mathbb{Q}(i)}}\right) = \left(\dfrac{p_2}{p_1}\right)$ and $(\frac{p_1}{p_2}) = -1$, then according to [6, p. 154] we find that $\left(\dfrac{2}{p_1}\right)_4\left(\dfrac{p_1}{2}\right)_4 = \left(\dfrac{2}{p_2}\right)_4\left(\dfrac{p_2}{2}\right)_4 = -1$. Finally from [7, p. 323 ] $(\frac{2}{a+b}) = 1$, which is false. ☐

**Numerical Examples 2.** $d$ is of the form (1).

| $d=p_1.p_2$ | $\left(\frac{p_1}{p_2}\right)$ | $a$ | $b$ | $\left(\frac{2}{a+b}\right)$ | $\mathbf{H}_0^{\frac{h(d)}{2}}$ | $\mathbf{H}_1$ | $\mathbf{H}_2$ |
|---|---|---|---|---|---|---|---|
| $697=17.41$ | $-1$ | 11 | 24 | $-1$ | $[3,1,0]$ | $[3,1,1]$ | $[0,1,0]$ |
| $3977=97.41$ | $-1$ | 56 | 29 | $-1$ | $[0,0,1]$ | $[7,0,1]$ | $[7,1,1]$ |

### 2.3. Generators of $\mathbf{C}_{\Bbbk,2}$ when $d=pq_1q_2$

**Theorem 3.** *Let $\Bbbk=\mathbb{Q}(\sqrt{d},i)$, where $d=pq_1q_2$ with $p$, $q_1$ and $q_2$ are prime numbers satisfying conditions $A$ and $B$ defined in the introduction. Denote by $\mathbf{C}_{\Bbbk,2}$ the 2-class group of $\Bbbk$. Put $p_1=\pi_1\pi_2$, with $\pi_1$, $\pi_2$ in $\mathbb{Z}[i]$, let $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be the prime ideals of $\Bbbk$ above $\pi_1$, $\pi_2$, $q_1$ and $q_2$ respectively, then:*

(1) *If $p$, $q_1$ and $q_2$ are satisfying conditions $B$ (I) or $B$ (II) and $\left(\frac{p}{q_1}\right)=-\left(\frac{p}{q_2}\right)=1$, then $\mathbf{C}_{\Bbbk,2}=\langle[\mathcal{H}_1],[\mathcal{H}_2],[\mathcal{Q}_2]\rangle$.*

(2) *Else, $\mathbf{C}_{\Bbbk,2}=\langle[\mathcal{H}_1],[\mathcal{H}_2],[\mathcal{Q}_1]\rangle$.*

*Proof.* As $q_1$, $q_2$ are congruent to 3 (mod 4), so they are ramified in $\Bbbk/\mathbb{Q}$; let $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be the ideals in $\Bbbk$ above $q_1$ and $q_2$ respectively. We know also that $\pi_j$ are ramified in $\Bbbk/\mathbb{Q}(i)$, then there exist ideals $\mathcal{H}_j$ in $\Bbbk$ such that: $(\pi_j)=\mathcal{H}_j^2$, moreover $\mathcal{H}_j$ is not principal in $\Bbbk$, in fact if we put $p_1=e^2+4f^2$, then $\mathcal{H}_j^2=(e\pm 2if)$ and as $\sqrt{p}\notin\mathbb{Q}(\sqrt{d})$, therefore Proposition 1 states the result, similar for $i,j\in\{1,2\}$ we have $\mathcal{H}_i\mathcal{Q}_j$ is not principal in $\Bbbk$ because $(\mathcal{H}_i\mathcal{Q}_j)^2=(\pi_iq_j)$ and $\pi_iq_j=q_j(e\pm 2if)$, as $\sqrt{(eq_j)^2+(2fq_j)^2}=q_j\sqrt{p}\notin\mathbb{Q}(\sqrt{d})$, hence Proposition 1 implies the result.

Let $\varepsilon_d=x+y\sqrt{d}$ be the fundamental unit of $\mathbb{Q}(\sqrt{d})$; as $d\equiv 1 \pmod 4$, then the unit index of $\Bbbk$ is 1 (corollary 3.2 in [3]), so according to [2] $x\pm 1$ is not a square in $\mathbb{N}$; therefore from Remark 1, if $\mathcal{H}$ is an ideal of $\Bbbk$ satisfy $\mathcal{H}^2=(l)$, where $l$ is a prime number in $\mathbb{N}$, then $\mathcal{H}$ is principal if and only if $l(x\pm 1)$ or $2l(x\pm 1)$ is a square in $\mathbb{N}$.

(1) For this first case the proof is of two points:

(i) Suppose $p$, $q_1$ and $q_2$ satisfy $B$ (I) and $\left(\frac{p}{q_1}\right)=-\left(\frac{p}{q_2}\right)=1$, so as $x^2-1=y^2p_1p_2q$, the only possible case is:
$\begin{cases} x\pm 1=2q_1y_1^2, \\ x\mp 1=2pq_2y_2^2; \end{cases}$ this yields that $2q_1(x\pm 1)$ is a square in $\mathbb{N}$ and $q_2(x\pm 1)$, $2q_2(x\pm 1)$ are not; as $\mathcal{Q}_1^2=(q_1)$ and $\mathcal{Q}_2^2=(q_2)$, therefore $\mathcal{Q}_1$ is principal in $\Bbbk$ and $\mathcal{Q}_2$ is not, the result derived.

(ii) Suppose $p$, $q_1$ and $q_2$ satisfy $B$ (II) and $\left(\frac{p}{q_1}\right) = -\left(\frac{p}{q_2}\right) = 1$, so as $x^2-1 = y^2p_1p_2q$, the only possible cases are:
$\begin{cases} x\pm 1 = q_1y_1^2, \\ x\mp 1 = pq_2y_2^2; \end{cases}$ or $\begin{cases} x\pm 1 = 2q_1y_1^2, \\ x\mp 1 = 2pq_2y_2^2; \end{cases}$ thus $q_1(x\pm 1)$ or $2q_1(x\pm 1)$ is a square in $\mathbb{N}$ and $q_2(x\pm 1)$, $2q_2(x\pm 1)$ are not; as $\mathcal{Q}_1^2 = (q_1)$ and $\mathcal{Q}_2^2 = (q_2)$, so $\mathcal{Q}_1$ is principal in $\Bbbk$ and $\mathcal{Q}_2$ is not, this ends the first case of theorem.

(2) In this case we have also two points to distinguish:

(i) Suppose that $p$, $q_1$ and $q_2$ satisfy $B$ (I) or $B$ (II) and $\left(\frac{p}{q_2}\right) = -\left(\frac{p}{q_1}\right) = 1$, we proceed as in the case (1) to prove that $\mathcal{Q}_2$ is principal in $\Bbbk$ and $\mathcal{Q}_1$ is not; so the result.

(ii) Suppose that $p$, $q_1$ et $q_2$ satisfy $B$ (III), then since $x^2-1 = y^2pq_1q_2$, the only possible case is: $\begin{cases} x\pm 1 = 2q_1q_2y_1^2, \\ x\mp 1 = 2py_2^2; \end{cases}$ so $2q_1q_2(x\pm 1)$ is square in $\mathbb{N}$ and $q_1(x\pm 1)$, $2q_1(x\pm 1)$, $q_2(x\pm 1)$ and $2q_2(x\pm 1)$ are not. This completes the proof. □

**Numerical Examples 3.** $d = pq_1q_2$ is of the form (4).

| $d = p.q_1.q_2$ | $\left(\frac{p}{q_1}\right)$ | $\left(\frac{p}{q_2}\right)$ | $\left(\frac{2}{q_1}\right)$ | $\left(\frac{2}{q_2}\right)$ |
|---|---|---|---|---|
| $357 = 17.7.3$ | $-1$ | $-1$ | $1$ | $-1$ |
| $969 = 17.19.3$ | $1$ | $-1$ | $-1$ | $-1$ |
| $3553 = 17.11.19$ | $-1$ | $1$ | $-1$ | $-1$ |

| $d = p.q_1.q_2$ | $\mathcal{Q}_1$ | $\mathcal{Q}_2$ | $\mathcal{Q}_1\mathcal{Q}_2$ | $\mathcal{H}_1$ | $\mathcal{H}_2$ |
|---|---|---|---|---|---|
| $357 = 17.7.3$ | $[1,1,0]$ | $[1,1,0]$ | $[0,0,0]$ | $[0,0,1]$ | $[0,0,1]$ |
| $969 = 17.19.3$ | $[0,0,0]$ | $[0,1,0]$ | $[0,1,0]$ | $[3,1,0]$ | $[3,0,0]$ |
| $3553 = 17.11.19$ | $[0,0,1]$ | $[0,0,0]$ | $[0,0,1]$ | $[3,1,1]$ | $[3,1,0]$ |

### 2.4. Generators of $\mathbf{C}_{\Bbbk,2}$ when $d = p_1p_2q$

Let $d = p_1p_2q$ satisfying the conditions of the form (5), then we have.

**Theorem 4.** *Let $\Bbbk = \mathbb{Q}(\sqrt{d}, i)$, where $d = p_1p_2q$ with $p_1$, $p_2$ and $q$ are prime numbers such that $p_1 \equiv p_2 \equiv -q \equiv 1 \pmod 4$, $p_1 \equiv 5$ or $p_2 \equiv 5 \pmod 8$ and at least two elements of $\{(\frac{p_1}{p_2}), (\frac{p_1}{q}), (\frac{p_2}{q})\}$ are equal to $-1$. Denote by $\mathbf{C}_{\Bbbk,2}$ the 2-class group of $\Bbbk$. Put $p_1 = \pi_1\pi_2$, $p_2 = \pi_3\pi_4$, with $\pi_1$, $\pi_2$, $\pi_3$ and $\pi_4$*

*are in $\mathbb{Z}[i]$, let $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{H}_3$ and $\mathcal{H}_4$ be the ideals in $\Bbbk$ above $\pi_1$, $\pi_2$, $\pi_3$ and $\pi_4$ respectively, then:*

(1) *If $p_1$, $p_2$ and $q$ are of type I, then $\mathbf{C}_{\Bbbk,2} = \langle[\mathcal{H}_1], [\mathcal{H}_3], [\mathcal{H}_4]\rangle$.*

(2) *If $p_1$, $p_2$ and $q$ are of type II or of type III, then $\mathbf{C}_{\Bbbk,2} = \langle[\mathcal{H}_1], [\mathcal{H}_2], [\mathcal{H}_3]\rangle$.*

*Proof.* We proceed as in the previous case to prove that:
• If $p_1$, $p_2$ and $q$ are of type I, then $\mathcal{H}_1\mathcal{H}_2$ is principal in $\Bbbk$ and $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{H}_3$, $\mathcal{H}_4$ and $\mathcal{H}_3\mathcal{H}_4$ are not.
• If $p_1$, $p_2$ and $q$ are of type II, then $\mathcal{H}_3\mathcal{H}_4$ is principal in $\Bbbk$ and $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{H}_3$, $\mathcal{H}_4$ and $\mathcal{H}_1\mathcal{H}_2$ are not.
• If $p_1$, $p_2$ and $q$ are of type III, then $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{H}_3$, $\mathcal{H}_4$, $\mathcal{H}_1\mathcal{H}_2$ and $\mathcal{H}_3\mathcal{H}_4$ are not principal in $\Bbbk$. □

**Numerical Examples 4.** The last case where $d = p_1p_2q$ is of the form (5).

| $p_1.p_2.q$ | $(\frac{2}{p_1})$ | $(\frac{2}{p_2})$ | $(\frac{p_1}{p_2})$ | $(\frac{p_1}{q})$ | $(\frac{p_2}{q})$ |
|---|---|---|---|---|---|
| 5.13.3 | $-1$ | $-1$ | $-1$ | $-1$ | 1 |
| 5.13.7 | $-1$ | $-1$ | $-1$ | $-1$ | $-1$ |
| 17.5.7 | 1 | $-1$ | $-1$ | $-1$ | $-1$ |
| 13.5.11 | $-1$ | $-1$ | $-1$ | $-1$ | 1 |
| 5.17.11 | $-1$ | 1 | $-1$ | 1 | $-1$ |

| $p_1.p_2.q$ | $\mathcal{H}_1$ | $\mathcal{H}_2$ | $\mathcal{H}_1\mathcal{H}_2$ | $\mathcal{H}_3$ | $\mathcal{H}_4$ | $\mathcal{H}_3\mathcal{H}_4$ |
|---|---|---|---|---|---|---|
| 5.13.3 | $[1,0,0]$ | $[0,1,1]$ | $[1,1,1]$ | $[0,1,0]$ | $[0,1,0]$ | $[0,0,0]$ |
| 5.13.7 | $[5,1,1]$ | $[5,0,1]$ | $[0,1,0]$ | $[0,1,1]$ | $[0,0,1]$ | $[0,1,0]$ |
| 17.5.7 | $[0,1,1]$ | $[0,1,1]$ | $[0,0,0]$ | $[0,1,0]$ | $[1,0,0]$ | $[1,1,0]$ |
| 13.5.11 | $[1,0,0]$ | $[0,1,0]$ | $[1,1,0]$ | $[0,1,1]$ | $[0,1,1]$ | $[0,0,0]$ |
| 5.17.11 | $[7,1,0]$ | $[7,0,0]$ | $[0,1,0]$ | $[7,1,1]$ | $[7,1,1]$ | $[0,0,0]$ |

## References

[1] A. Azizi, Construction de la tour des 2-corps de classes de Hilbert de certains corps biquadratiques, *Pac. J. Math*, **208**, No. 1 (2003), 1-10.

[2] A. Azizi, Sur les unités de certains corps de nombres de degré 8 sur $\mathbb{Q}$, *Ann. Sci. Math. Québec*, **29**, No. 2 (2005), 111-129.

[3] A. Azizi, M. Taous, Determination des corps $\mathbb{Q}(\sqrt{d}, i)$ dont le 2-groupes de classes est de type $(2, 4)$ ou $(2, 2, 2)$, *Rend. Istit. Mat. Univ. Trieste*, **40**, No. XL (2009), 93-116.

[4] A. Scholz, Über die Löbarkeit der Gleichung $t^2 - Du^2 = -4$, *Math. Z*, **39** (1934), 95-111.

[5] C. Batut, K. Belabas, D. Bernadi, H. Cohen, M. Olivier, *GP/PARI Calculator*, Version 2.2.6 (2003).

[6] F. Lemmermeyer, *Reciprocity Laws*, Springer-Verlag Berlin (2000).

[7] P. Kaplan, Sur le 2-groupe de classes d'idéaux des corps quadratiques, *J. Reine Angew. Math.*, **1976**, No. 283-284 (1976), 313-363.