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SAIGO OPERATOR OF FRACTIONAL INTEGRATION OF HYPERGEOMETRIC FUNCTIONS

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Abstract: The aim of this paper is to obtain derivations of some hypergeometric functions for Saigo operator by the application of the generalized fractional integration due to Saigo involving the quadratic transformation formula.

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Key Words: Saigo operator of fractional integration, Gauss hypergeometric function, Pochhammer symbol

1. Introduction

The Gauss Hypergeometric function is

$$ {}_2F_1[a,b;c:x] = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!} \tag{1.1}$$

where c is nether zero nor a negative integer, $|x| < 1; x = 1$ and $\mathrm{Re}(c-a-b) > 0, x =$ - 1 and $\mathrm{Re}(c-a-b) > -1$, $(\alpha)_n$ is the usual Pochhammer symbol

$$(\alpha)_n = \alpha(\alpha+1)(\alpha+2)....(\alpha+n-1) = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}, (\alpha)_0 = 1, \tag{1.2}$$

$n \geq 1, \alpha \neq 0$.

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Let $\alpha, \beta, \eta \in C$ and $x \in R^{+}$. Then the generalized fractional integration due to Saigo [12] is defined as

$$I_{0,x}^{\alpha,\beta,\eta}[f(x)]$$

$$= \frac{x^{-\alpha-\beta}}{\Gamma\alpha} \int_0^x (x-t)^{\alpha-1} {}_2F_1\left[\alpha+\beta, -\eta, \alpha; 1-\frac{t}{x}\right] f(t)dt, Re(\alpha) > 0 \quad (1.3)$$

$$= \frac{d^n}{dx^n} I_{0,x}^{\alpha+n, \beta-n, \eta-n}[f(x)] \quad (1.4)$$

where $0 < Re(\alpha) + n \leq 1$; $n = 1, 2, ...$ Also for ρ being real, the function $x^{\rho-1}$ has the integral formula

$$I_{0,x}^{\alpha,\beta,\eta}\left[x^{\rho-1}\right] = \frac{\Gamma(\rho)\Gamma(\rho+\eta-\beta)}{\Gamma(\rho-\beta)\Gamma(\rho+\eta+\alpha} x^{\rho-\beta-1} \quad (1.5)$$

where $Re(\rho) > \max\,[0, Re(\beta-\eta)]$.

2. Hypergeometric Series Identities

In this section we obtain derivations of some hypergeometric functions for Saigo operator.

Theorem 2.1.

$$I_{0,x}^{\alpha,\beta,\eta}\left[x^{\sigma-1}(1-x)^{-a} {}_2F_1\left[\begin{matrix}\frac{a}{2}, \frac{1}{2}+\frac{a}{2} \\ 1+a-b\end{matrix}; \frac{-4x}{(1-x)^2}\right]\right]$$
$$= \frac{\Gamma\sigma\ \Gamma(\sigma+\eta-\beta)}{\Gamma(\sigma-\beta)\ \Gamma(\sigma+\eta+\alpha)} x^{\sigma-\beta-1} {}_4F_3\left[\begin{matrix}\sigma, \sigma+\eta-\beta, a, b \\ \sigma-\beta, \sigma+\beta+\alpha, 1+a-b\end{matrix}; -x\right], \quad (2.1)$$

where $Re(\alpha)$, $Re(\beta) > 0$, $Re(\sigma) = \max[0, Re(\beta-\gamma)]$; a is a non positive integer.

Proof. We start with the quadratic transformation formula [[11], (3.2)]

$$(1-x)^{-a} {}_2F_1\left[\begin{matrix}\frac{a}{2}, \frac{1}{2}+\frac{a}{2} \\ 1+a-b\end{matrix}; \frac{-4x}{(1-x)^2}\right] = {}_2F_1\left[\begin{matrix}a, b \\ 1+a-b\end{matrix}; -x\right], \quad (2.2)$$

where a is a non positive integer. Operating both the sides by the functional integrating operator $I_{0,x}^{\alpha,\beta,\eta}\left[x^{\sigma-1}\right]$ and using 1.1 , we have

$$I_{0,x}^{\alpha,\beta,\eta}\left[x^{\sigma-1}(1-x)^{-a}{}_2F_1\left[\begin{matrix}\frac{a}{2},\frac{1}{2}+\frac{a}{2}\\1+a-b\end{matrix};\frac{-4x}{(1-x)^2}\right]\right]$$

$$=I_{0,x}^{\alpha,\beta,\eta}\left[\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(1+a-b)_n}\frac{(-1)^n x^{n+\sigma-1}}{n!}\right]$$

Using Equation 1.3 and changing the order of the integration and summation which is valid under the condition stated above, the above expression reduced to the form

$$=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n(-1)^n}{(1+a-b)_n n!}I_{0,x}^{\alpha,\beta,\eta}\left[x^{n+\sigma-1}\right]$$

applying Equation 1.5, we obtain

$$=\frac{\Gamma\sigma\ \Gamma(\sigma+\eta-\beta)}{\Gamma(\sigma-\beta)\ \Gamma(\sigma+\eta+\alpha)}x^{\sigma-\beta-1}{}_4F_3\left[\begin{matrix}\sigma,\sigma+\eta-\beta,a,b,\\\sigma-\beta,\sigma+\beta+\alpha,1+a-b\end{matrix};-x\right].$$

□

Theorem 2.2. *If we start with the quadratic transformation formula* $\frac{1}{(1-x)^{3a}}{}_3F_2\left[\begin{matrix}a,\frac{1}{3}+a,\frac{2}{3}+a\\b,\frac{3}{2}+3a-b\end{matrix};\frac{27x^2}{4(1-x)^3}\right]={}_3F_2\left[\begin{matrix}3a,\frac{-1}{2}+b,1+3a-b\\2b-1,2+6a-2b\end{matrix};4x\right]$, *then we obtain*

$$I_{0,x}^{\alpha,\beta,\eta}\left[x^{\sigma-1}\frac{1}{(1-x)^{3a}}{}_3F_2\left[\begin{matrix}a,\frac{1}{3}+a,\frac{2}{3}+a\\b,\frac{3}{2}+3a-b\end{matrix};\frac{27x^2}{4(1-x)^3}\right]\right]$$

$$=\frac{\Gamma\sigma\ \Gamma(\sigma+\eta-\beta)}{\Gamma(\sigma-\beta)\ \Gamma(\sigma+\eta+\alpha)}x^{\sigma-\beta-1}$$

$${}_5F_4\left[\begin{matrix}\sigma,\sigma+\eta-\beta,3a,\frac{-1}{2}+b,1+3a-b\\\sigma-\beta,\sigma+\beta+\alpha,2b-1,2+6a-2b\end{matrix};4x\right],$$

where $Re(\alpha)$, $Re(\beta)>0$, $Re(\sigma)=\max[0,Re(\beta-\gamma)]$; *a is a non positive integer.*

Theorem 2.3. *If we start with the quadratic transformation formula* $(1-x)^a{}_2F_1\left[\begin{matrix}a,c-b\\c\end{matrix};\frac{-x}{1-x}\right]={}_2F_1\left[\begin{matrix}a,b\\c\end{matrix};x\right]$, *then we obtain*

$$I_{0,x}^{\alpha,\beta,\eta}\left[x^{\sigma-1}(1-x)^a{}_2F_1\left[\begin{matrix}a,c-b\\c\end{matrix};\frac{-x}{1-x}\right]\right]$$

$=\frac{\Gamma\sigma\ \Gamma(\sigma+\eta-\beta)}{\Gamma(\sigma-\beta)\ \Gamma(\sigma+\eta+\alpha)}x^{\sigma-\beta-1}{}_4F_3\left[\begin{matrix}\sigma,\sigma+\eta-\beta,a,b\\\sigma-\beta,\sigma+\beta+\alpha,c\end{matrix};x\right]$ *where* $Re(\alpha)$, $Re(\beta)>0$, $Re(\sigma)=\max[0,Re(\beta-\gamma)]$; *a is a non positive integer.*

Theorem 2.4. *If we start with the quadratic transformation formula of GauB* $\left[[3], Ex.\ 4(iii), p.97, with \alpha\to\frac{a}{2},\beta\to\frac{b}{2},x\to x\right]$

${}_2F_1\left[\begin{matrix}\frac{a}{2},\frac{b}{2}\\ \frac{1}{2}+\frac{a}{2}+\frac{b}{2}\end{matrix};4x(1-x)\right]={}_2F_1\left[\begin{matrix}a,b\\ \frac{1}{2}+\frac{a}{2}+\frac{b}{2}\end{matrix};x\right]$ *and assume that a is a non positive integer, then we obtain*

$$I^{\alpha,\beta,\eta}_{0,x}\left[x^{\sigma-1}{}_2F_1\left[\begin{matrix}\frac{a}{2},\frac{b}{2}\\ \frac{1}{2}+\frac{a}{2}+\frac{b}{2}\end{matrix};4x(1-x)\right]\right]$$
$$=\frac{\Gamma\sigma\;\Gamma(\sigma+\eta-\beta)}{\Gamma(\sigma-\beta)\;\Gamma(\sigma+\eta+\alpha)}x^{\sigma-\beta-1}{}_4F_3\left[\begin{matrix}\sigma,\sigma+\eta-\beta,a,b\\ \sigma-\beta,\sigma+\beta+\alpha,\frac{1}{2}+\frac{a}{2}+\frac{b}{2}\end{matrix};x\right],$$

where $Re(\alpha)$, $Re(\beta)>0$, $Re(\sigma)=\max[0,Re(\beta-\gamma)]$; a is a non positive integer.

Theorem 2.5. *If we start with the quadratic transformation formula [[11], (5.12) with $z\to x$] :*

$$(1-x)^{\frac{-a}{2}}{}_2F_1\left[\begin{matrix}\frac{a}{2},\frac{-a}{2}+b\\ \frac{1}{2}+b\end{matrix};\frac{x^2}{4(x-1)}\right]={}_2F_1\left[\begin{matrix}a,b\\ 2b\end{matrix};x\right],$$

and assume that a is an even non positive integer, then we obtain

$$I^{\alpha,\beta,\eta}_{0,x}\left[x^{\sigma-1}(1-x)^{\frac{-a}{2}}{}_2F_1\left[\begin{matrix}\frac{a}{2},\frac{-a}{2}+b\\ \frac{1}{2}+b\end{matrix};\frac{x^2}{4(x-1)}\right]\right]$$
$$=\frac{\Gamma\sigma\;\Gamma(\sigma+\eta-\beta)}{\Gamma(\sigma-\beta)\;\Gamma(\sigma+\eta+\alpha)}x^{\sigma-\beta-1}{}_4F_3\left[\begin{matrix}\sigma,\sigma+\eta-\beta,a,b\\ \sigma-\beta,\sigma+\beta+\alpha,2b\end{matrix};x\right],$$

where $Re(\alpha)$, $Re(\beta)>0$, $Re(\sigma)=\max[0,Re(\beta-\gamma)]$; a is an even non positive integer.

Theorem 2.6. *If we start with the transformation formula [[11], (3.31)]*

$$(1-x)^{c-1}{}_2F_1\left[\begin{matrix}\frac{-a}{2}+\frac{c}{2},\frac{-1}{2}+\frac{a}{2}+\frac{c}{2}\\ c\end{matrix};4x(1-x)\right]={}_2F_1\left[\begin{matrix}a,1-a\\ c\end{matrix};x\right],$$

then we obtain

$$I^{\alpha,\beta,\eta}_{0,x}\left[x^{\sigma-1}(1-x)^{c-1}{}_2F_1\left[\begin{matrix}\frac{-a}{2}+\frac{c}{2},\frac{-1}{2}+\frac{a}{2}+\frac{c}{2}\\ c\end{matrix};4x(1-x)\right]\right]$$
$$=\frac{\Gamma\sigma\;\Gamma(\sigma+\eta-\beta)}{\Gamma(\sigma-\beta)\;\Gamma(\sigma+\eta+\alpha)}x^{\sigma-\beta-1}{}_4F_3\left[\begin{matrix}\sigma,\sigma+\eta-\beta,a,1-a\\ \sigma-\beta,\sigma+\beta+\alpha,c\end{matrix};x\right],$$

where $Re(\alpha)$, $Re(\beta)>0$, $Re(\sigma)=\max[0,Re(\beta-\gamma)]$.

Theorem 2.7. *If we start with the transformation formula* $[[4], (4.22)$, *with* $\alpha \to a, \beta \to b, x \to z$ *and [11], (5.12), with*

$$z \to \frac{x}{(2-x)}] : (1-x)^{-a}{}_2F_1\left[\begin{matrix} a, b \\ \frac{1}{2}+a+b \end{matrix}; \frac{x^2}{4(x-1)}\right] = {}_2F_1\left[\begin{matrix} 2a, a+b \\ 2a+2b \end{matrix}; x\right]$$

and assume that a *is a non positive integer, then we obtain*

$$I^{\alpha,\beta,\eta}_{0,x}\left[x^{\sigma-1}(1-x)^{-a}{}_2F_1\left[\begin{matrix} a, b \\ \frac{1}{2}+a+b \end{matrix}; \frac{x^2}{4(x-1)}\right]\right]$$
$$= \frac{\Gamma\sigma\ \Gamma(\sigma+\eta-\beta)}{\Gamma(\sigma-\beta)\ \Gamma(\sigma+\eta+\alpha)} x^{\sigma-\beta-1}{}_4F_3\left[\begin{matrix} \sigma, \sigma+\eta-\beta, 2a, a+b \\ \sigma-\beta, \sigma+\beta+\alpha, 2a+2b \end{matrix}; x\right],$$

where Re(α*), Re(*β*)* > 0*, Re(*σ*)* $= \max[0, Re(\beta-\gamma)]$.

Theorem 2.8. *If we start with the transformation formula* $[[11], (3.31)$, *with* $z \to \frac{4z}{(1-z)}, a \to a+c+\frac{1}{2}, c \to b] :$
$(1-x)^{a+b-\frac{1}{2}}{}_2F_1\left[\begin{matrix} a, b \\ \frac{1}{2}+a+b \end{matrix}; 4x(1-x)\right] = {}_2F_1\left[\begin{matrix} \frac{1}{2}+a-b, \frac{1}{2}-a+b \\ \frac{1}{2}+a+b \end{matrix}; x\right]$ *and assume that* a *is a non positive integer, then we obtain*

$$I^{\alpha,\beta,\eta}_{0,x}\left[x^{\sigma-1}(1-x)^{a+b-\frac{1}{2}}{}_2F_1\left[\begin{matrix} a, b \\ \frac{1}{2}+a+b \end{matrix}; 4x(1-x)\right]\right]$$
$$= \frac{\Gamma\sigma\ \Gamma(\sigma+\eta-\beta)}{\Gamma(\sigma-\beta)\ \Gamma(\sigma+\eta+\alpha)} x^{\sigma-\beta-1}{}_4F_3\left[\begin{matrix} \sigma, \sigma+\eta-\beta, \frac{1}{2}+a-b, \frac{1}{2}-a+b \\ \sigma-\beta, \sigma+\beta+\alpha, \frac{1}{2}+a+b \end{matrix}; x\right],$$

where Re(α*), Re(*β*)* > 0*, Re(*σ*)* $= \max[0, Re(\beta-\gamma)]$*;* a *is a non positive integer.*

Theorem 2.9. *If we start with the transformation formula* $[[8], (3.4.8), q \to 1] : \frac{1+x}{(1-x)^{2a}}{}_2F_1\left[\begin{matrix} a, \frac{1}{2}+a \\ b \end{matrix}; \frac{-4x}{(1-x)^2}\right] = {}_3F_2\left[\begin{matrix} 2a-1, \frac{-1}{2}+a, 2a-b \\ \frac{-1}{2}+a, b \end{matrix}; -x\right]$ *and assume that* a *is a non positive integer, then we obtain*

$$I^{\alpha,\beta,\eta}_{0,x}\left[x^{\sigma-1}\frac{1+x}{(1-x)^{2a}}{}_2F_1\left[\begin{matrix} a, \frac{1}{2}+a \\ b \end{matrix}; \frac{-4x}{(1-x)^2}\right]\right]$$
$$= \frac{\Gamma\sigma\ \Gamma(\sigma+\eta-\beta)}{\Gamma(\sigma-\beta)\ \Gamma(\sigma+\eta+\alpha)} x^{\sigma-\beta-1}{}_5F_4\left[\begin{matrix} \sigma, \sigma+\eta-\beta, 2a-1, \frac{-1}{2}+a, 2a-b \\ \sigma-\beta, \sigma+\beta+\alpha, \frac{-1}{2}+a, b \end{matrix}; -x\right],$$

where Re(α*), Re(*β*)* > 0*, Re(*σ*)* $= \max[0, Re(\beta-\gamma)]$*;* a *is a non positive integer.*

Theorem 2.10. *If we start with the transformation formula* $[[4], p.94, Ex.4.(iv) \text{with } x \to z] :$

$$(1-x)^{-a}{}_3F_2\left[\begin{matrix}\frac{a}{2},\frac{1}{2}+\frac{a}{2},1+a-b-c\\ 1+a-b,1+a-c\end{matrix};\frac{-4x}{(1-x)^2}\right]={}_3F_2\left[\begin{matrix}a,b,c\\ 1+a-b,1+a-c\end{matrix};x\right]$$

and assume that a is a non positive integer, then we obtain

$$I\begin{matrix}\alpha,\beta,\eta\\ 0,x\end{matrix}\left[x^{\sigma-1}(1-x)^{-a}{}_3F_2\left[\begin{matrix}\frac{a}{2},\frac{1}{2}+\frac{a}{2},1+a-b-c\\ 1+a-b,1+a-c\end{matrix};\frac{-4x}{(1-x)^2}\right]\right]$$
$$=\frac{\Gamma\sigma\ \Gamma(\sigma+\eta-\beta)}{\Gamma(\sigma-\beta)\ \Gamma(\sigma+\eta+\alpha)}x^{\sigma-\beta-1}{}_5F_4\left[\begin{matrix}\sigma,\sigma+\eta-\beta,a,b,c\\ \sigma-\beta,\sigma+\beta+\alpha,1+a-b,1+a-c\end{matrix};x\right],$$

where $Re(\alpha)$, $Re(\beta)>0$, $Re(\sigma)=\max[0,Re(\beta-\gamma)]$; a is a non positive integer.

Theorem 2.11. *If we start with the transformation formula $[[3],p.97,$ $Ex.6$ with $b\to 1+a-b,c\to 1+a-c,x\to x]$:*

$$\frac{1+x}{(1-x)^{1+a}}{}_3F_2\left[\begin{matrix}\frac{1}{2}+\frac{a}{2},1+\frac{a}{2},-a+b+c-1\\ b,c\end{matrix};\frac{-4x}{(x-1)^2}\right]$$
$$={}_4F_3\left[\begin{matrix}a,1+\frac{a}{2},1+a-b,1+a-c\\ \frac{a}{2},b,c\end{matrix};x\right]$$

and assume that a is a non positive integer, then we obtain

$$I\begin{matrix}\alpha,\beta,\eta\\ 0,x\end{matrix}\left[x^{\sigma-1}\frac{1+x}{(1-x)^{1+a}}{}_3F_2\left[\begin{matrix}\frac{1}{2}+\frac{a}{2},1+\frac{a}{2},-a+b+c-1\\ b,c\end{matrix};\frac{-4x}{(x-1)^2}\right]\right]$$
$$=\frac{\Gamma\sigma\ \Gamma(\sigma+\eta-\beta)}{\Gamma(\sigma-\beta)\ \Gamma(\sigma+\eta+\alpha)}x^{\sigma-\beta-1}$$
$${}_6F_5\left[\begin{matrix}\sigma,\sigma+\eta-\beta,a,1+\frac{a}{2},1+a-b,1+a-c\\ \sigma-\beta,\sigma+\beta+\alpha,\frac{a}{2},b,c\end{matrix};-x\right],$$

where $Re(\alpha)$, $Re(\beta)>0$, $Re(\sigma)=\max[0,Re(\beta-\gamma)]$; a is a non positive integer.

Theorem 2.12. *If we start with the transformation formula $[[4],(4.05)$ with $\rho_1\to b,\rho_2\to 3a-b+\frac{3}{2},x\to x]$:*

$$(1-x)^{-3a}{}_3F_2\left[\begin{matrix}a,\frac{1}{3}+a,\frac{2}{3}+a\\ b,\frac{3}{2}+3a-b\end{matrix};\frac{-27x}{4(1-x)^3}\right]$$
$$={}_3F_2\left[\begin{matrix}3a,-3a+2b-1,2+3a-2b\\ b,\frac{3}{2}+3a-b\end{matrix};\frac{x}{4}\right],$$

and assume that a is a non positive integer, then we obtain

$$I\begin{matrix}\alpha,\beta,\eta\\ 0,x\end{matrix}\left[x^{\sigma-1}(1-x)^{-3a}{}_3F_2\left[\begin{matrix}a,\frac{1}{3}+a,\frac{2}{3}+a\\ b,\frac{3}{2}+3a-b\end{matrix};\frac{-27x}{4(1-x)^3}\right]\right]$$

$$=\frac{\Gamma\sigma\ \Gamma(\sigma+\eta-\beta)}{\Gamma(\sigma-\beta)\ \Gamma(\sigma+\eta+\alpha)}x^{\sigma-\beta-1}{}_5F_4\left[\begin{matrix}\sigma,\sigma+\eta-\beta,3a,-3a+2b-1,2+3a-2b\\ \sigma-\beta,\sigma+\beta+\alpha,b,\frac{3}{2}+3a-b\end{matrix};\frac{x}{4}\right]$$

where Re(α), Re(β) > 0, Re(σ) = max[0, Re(β − γ)]; a is a non positive integer.

Theorem 2.13. *If we start with the second cubic transformation formula of Bailey* $\left[[4],(4.06) \text{with } \rho_1\to b,\rho_2\to 3a-b+\frac{3}{2},x\to x\right]$:

$$(1-x)^{-3a}{}_3F_2\left[\begin{matrix}a,\frac{1}{3}+a,\frac{2}{3}+a\\ b,\frac{3}{2}+3a-b\end{matrix};\frac{27x^2}{4(1-x)^3}\right]={}_3F_2\left[\begin{matrix}3a,\frac{-1}{2}+b,1+3a-b\\ 2b-1,2+6a-2b\end{matrix};4x\right]$$

and assume that a is a non positive integer, then we obtain

$$I\begin{matrix}\alpha,\beta,\eta\\ 0,x\end{matrix}\left[x^{\sigma-1}(1-x)^{-3a}{}_3F_2\left[\begin{matrix}a,\frac{1}{3}+a,\frac{2}{3}+a\\ b,\frac{3}{2}+3a-b\end{matrix};\frac{27x}{4(1-x)^3}\right]\right]$$
$$=\frac{\Gamma\sigma\ \Gamma(\sigma+\eta-\beta)}{\Gamma(\sigma-\beta)\ \Gamma(\sigma+\eta+\alpha)}x^{\sigma-\beta-1}$$
$${}_5F_4\left[\begin{matrix}\sigma,\sigma+\eta-\beta,3a,\frac{-1}{2}+b,1+3a-b\\ \sigma-\beta,\sigma+\beta+\alpha,2b-1,2+6a-2b\end{matrix};4x\right],$$

where Re(α), Re(β) > 0, Re(σ) = max[0, Re(β − γ)]; a is a non positive integer.

Theorem 2.14. *If we start with the transformation formula [[5], Entry 4 of Ramanujan, Ch. 11, p. 50]:*

$$(1-x)^{-a}{}_2F_1\left[\begin{matrix}\frac{a}{2},\frac{1}{2}+\frac{a}{2}\\ \frac{1}{2}+b\end{matrix};\frac{-4x}{(1-x)^2}\right]={}_2F_1\left[\begin{matrix}a,\frac{1}{2}+a-b\\ \frac{1}{2}+b\end{matrix};-x\right]$$

and assume that a is a non positive integer, then we obtain

$$I\begin{matrix}\alpha,\beta,\eta\\ 0,x\end{matrix}\left[x^{\sigma-1}(1-x)^{-a}{}_2F_1\left[\begin{matrix}\frac{a}{2},\frac{1}{2}+\frac{a}{2}\\ \frac{1}{2}+b\end{matrix};\frac{-4x}{(1-x)^2}\right]\right]$$
$$=\frac{\Gamma\sigma\ \Gamma(\sigma+\eta-\beta)}{\Gamma(\sigma-\beta)\ \Gamma(\sigma+\eta+\alpha)}x^{\sigma-\beta-1}{}_4F_3\left[\begin{matrix}\sigma,\sigma+\eta-\beta,a,\frac{1}{2}+a-b\\ \sigma-\beta,\sigma+\beta+\alpha,\frac{1}{2}+b\end{matrix};-x\right],$$

where Re(α), Re(β) > 0, Re(σ) = max[0, Re(β − γ)]; a is a non positive integer.

Theorem 2.15. *If we start with the transformation formula which follows from Theorem VIII of* [[14], (2.5.32)] :

$${}_2F_1\left[\begin{matrix}a,b\\ c\end{matrix};x\right]{}_2F_1\left[\begin{matrix}a,-b+c\\ c\end{matrix};x\right]$$
$$=(1-x)^{-a}{}_4F_3\left[\begin{matrix}a,b,-a+c,-b+c\\ c,\frac{c}{2},\frac{1}{2}+\frac{c}{2}\end{matrix};\frac{-x^2}{4(1-x)}\right]$$

and assume that a is a non positive integer, then we obtain

$$I_{0,x}^{\alpha,\beta,\eta}\left[x^{\sigma-1}(1-x)^{-a}{}_2F_1\left[\begin{matrix}\frac{a}{2},\frac{1}{2}+\frac{a}{2}\\ \frac{1}{2}+b\end{matrix};\frac{-4x}{(1-x)^2}\right]\right]$$
$$=\frac{\Gamma\sigma\;\Gamma(\sigma+\eta-\beta)}{\Gamma(\sigma-\beta)\;\Gamma(\sigma+\eta+\alpha)}x^{\sigma-\beta-1}{}_4F_3\left[\begin{matrix}\sigma,\sigma+\eta-\beta,a,\frac{1}{2}+a-b\\ \sigma-\beta,\sigma+\beta+\alpha,\frac{1}{2}+b\end{matrix};-x\right],$$

where $Re(\alpha)$, $Re(\beta)>0$, $Re(\sigma)=\max[0,Re(\beta-\gamma)]$; a is a non positive integer.

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