International Journal of Pure and Applied Mathematics
Volume 81 No. 6 2012, 871-875
ISSN: 1311-8080 (printed version)
url: http://www.ijpam.eu

ijpam.eu

POINTS FROM WHICH THE PROJECTION OF A CURVE IS VERY STRANGE

E. Ballico
Department of Mathematics
University of Trento
38 123 Povo (Trento) - Via Sommarive, 14, ITALY
e-mail: ballico@science.unitn.it

Abstract: Let $C \subset \mathbb{P}^n$ be an integral curve. Fix an integer $k > 0$. We study the set of all $O \in \mathbb{P}^n$ such that for general $(P_1, \cdots, P_k) \in C^k$ the linear span of $\{O, P_1, \dots, P_k\}$ contains a point $\neq O$ and linearly independent from $\{P_1, \dots, P_k\}$.

AMS Subject Classification: 14N05, 14H50
Key Words: very strange curve, strange curve, linear projection

1. Introduction

Let $C \subset \mathbb{P}^n$, $n \geq 3$, be an integral and non-degenerate curve defined over an algebraically closed field $\mathbb{K}$. For each set $S \subset \mathbb{P}^n$ let $\langle S \rangle$ denote its linear span. For each integer $k \in \{1, \dots, n-1\}$ let $\mathcal{E}(C,k)$ (resp. $\mathcal{E}'(C,k)$, resp. $\mathcal{E}''(C,k)$) denote the set of all $O \in \mathbb{P}^n$ (resp. (resp. $O \in \mathbb{P}^n \setminus C$, resp. $O \in C$) such that for a general $S \subset C$ with $\sharp(S) = k$, the linear space $\langle S \cup \{O\} \rangle$ contains a point not in $\langle S \rangle \cup \{O\}$. For each $O \in \mathbb{P}^n$ let $\ell_O : \mathbb{P}^n \setminus \{O\} \to \mathbb{P}^{n-1}$ denote the linear projection from O. The case $k = 1$ is classical (see [3] and references therein). Let $C \subset \mathbb{P}^n$, $n \geq 3$, be an integral and non-degenerate curve. If $O \notin C$, then $\ell_O|C$ is a morphism and ℓ_O is called an outer projection of C. Both C and $\ell_O(C)$ are irreducible curves and the degree (resp. separable degree) of the

Received: November 16, 2012

map $\ell_O|C : C \to \ell_O(C)$ is called the degree (or separable degree) of the outer projection ℓ_O of C. If $O \in C$, then ℓ_O is called an inner projection of C; the closed subscheme $C_O := \overline{\ell_O(C \setminus \{O\})} \subset \mathbb{P}^{n-1}$ is also called an inner projection of C. The morphism $\ell_O|(C \setminus \{O\})$ induces a rational map $C \dashrightarrow C_O$ and the degree (resp. separable degree) of this rational map is called the degree (resp. separable degree) of the inner projection ℓ_O of C. The separable degree of the inner projection ℓ_O is $a - 1$, where $a := \sharp(D \cap C)$, where D is the line spanned by O and a general point of C. Let $\mathcal{U}(C)$ (resp. $\mathcal{U}'(C)$, resp. $\mathcal{U}''(C)$) denote the set of all $O \in \mathbb{P}^n$ (resp. $O \in \mathbb{P}^n \setminus C$, resp. $O \in C$) such that ℓ_O has separable degree ≥ 1. Hence $\mathcal{U}(C) = \mathcal{E}(C,1)$, $\mathcal{U}'(C) = \mathcal{E}'(C,1)$ and $\mathcal{U}''(C) = \mathcal{E}''(C,1)$. It is an easy exercise to check that $\mathcal{U}'(C)$ is finite if C is not strange (Lemma 1) and that $\mathcal{U}''(C)$ is infinite if and only if C is very strange in the sense of [5] (Remark 1) (in characteristic zero this is the one-dimensional case of [3] (for outer projections) and of [1] and [2] (for inner projections)). In this note we prove the following results:

- If $\mathcal{E}(C,k) \neq \emptyset$ and $k \geq 2$, then $\mathrm{char}(\mathbb{K}) > 0$ (Proposition 2).
- If $\mathcal{E}(C,k) \neq \emptyset$, $k \geq 2$ and C is not strange, then $\mathcal{E}(C,k)$ is contained in a line (Proposition 2).
- For all $n \geq k+2 \geq 4$ there is a non-strange curve $C \subset \mathbb{P}^n$ such that $\mathcal{E}(C,k) \neq \emptyset$ (Proposition 5).

In several cases the curve C_O is very strange in the sense of [5].

2. Statements and Proofs

Proposition 1. *Let $C \subset \mathbb{P}^n$, $n \geq 3$, be an integral and non-degenerate curve.*

(a) If C is not strange, then $\mathcal{U}'(C)$ is finite.

(b) Assume that C is strange with strange point E and that $\mathcal{U}'(C)$ is not finite. Let T be any irreducible component with positive dimension of the projective set $\overline{\mathcal{U}'(C)}$. Then $\dim(T) = 1$, T is not a line, and T is a strange curve with E as its strange point.

Proof. Assume that $\mathcal{U}'(C)$ is infinite. The set $\mathcal{U}'(C)$ is constructible. Hence $\overline{\mathcal{U}'(C)}$ has an irreducible component, T, with positive dimension. Since C has only a two-dimensional family of secant lines, we have $\dim(T) \leq 1$.

Hence $\dim(T) = 1$. Fix a general $O \in T$. Since O is general, we have $O \in \mathcal{U}'(C)$. Set $f := \deg(\ell_O|C)$. Fix a general $P \in C$. Since P is general, we have $P \in C_{reg}$, $\ell_O(P) \in \ell_O(C)_{reg}$ and $\sharp((\ell_O|C)^{-1}(\ell_O(P))) = f$. Since $\ell_O(T_UC) = T_{\ell_O(P)}(\ell_O(P))$ for every $U \in (\ell_O|C)^{-1}(\ell_O(P))$ we get that $f-1$ tangent lines of C meets the tangent line T_PC. Fixing O, but varying O in T, we get that any two general tangent lines of C meets. Hence C is strange (see [4], Proposition IV.3.8, whose proof works even for singular curves). Call E the strange point of C. Fix a general $P \in C$. Since P is general, we have $P \notin T$, $P \in C_{reg}$ and a general secant line of C through P intersects C in $e-1$ further points. Since $P \neq E$, the map $\ell_P|(C \setminus \{P\})$ is separable. Since every secant line of C meets T, we get $C_P \supseteq \ell_P(T)$. Hence $C_P = \ell_P(T)$. Since $\ell_P(E)$ is a strange point of C_P, it is a strange point of $\ell_P(T)$. Hence for any $F, G \in T_{reg}$, the line T_FT is contained in the plane spanned by P and T_GT. Hence $T_GT \cap T_FT \neq \emptyset$. Hence either T is contained in a plane or T is strange. Assume for the moment that Π is a plane containing T. Taking $P \notin \Pi$. Since $\ell_P(E)$ is a strange point of $C_P = \ell_P(T)$, we get that T is strange. Hence T is strange (but perhaps a line). Since $\ell_P(E)$ is a strange point of $\ell_P(T)$ for a general $P \in C$, we get that E is the strange point of T. Assume that T is a line. Let $\ell_T : \mathbb{P}^n \setminus T \to \mathbb{P}^{n-2}$ denote the linear projection from T. Since C is non-degenerate, the irreducible set $\overline{\ell_T(C \setminus C \cap T)}$ spans $\mathbb{P}^2$. Since every secant line of C meets T, $\overline{\ell_T(C \setminus C \cap T)}$ is a point. Hence $n = 2$, a contradiction. □

Remark 1. The set $\mathcal{U}''(C)$ is infinite if and only if a general secant line of C is a multisecant line of C. Hence if $\mathcal{U}''(C)$ is infinite, then a general hyperplane section of C is not in linearly general position. In this case $p := \mathrm{char}(\mathbb{K}) > 0$ and C is a strange curve (see [5], Lemma 1.1).

Proposition 2. *Assume $n - 1 > k \geq 2$. Assume that $C \subset \mathbb{P}^n$ is not strange and that $\mathcal{E}(C, k) \neq \emptyset$. Then* $\mathrm{char}(\mathbb{K}) > 0$, *there is a line $D \subset \mathbb{P}^n$ such that $\mathcal{E}(C, k) \subseteq D$ and every tangent line of C_{reg} meets D.*

Proof. Fix $O \in \mathcal{E}(C, k)$. Set $C_O := \overline{\ell_O(C \setminus \{O\})} \subset \mathbb{P}^{n-1}$. Since $=\in$ $\mathcal{E}(C, k)$, the linear span of k general points of C_O contains at least another point of C, i.e. C_O is very strange in the sense of [5]. Hence C_O is strange (see [5], Lemma 1.1). Hence $\mathrm{char}(\mathbb{K}) > 0$. Let $D \subset \mathbb{P}^n$ be the only line containing O such that $\ell_O(D \setminus \{O\})$ is the strange point of C_O. Since $\ell_O(D \setminus \{O\})$ is the strange point of C_O, every tangent line of $(C_O)_{reg}$ contains $\ell_O(D \setminus \{O\})$. Hence a general tangent lines of C meets D. Since C is not strange, a general point of D is contained in infinitely many tangent lines of C. Since $n \geq 4$ and C is not strange, there is no line $D' \neq D$ such that a general tangent line of C meets

D'. Hence $\mathcal{E}(C,k) \subseteq D$. □

Corollary 1. *Assume* $n-1 > k \ge 2$ *and that* $C \subset \mathbb{P}^n$ *is not strange. Then* $\sharp(\mathcal{E}''(C,k)) \le \deg(C) - n + 2$.

Proof. Take D as in Proposition 2. Since C is non-degenerate, we have $\sharp(C \cap D) \le \deg(C) - n + 2$. □

Proposition 3. *Assume* $n \ge 5$, $\deg(C) > 24$ *and that* C *is very strange. Then a general* $O \in C$ *is contained in* $\mathcal{E}''(C,k)$ *for each integer* $k \in \{3,\dots,n-2\}$.

Proof. Fix $k \in \{3,\dots,n-2\}$ and a general $(O, P_1,\dots,P_k) \in C^{k+1}$. To prove Proposition 3 we need to prove the existence $P \in C \cap \langle\{O, P_1,\dots,P_k\}\rangle$ such that $P \notin \langle\{P_1,\dots,P_k\}\rangle$ and $P \ne O$. Since $k+1 \ge 4$, there is $P \in (C \setminus \{O, P_1,\dots,P_k\}) \cap \langle\{O, P_1,\dots,P_k\}\rangle$ (see [5], Theorem 2.5). Hence to prove Proposition 3 it is sufficient to prove the existence of P as above with the further condition that $P \notin \langle\{P_1,\dots,P_k\}\rangle$ and $P \notin O$. Let s be the minimal integer ≥ 2 such that the linear span of s general points of C contains another point of C. If $s = k+1$, then we are done. Hence we may assume $s \le k$. For each $x \in \{1,\dots,k+1\}$ let $a(x)$ the number of points of C in the linear span of x general points of C. We need to prove that $a(k+1) \ge a(k)+2$. This is an easy exercise using that either $a(2) \ge 3$ or $a(2) = 2$ and $a(3) \ge 4$ by [5], Theorem 2.5. □

Proposition 4. *Take* C *as in Proposition 2 and fix* $O \in \mathcal{E}(C,k)$. *If* $\deg(C_O) > 24$, *then the minimal integer* $y \in \{2,\dots,n-2\}$ *such that* $O \in \mathcal{E}(C,y)$ *is either* 2 *or* 3 *and* $O \in \mathcal{E}(C,x) \ne \emptyset$ *for all* $x \in \{y,\dots,n-2\}$.

Proof. We saw in the proof of Proposition 2 that C_O is very strange. For each $x \in \{2,\dots,n-3\}$, let $b(x)$ the number of points of C contained in the linear span of x general point of C_O. By [5], Theorem 2.5, we have $2 \le s \le 3$. It is easy to check that $b(x+1) \ge b(x)+2$ for all $x \in \{s,\dots,n-3\}$. By the definition of C_O we get $y = s$ and $O \in \mathcal{E}(C,x)$ for all $x \in \{s,\dots,n-2\}$. □

Proposition 5. *Assume* $p := \mathrm{char}(\mathbb{K}) > 0$ *and fix integers* $n \ge k+2 \ge 4$. *Then there is a non-strange non-degenerate curve* $C \subset \mathbb{P}^n$ *such that* $\mathcal{E}'(C,k) \ne \emptyset$.

Proof. Fix a p-power $q > 2$ and let $Y \subset \mathbb{P}^{n-1}$ the complete interrsection curve of $n-2$ hypersurfaces of degree q constructed in [5], Example 1.2. The curve Y is rational, i.e. it is the linear projection of a rational normal curve $D \subset \mathbb{P}^r$, $r := q^{n-2}$, by a linear subspace $W \subset \mathbb{P}^r$ such that $\dim(W) = r-n$ and

$W \cap D = \emptyset$. Since Y is strange, there is an $(r-n)$-dimensional linear subspace $V \subset \mathbb{P}^r$ such that $W \subset V$ and every tangent line of D meets V. Let $V' \subseteq V$ be the minimal subspace of V such that a general tangent line meets V'. Fix a general codimension 1 linear subspace M of W and let $C \subset \mathbb{P}^n$ be the image of D by the linear projection $\ell_M : \mathbb{P}^r \setminus M \to \mathbb{P}^n$ from M. Since $\deg(Y) = \deg(D)$, we have $\deg(C) = \deg(D)$. Since M is a codimension 1-subspace of W, $\ell_M(W \setminus W)$ is a single point, O and $C_O = Y$. Since $\deg(Y) = \deg(D)$, we have $O \notin C$. Since ℓ_W is birational onto its image, V' is not contained in W. Since M is a general codimension 1 linear subspace of W, V' is not contained in M. Hence C is not strange.For each $x \in \{2, \dots, n-2\}$, the linear span of x general points of Y contains exactly q^x points of Y. Since $Y = C_O$, we have $O \in \mathcal{E}(C,k)$ for all $k \in \{2, \dots, n-2\}$. $\square$

Acknowledgements

The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

References

[1] E. Ballico, Special projections of projective varieties, *International Mathematical Journal*, **3**, No. 1 (2003), 11-12.

[2] E. Ballico, Special inner projections of projective varieties, *Ann. Univ. Ferrara, Sez. VII – Sc. Mat.*, **50** (2004), 23-26.

[3] A. Calabri, C. Ciliberto, On special projections of varieties: Epitome to a theorem of Beniamino Segre, *Adv. Geom.*, **1**, No. 1 (2001), 97-106.

[4] R. Hartshorne, *Algebraic Geometry*, Springer-Verlag, Berlin (1977).

[5] J. Rathmann, The uniform position principle for curves in characteristic p, *Math. Ann.*, **276**, No. 4 (1987), 565-579.