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DIFFERENTIAL GEOMETRY OF MICROLINEAR FRÖLICHER SPACES IV-2

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Abstract: This paper is the sequel to our previous paper (Differetial Geometry of Microlinear Frölicher spaces IV-1), where three approaches to jet bundles are presented and compared. The first objective in this paper is to give the affine bundle theorem for the second and third approaches to jet bundles. The second objective is to deal with the three approaches to jet bundles in the context where coordinates are available. In this context all the three approaches are shown to be equivalent.

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1. Introduction

The principal objectives in this second part of the paper are firstly to deal with the affine bundle theorem in the second and third approaches to jet bundles and secondly to treat the three approaches to jet bundles in [4] within the context where coordinates are available, namely, $E = \mathbb{R}^{p+q}$, $M = \mathbb{R}^p$, and π is the canonical projection. §3 is devoted to the affine bundle theorem. We let i (j, resp.) range over the natural numbers between 1 and p (between 1 and q, resp.), including the endpoints. It is shown that, within this traditional

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context, the three approaches are essentially equivalent. The traditional coordinate approach to jet bundles is given a noble description after the manner of [1] in §5, where the affine bundle theorem is established on these lines. Our three approaches are related to this traditional approach in §6, §7 and §8 in order. In particular, the affine bundles in the second and third approaches are shown to be isomorphic to that in the traditional approach.

2. Previous Results

We collect a few results of our previous paper [4] to be quoted in this paper.

Definition 1. Let n be a natural number. A D^n-pseudotangential *over* the bundle $\pi : E \to M$ *at* $x \in E$ is a mapping $\nabla_x : (M \otimes \mathcal{W}_{D^n})_{\pi(x)} \to (E \otimes \mathcal{W}_{D^n})_x$ abiding by the following conditions:

1. We have
$$(\pi \otimes \mathrm{id}_{\mathcal{W}_{D^n}})(\nabla_x(\gamma)) = \gamma$$
for any $\gamma \in (M \otimes \mathcal{W}_{D^n})_{\pi(x)}$.

2. We have
$$\nabla_x(\alpha \underset{i}{\cdot} \gamma) = \alpha \underset{i}{\cdot} \nabla_x(\gamma) \qquad (1 \leq i \leq n)$$
for any $\gamma \in (M \otimes \mathcal{W}_{D^n})_{\pi(x)}$ and any $\alpha \in \mathbb{R}$.

3. The diagram
$$\begin{array}{ccc} (M \otimes \mathcal{W}_{D^n})_{\pi(x)} & \to & (M \otimes \mathcal{W}_{D^n})_{\pi(x)} \otimes \mathcal{W}_{D_m} \\ \nabla_x \downarrow & & \downarrow \nabla_x \otimes \mathrm{id}_{\mathcal{W}_{D_m}} \\ (E \otimes \mathcal{W}_{D^n})_x & \to & (E \otimes \mathcal{W}_{D^n})_x \otimes \mathcal{W}_{D_m} \end{array}$$
is commutative, where m is an arbitrary natural number, the upper horizontal arrow is
$$\mathrm{id}_M \otimes \mathcal{W}_{(d_1,\ldots,d_n,e) \in D^n \times D_m \mapsto (d_1,\ldots,d_{i-1},ed_i,d_{i+1},\ldots d_n) \in D^n},$$
and the lower horizontal arrow is
$$\mathrm{id}_E \otimes \mathcal{W}_{(d_1,\ldots,d_n,e) \in D^n \times D_m \mapsto (d_1,\ldots,d_{i-1},ed_i,d_{i+1},\ldots d_n) \in D^n}.$$

4. We have
$$\nabla_x(\gamma^\sigma) = (\nabla_x(\gamma))^\sigma$$
for any $\gamma \in (M \otimes \mathcal{W}_{D^n})_{\pi(x)}$ and for any $\sigma \in \mathbf{S}_n$.

Definition 2. The notion of a D^n-tangential *over* the bundle $\pi : E \to M$ *at* x is defined by induction on n. The notion of a D-tangential *over* the bundle $\pi : E \to M$ *at* x shall be identical with that of a D-pseudotangential *over* the bundle $\pi : E \to M$ *at* x . Now we proceed inductively. A D^{n+1}-pseudotangential

$$\nabla_x : (M \otimes \mathcal{W}_{D^{n+1}})_{\pi(x)} \to (E \otimes \mathcal{W}_{D^{n+1}})_x$$

over the bundle $\pi : E \to M$ at $x \in E$ is called a D^{n+1}-tangential *over* the bundle $\pi : E \to M$ *at* x if it acquiesces in the following two conditions:

1. $\widehat{\pi}_{n+1,n}(\nabla_x)$ is a D^n-tangential *over* the bundle $\pi : E \to M$ *at* x.

2. For any $\gamma \in (M \otimes \mathcal{W}_{D^n})_{\pi(x)}$, we have
$$\begin{aligned}&\nabla_x \left(\left(\mathrm{id}_M \otimes \mathcal{W}_{(d_1,\ldots,d_n,d_{n+1}) \in D^{n+1} \mapsto (d_1,\ldots,d_n d_{n+1}) \in D^n}\right)(\gamma)\right)\\ &= \left(\mathrm{id}_E \otimes \mathcal{W}_{(d_1,\ldots,d_n,d_{n+1}) \in D^{n+1} \mapsto (d_1,\ldots,d_n d_{n+1}) \in D^{n+1}}\right)\left(\left(\widehat{\pi}_{n+1,n}(\nabla_x)\right)(\gamma)\right)\end{aligned}$$

Proposition 3. *Let m, n be natural numbers with $m \leq n$. Let $k_1, ..., k_m$ be positive integers with $k_1 + ... + k_m = n$. For any $\nabla_x \in \mathbb{J}^{D^n}(\pi)$, any $\gamma \in (M \otimes \mathcal{W}_{D^m})_{\pi(x)}$ and any $\sigma \in \mathbf{S}_n$, we have*

$$\begin{aligned}&\nabla_x \left(\left(\mathrm{id}_M \otimes \mathcal{W}_{(d_1,\ldots,d_n) \in D^n \mapsto \left(d_{\sigma(1)} \ldots d_{\sigma(k_1)}, d_{\sigma(k_1+1)} \ldots d_{\sigma(k_1+k_2)}, \ldots, d_{\sigma(k_1+\ldots+k_{m-1}+1)} \ldots d_{\sigma(n)}\right)}\right)(\gamma)\right)\\ &= \left(\mathrm{id}_E \otimes \mathcal{W}_{(d_1,\ldots,d_n) \in D^n \mapsto \left(d_{\sigma(1)} \ldots d_{\sigma(k_1)}, d_{\sigma(k_1+1)} \ldots d_{\sigma(k_1+k_2)}, \ldots, d_{\sigma(k_1+\ldots+k_{m-1}+1)} \ldots d_{\sigma(n)}\right)}\right)\\ &\qquad\left(\left(\pi_{n,m}(\nabla_x)\right)(\gamma)\right).\end{aligned}$$

Proposition 4. *The diagram*

$$\begin{array}{ccc} \hat{\mathbb{J}}_x^{D^{n+1}}(\pi) & \xrightarrow{\widehat{\psi}_{n+1}} & \hat{\mathbb{J}}_x^{D_{n+1}}(\pi) \\ \widehat{\pi}_{n+1,n} \downarrow & & \downarrow \widehat{\pi}_{n+1,n} \\ \hat{\mathbb{J}}_x^{D^n}(\pi) & \overrightarrow{\widehat{\psi}_n} & \hat{\mathbb{J}}_x^{D_n}(\pi) \end{array}$$

commutes.

3. The Affine Bundle Theorem

3.1. The Theorem in the Second Approach

3.1.1. Affine Bundles

Lemma 5. *The diagram*

$$\begin{array}{ccc} D\{n\}_{n-1} & \xrightarrow{i_{D\{n\}_{n-1}\to D^n}} & D^n \\ i_{D\{n\}_{n-1}\to D^n}\downarrow & & \downarrow \Psi_{D^n} \\ D^n & \xrightarrow{\Phi_{D^n}} & D^n \oplus D \end{array}$$

is a quasi-colimit diagram, where $i_{D\{n\}_{n-1}\to D^n}$ *is the canonical injection of* $D\{n\}_{n-1}$ *into* D^n*, and*

$$\Phi_{D^n}(d_1, ..., d_n) = (d_1, ..., d_n, 0)$$
$$\Psi_{D^n}(d_1, ..., d_n) = (d_1, ..., d_n, d_1...d_n).$$

This implies directly that

Proposition 6. *Given* $\gamma_+, \gamma_- \in M \otimes \mathcal{W}_{D^n}$ *with*

$$\left(\mathrm{id}_M \otimes \mathcal{W}_{i_{D\{n\}_{n-1}\to D^n}}\right)(\gamma_+) = \left(\mathrm{id}_M \otimes \mathcal{W}_{i_{D\{n\}_{n-1}\to D^n}}\right)(\gamma_-),$$

there exists unique $\gamma \in M \otimes \mathcal{W}_{D^n \oplus D}$ *with*

$$(\mathrm{id}_M \otimes \mathcal{W}_{\Psi_{D^n}})(\gamma) = \gamma_+ \text{ and}$$
$$(\mathrm{id}_M \otimes \mathcal{W}_{\Phi_{D^n}})(\gamma) = \gamma_-$$

Notation 7. Under the same notation as in the above proposition, we denote

$$(\mathrm{id}_M \otimes \mathcal{W}_{\Xi_{D^n}})(\gamma)$$

by $\gamma_+ \dot{-} \gamma_-$, where $\Xi_{D^n} : D \to D^n \oplus D$ is the mapping

$$d \in D \mapsto (0, ..., 0, d) \in D^n \oplus D$$

From the very definition of $\dot{-}$, we have

Proposition 8. *Let F be a mapping of M into M'. Given $\gamma_+, \gamma_- \in M \otimes \mathcal{W}_{D^n}$ with*

$$\left(\mathrm{id}_M \otimes \mathcal{W}_{i_{D\{n\}_{n-1} \to D^n}}\right)(\gamma_+) = \left(\mathrm{id}_M \otimes \mathcal{W}_{i_{D\{n\}_{n-1} \to D^n}}\right)(\gamma_-),$$

we have

$$\left(\mathrm{id}_{M'} \otimes \mathcal{W}_{i_{D\{n\}_{n-1} \to D^n}}\right)((F \otimes \mathrm{id}_{\mathcal{W}_{D^n}})(\gamma_+)) = \left(\mathrm{id}_{M'} \otimes \mathcal{W}_{i_{D\{n\}_{n-1} \to D^n}}\right)((F \otimes \mathrm{id}_{\mathcal{W}_{D^n}})(\gamma_-))$$

and

$$(F \otimes \mathrm{id}_{\mathcal{W}_D})\left(\gamma_+ \dot{-} \gamma_-\right) = (F \otimes \mathrm{id}_{\mathcal{W}_{D^n}})(\gamma_+) \dot{-} (F \otimes \mathrm{id}_{\mathcal{W}_{D^n}})(\gamma_-)$$

Lemma 9. *The diagram*

$$\begin{array}{ccc} 1 & \xrightarrow{i_{1 \to D}} & D \\ i_{1 \to D^n} \downarrow & & \downarrow \Xi_{D^n} \\ D^n & \xrightarrow{\Phi_{D^n}} & D^n \oplus D \end{array}$$

is a quasi-colimit diagram, where $i_{1 \to D^n}$ is the canonical injection of 1 into D^n and $i_{1 \to D}$ is the canonical injection of 1 into D.

This implies directly that

Proposition 10. *Given $t \in M \otimes \mathcal{W}_D$ and $\gamma \in M \otimes \mathcal{W}_{D^n}$ with*

$$(\mathrm{id}_M \otimes \mathcal{W}_{i_{1 \to D}})(t) = (\mathrm{id}_M \otimes \mathcal{W}_{i_{1 \to D^n}})(\gamma),$$

there exists unique $\gamma' \in M \otimes \mathcal{W}_{D^n \oplus D}$ with

$$(\mathrm{id}_M \otimes \mathcal{W}_{\Xi_{D^n}})\left(\gamma'\right) = t \text{ and}$$
$$(\mathrm{id}_M \otimes \mathcal{W}_{\Phi_{D^n}})\left(\gamma'\right) = \gamma.$$

Notation 11. Under the same notation as in the above proposition, we denote

$$(\mathrm{id}_M \otimes \mathcal{W}_{\Psi_{D^n}})\left(\gamma'\right)$$

by $t \dot{+} \gamma$, where Ψ_{D^n} is as in Lemma 5

From the very definition of $\dot{+}$, we have

Proposition 12. *Let F be a mapping of M into M'. Given $t \in M \otimes \mathcal{W}_D$ and $\gamma \in M \otimes \mathcal{W}_{D^n}$ with*

$$\left(\mathrm{id}_M \otimes \mathcal{W}_{i_{1\to D}}\right)(t) = \left(\mathrm{id}_M \otimes \mathcal{W}_{i_{1\to D^n}}\right)(\gamma),$$

we have

$$\left(\mathrm{id}_{M'} \otimes \mathcal{W}_{i_{1\to D}}\right)\left(\left(F \otimes \mathrm{id}_{\mathcal{W}_D}\right)(t)\right) = \left(\mathrm{id}_{M'} \otimes \mathcal{W}_{i_{1\to D^n}}\right)\left(\left(F \otimes \mathrm{id}_{\mathcal{W}_{D^n}}\right)(\gamma)\right)$$

and

$$\left(F \otimes \mathrm{id}_{\mathcal{W}_{D^n}}\right)\left(t \dot{+} \gamma\right) = \left(F \otimes \mathrm{id}_{\mathcal{W}_D}\right)(t) \dot{+} \left(F \otimes \mathrm{id}_{\mathcal{W}_{D^n}}\right)(\gamma).$$

We can proceed as in §§3.4 of [3] to get

Theorem 13. *The canonical projection $\mathrm{id}_M \otimes \mathcal{W}_{i_{D\{n\}_{n-1}\to D^n}} : M \otimes \mathcal{W}_{D^n} \to M \otimes \mathcal{W}_{D\{n\}_{n-1}}$ is an affine bundle over the vector bundle $(M \otimes \mathcal{W}_D) \underset{M}{\times} \left(M \otimes \mathcal{W}_{D\{n\}_{n-1}}\right) \to M \otimes \mathcal{W}_{D\{n\}_{n-1}}$.*

We have the following n-dimensional counterparts of Propositions 5, 6 and 7 in §§3.4 of [3].

Proposition 14. *For any $\alpha \in \mathbb{R}$, any $\gamma_+, \gamma_-, \gamma \in M \otimes \mathcal{W}_{D^n}$ and any $t \in M \otimes \mathcal{W}_D$ with*

$$\left(\mathrm{id}_M \otimes \mathcal{W}_{i_{D\{n\}_{n-1}\to D^n}}\right)(\gamma_+) = \left(\mathrm{id}_M \otimes \mathcal{W}_{i_{D\{n\}_{n-1}\to D^n}}\right)(\gamma_-)$$

and

$$\left(\mathrm{id}_M \otimes \mathcal{W}_{i_{1\to D}}\right)(t) = \left(\mathrm{id}_M \otimes \mathcal{W}_{i_{1\to D^n}}\right)(\gamma),$$

we have

$$\alpha(\gamma_+ \dot{-} \gamma_-) = (\alpha \underset{i}{\cdot} \gamma_+) \dot{-} (\alpha \underset{i}{\cdot} \gamma_-)$$
$$\alpha \underset{i}{\cdot} (t \dot{+} \gamma) = \alpha t \dot{+} \alpha \underset{i}{\cdot} \gamma$$

Proposition 15. *The diagrams*

$$\begin{array}{ccc}(M\otimes\mathcal{W}_{D^n})\underset{M\otimes\mathcal{W}_{D\{n\}_{n-1}}}{\times}(M\otimes\mathcal{W}_{D^n}) & \rightarrow & M\otimes\mathcal{W}_D\\ \downarrow_i & & \downarrow\\ \left((M\otimes\mathcal{W}_{D^n})\underset{M\otimes\mathcal{W}_{D\{n\}_{n-1}}}{\times}(M\otimes\mathcal{W}_{D^n})\right)\otimes\mathcal{W}_{D_m} & \rightarrow & (M\otimes\mathcal{W}_D)\otimes\mathcal{W}_{D_m}\end{array}\quad(1\leq i\leq n)$$

$$\begin{array}{ccc}(M\otimes\mathcal{W}_D)\underset{M}{\times}(M\otimes\mathcal{W}_{D^n}) & \rightarrow & M\otimes\mathcal{W}_{D^n}\\ \downarrow_i & & \downarrow_i\\ \left((M\otimes\mathcal{W}_D)\underset{M}{\times}(M\otimes\mathcal{W}_{D^n})\right)\otimes\mathcal{W}_{D_m} & \rightarrow & (M\otimes\mathcal{W}_{D^n})\otimes\mathcal{W}_{D_m}\end{array}\quad(1\leq i\leq n)$$

are commutative, where

1. *In the former diagram, the lower horizontal arrow represents*

$$\left((\gamma_+,\gamma_-)\in(M\otimes\mathcal{W}_{D^n})\underset{M\otimes\mathcal{W}_{D\{n\}_{n-1}}}{\times}(M\otimes\mathcal{W}_{D^n})\mapsto(\gamma_+\dot{-}\gamma_-)\in M\otimes\mathcal{W}_D\right)$$
$$\otimes\,\mathrm{id}_{\mathcal{W}_{D_m}},$$

the upper horizontal arrow represents

$$(\gamma_+,\gamma_-)\in(M\otimes\mathcal{W}_{D^n})\underset{M\otimes\mathcal{W}_{D\{n\}_{n-1}}}{\times}(M\otimes\mathcal{W}_{D^n})\mapsto(\gamma_+\dot{-}\gamma_-)\in M\otimes\mathcal{W}_D,$$

the left vertical arrow represents the composition of mappings

$$(M\otimes\mathcal{W}_{D^n})\underset{M\otimes\mathcal{W}_{D\{n\}_{n-1}}}{\times}(M\otimes\mathcal{W}_{D^n})$$
$$\xrightarrow{\frac{\left(\mathrm{id}_M\otimes\mathcal{W}_{(d_1,\ldots,d_n,e)\in D^n\times D_m\mapsto(d_1,\ldots,ed_i,\ldots,d_n)\in D^n}\right)\times}{\left(\mathrm{id}_M\otimes\mathcal{W}_{(d_1,\ldots,d_n,e)\in D^n\times D_m\mapsto(d_1,\ldots,ed_i,\ldots,d_n)\in D^n}\right)}}$$
$$(M\otimes\mathcal{W}_{D^n\times D_m})\underset{M\otimes\mathcal{W}_{D\{n\}_{n-1}\times D_m}}{\times}(M\otimes\mathcal{W}_{D^n\times D_m})$$
$$=((M\otimes\mathcal{W}_{D^n})\otimes\mathcal{W}_{D_m})\underset{\left(M\otimes\mathcal{W}_{D\{n\}_{n-1}}\right)\otimes\mathcal{W}_{D_m}}{\times}((M\otimes\mathcal{W}_{D^n})\otimes\mathcal{W}_{D_m})$$
$$=\left((M\otimes\mathcal{W}_{D^n})\underset{M\otimes\mathcal{W}_{D\{n\}_{n-1}}}{\times}(M\otimes\mathcal{W}_{D^n})\right)\otimes\mathcal{W}_{D_m},$$

and the right vertical arrow represents the composition of mappings

$$M\otimes\mathcal{W}_D\xrightarrow{\mathrm{id}_M\otimes\mathcal{W}_{(d,e)\in D\times D_m\mapsto de\in D}}M\otimes\mathcal{W}_{D\times D_m}=(M\otimes\mathcal{W}_D)\otimes\mathcal{W}_{D_m};$$

2. *In the latter diagram, the lower horizontal arrow represents*

$$\left((t,\gamma)\in(M\otimes\mathcal{W}_D)\underset{M}{\times}(M\otimes\mathcal{W}_{D^n})\mapsto t\dot{+}\gamma\in M\otimes\mathcal{W}_{D^n}\right)$$
$$\otimes\,\mathrm{id}_{\mathcal{W}_{D_m}},$$

the upper horizontal arrow represents

$$(t,\gamma)\in(M\otimes\mathcal{W}_D)\underset{M}{\times}(M\otimes\mathcal{W}_{D^n})\mapsto t\dot{+}\gamma\in M\otimes\mathcal{W}_{D^n},$$

the left vertical arrow represents the composition of mappings

$$(M\otimes\mathcal{W}_D)\underset{M}{\times}(M\otimes\mathcal{W}_{D^n})$$
$$\xrightarrow{\left(\mathrm{id}_M\otimes\mathcal{W}_{(d,e)\in D\times D_m\mapsto ed\in D}\right)\times\left(\mathrm{id}_M\otimes\mathcal{W}_{(d_1,\dots,d_n,e)\in D^n\times D_m\mapsto(d_1,\dots,ed_i,\dots,d_n)\in D^n}\right)}$$
$$(M\otimes\mathcal{W}_{D\times D_m})\underset{M\otimes\mathcal{W}_{D\{n\}_{n-1}\times D_m}}{\times}(M\otimes\mathcal{W}_{D^n\times D_m})$$
$$=((M\otimes\mathcal{W}_D)\otimes\mathcal{W}_{D_m})\underset{\left(M\otimes\mathcal{W}_{D\{n\}_{n-1}}\right)\otimes\mathcal{W}_{D_m}}{\times}((M\otimes\mathcal{W}_{D^n})\otimes\mathcal{W}_{D_m})$$
$$=\left((M\otimes\mathcal{W}_D)\underset{M\otimes\mathcal{W}_{D\{n\}_{n-1}}}{\times}(M\otimes\mathcal{W}_{D^n})\right)\otimes\mathcal{W}_{D_m},$$

and the right vertical arrow represents the composition of mappings

$$M\otimes\mathcal{W}_{D^n}\xrightarrow{\mathrm{id}_M\otimes\mathcal{W}_{(d_1,\dots,d_n,e)\in D^n\times D_m\mapsto(d_1,\dots,ed_i,\dots,d_n)\in D^n}}M\otimes\mathcal{W}_{D^n\times D_m}$$
$$=(M\otimes\mathcal{W}_{D^n})\otimes\mathcal{W}_{D_m}.$$

Proposition 16. *For any $\sigma\in\mathbf{S}_n$, any $\gamma_+,\gamma_-,\gamma\in M\otimes\mathcal{W}_{D^n}$ and any $t\in M\otimes\mathcal{W}_D$ with*

$$\left(\mathrm{id}_M\otimes\mathcal{W}_{i_{D\{n\}_{n-1}\to D^n}}\right)(\gamma_+)=\left(\mathrm{id}_M\otimes\mathcal{W}_{i_{D\{n\}_{n-1}\to D^n}}\right)(\gamma_-)$$

and

$$(\mathrm{id}_M\otimes\mathcal{W}_{i_{1\to D}})(t)=(\mathrm{id}_M\otimes\mathcal{W}_{i_{1\to D^n}})(\gamma),$$

we have

$$(\gamma_+)^\sigma\dot{-}(\gamma_-)^\sigma=\gamma_+\dot{-}\gamma_-$$
$$(t\dot{+}\gamma)^\sigma=t\dot{+}\gamma^\sigma.$$

Proposition 17. *For $\gamma_+, \gamma_- \in M \otimes \mathcal{W}_{D^n}$ with*

$$\left(\mathrm{id}_M \otimes \mathcal{W}_{i_{D\{n\}_{n-1} \to D^n}}\right)(\gamma_+) = \left(\mathrm{id}_M \otimes \mathcal{W}_{i_{D\{n\}_{n-1} \to D^n}}\right)(\gamma_-),$$

we have

$$\begin{aligned}&\left(\mathrm{id}_M \otimes \mathcal{W}_{(d_1,...,d_n) \in D^n \mapsto d_1 ... d_n \in D}\right)\left(\gamma_+ \dot{-} \gamma_-\right) \\ &= (...(\gamma_+ \underset{1}{-} \gamma_-) \underset{2}{-} \mathbf{s}_1 \circ \mathbf{d}_1(\gamma_+)) \underset{3}{-} \mathbf{s}_1^2 \circ \mathbf{d}_1^2(\gamma_+))... \underset{n}{-} \mathbf{s}_1^{n-1} \circ \mathbf{d}_1^{n-1}(\gamma_+))\end{aligned}$$

3.1.2. Symmetric Forms

Definition 18. A *symmetric D^n-form at $x \in E$* is a mapping $\omega_x : (M \otimes \mathcal{W}_{D^n})_{\pi(x)} \to (E \otimes \mathcal{W}_D)_x^{\perp}$ subject to the following conditions:

1. We have
$$\omega_x(\alpha \underset{i}{\cdot} \gamma) = \alpha \omega_x(\gamma) \qquad (1 \leq i \leq n)$$
for any $\gamma \in (M \otimes \mathcal{W}_{D^n})_{\pi(x)}$ and any $\alpha \in \mathbb{R}$.

2. The diagram
$$\begin{array}{ccc} (M \otimes \mathcal{W}_{D^n})_{\pi(x)} & \rightarrow & (M \otimes \mathcal{W}_{D^n})_{\pi(x)} \otimes \mathcal{W}_{D_m} \\ \omega_x \downarrow & & \downarrow \omega_x \otimes \mathrm{id}_{\mathcal{W}_{D_m}} \\ (E \otimes \mathcal{W}_D)_x & \xrightarrow[\mathrm{id}_E \otimes \mathcal{W}_{\times_{D \times D_m \to D}}]{} & (E \otimes \mathcal{W}_D)_x \otimes \mathcal{W}_{D_m} \end{array} \qquad (1 \leq i \leq n)$$
is commutative, where the upper horizontal arrow is
$$\mathrm{id}_M \otimes \mathcal{W}_{(d_1,...,d_n,e) \in D^n \times D_m \mapsto (d_1,...,d_{i-1},ed_i,d_{i+1},...d_n) \in D^n}.$$

3. We have
$$\omega_x(\gamma^\sigma) = \omega_x(\gamma)$$
for any $\gamma \in (M \otimes \mathcal{W}_{D^n})_{\pi(x)}$ and any $\sigma \in \mathbf{S}_n$.

4. We have
$$\omega_x\left(\left(\mathrm{id}_M \otimes \mathcal{W}_{(d_1,...,d_n) \in D^n \longmapsto (d_1,...,d_{n-2},d_{n-1}d_n) \in D^{n-1}}\right)(\gamma)\right) = 0$$
for any $\gamma \in (M \otimes \mathcal{W}_{D^{n-1}})_{\pi(x)}$.

Notation 19. We denote by $\mathbb{S}_x^{D^n}(\pi)$ the totality of symmetric D^n-*forms* at $x \in E$. We denote by $\mathbb{S}^{D^n}(\pi)$ the set-theoretic union of $\mathbb{S}_x^{D^n}(\pi)$'s for all $x \in E$. The canonical projection $\mathbb{S}^{D^n}(\pi) \to E$ is obviously a vector bundle.

Proposition 20. *Let* $\omega \in \mathbb{S}_x^{D^{n+1}}(\pi)$*. Then we have*

$$\omega(\mathbf{s}_i(\gamma)) = 0 \qquad (1 \leq i \leq n+1)$$

for any $\gamma \in (M \otimes \mathcal{W}_{D^n})_{\pi(x)}$.

Proof. For any $\alpha \in \mathbb{R}$, we have

$$\omega(\mathbf{s}_i(\gamma)) = \omega(\alpha \underset{i}{\cdot} \mathbf{s}_i(\gamma)) = \alpha\omega(\mathbf{s}_i(\gamma))$$

Letting $\alpha = 0$, we have the desired conclusion. □

3.1.3. The Theorem

The following proposition will be used in the proof of Proposition 3.6.

Proposition 21. *Let* $\nabla_x \in \mathbb{J}_x^{D^n}(\pi)$*,* $t \in (M \otimes \mathcal{W}_D)_{\pi(x)}$ *and* $\gamma, \gamma_+, \gamma_- \in (M \otimes \mathcal{W}_{D^n})_{\pi(x)}$ *with*

$$\left(\mathrm{id}_M \otimes \mathcal{W}_{i_{D\{n\}_{n-1} \to D^n}}\right)(\gamma_+) = \left(\mathrm{id}_M \otimes \mathcal{W}_{i_{D\{n\}_{n-1} \to D^n}}\right)(\gamma_-).$$

Then we have

$$\nabla_x(\gamma_+) \dot{-} \nabla_x(\gamma_-) = \left(\underline{\pi}_{n,1}(\nabla_x)\right)(\gamma_+ \dot{-} \gamma_-)$$
$$(\pi_{n,1}(\nabla_x))(t) \dot{+} \nabla_x(\gamma) = \nabla_x(t \dot{+} \gamma)$$

Proof. It is an easy exercise of affine geometry to show that the coveted two formulas are equivalent. Here we deal only with the former in case of $n = 2$, leaving the general treatment safely to the reader. We have

$$\left(\mathrm{id}_E \otimes \mathcal{W}_{(d_1,d_2) \in D^2 \mapsto d_1 d_2 \in D}\right)(\nabla_x(\gamma_+) \dot{-} \nabla_x(\gamma_-))$$
$$= \left(\nabla_x(\gamma_+) \underset{1}{-} \nabla_x(\gamma_-)\right) \underset{2}{-} (\mathbf{s}_1 \circ \mathbf{d}_1)(\nabla_x(\gamma_+))$$

[By Proposition 17]

$$= \nabla_x((\gamma_+ \underset{1}{-} \gamma_-) \underset{2}{-} (\mathbf{s}_1 \circ \mathbf{d}_1)(\gamma_+))$$
$$= \nabla_x\left(\left(\mathrm{id}_M \otimes \mathcal{W}_{(d_1,d_2) \in D^2 \mapsto d_1 d_2 \in D}\right)(\gamma_+ \dot{-} \gamma_-)\right)$$

[By Proposition 17]

$$= \left(\mathrm{id}_E \otimes \mathcal{W}_{(d_1,d_2) \in D^2 \mapsto d_1 d_2 \in D}\right)\left(\pi_{2,1}(\nabla_x)(\gamma_+ \dot{-} \gamma_-)\right)$$

[By Proposition 3]

□

Proposition 22. *Let* $\nabla_x^+, \nabla_x^- \in \mathbb{J}_x^{n+1}(\pi)$ *with*

$$\pi_{n+1,n}(\nabla_x^+) = \pi_{n+1,n}(\nabla_x^-).$$

Then the assignment $\gamma \in (M \otimes \mathcal{W}_{D^{n+1}})_{\pi(x)} \longmapsto \nabla_x^+(\gamma) \dot{-} \nabla_x^-(\gamma)$ *belongs to* $\mathbb{S}_x^{D^{n+1}}(\pi)$.

Proof. 1. Since

$$\begin{aligned} &(\pi \otimes \mathrm{id}_{\mathcal{W}_D}) \left(\nabla_x^+(\gamma) \dot{-} \nabla_x^-(\gamma)\right) \\ &= \left(\pi \otimes \mathrm{id}_{\mathcal{W}_{D^{n+1}}}\right) \left(\nabla_x^+(\gamma)\right) \dot{-} \left(\pi \otimes \mathrm{id}_{\mathcal{W}_{D^{n+1}}}\right) \left(\nabla_x^-(\gamma)\right) \\ &\text{[By Proposition 8]} \\ &= 0, \end{aligned}$$

$\nabla_x^+(\gamma) \dot{-} \nabla_x^-(\gamma)$ belongs in $(E \otimes \mathcal{W}_D)_x^\perp$.

2. For any $\alpha \in \mathbb{R}$ and any natural number i with $1 \leq 1 \leq n+1$, we have

$$\begin{aligned} &\nabla_x^+(\alpha \underset{i}{\cdot} \gamma) \dot{-} \nabla_x^-(\alpha \underset{i}{\cdot} \gamma) \\ &= \alpha \underset{i}{\cdot} \nabla_x^+(\gamma) \dot{-} \alpha \underset{i}{\cdot} \nabla_x^-(\gamma) \\ &= \alpha(\nabla_x^+(\gamma) \dot{-} \nabla_x^-(\gamma)), \end{aligned}$$

which implies that the assignment

$$\gamma \in (M \otimes \mathcal{W}_{D^{n+1}})_{\pi(x)} \longmapsto \nabla_x^+(\gamma) \dot{-} \nabla_x^-(\gamma) \in (E \otimes \mathcal{W}_D)_x^\perp$$

abides by the first condition in Definition 18.

3. To see that the assignment abides by the second condition in Definition 18, it suffices to note that the diagram

$$\begin{array}{ccc} (M \otimes \mathcal{W}_{D^{n+1}})_{\pi(x)} & \rightarrow_i & (M \otimes \mathcal{W}_{D^{n+1}})_{\pi(x)} \otimes \mathcal{W}_{D_m} \\ \downarrow & & \downarrow \\ (E \otimes \mathcal{W}_{D^{n+1}})_x \underset{E \otimes \mathcal{W}_{D\{n+1\}_n}}{\times} (E \otimes \mathcal{W}_{D^{n+1}})_x & \rightarrow_i & \left((E \otimes \mathcal{W}_{D^{n+1}})_x \underset{E \otimes \mathcal{W}_{D\{n+1\}_n}}{\times} (E \otimes \mathcal{W}_{D^{n+1}})_x \right) \otimes \mathcal{W}_{D_m} \\ \downarrow & & \downarrow \\ (E \otimes \mathcal{W}_D)_x & \rightarrow & (E \otimes \mathcal{W}_D)_x \otimes \mathcal{W}_{D_m} \end{array}$$

$(1 \leq i \leq n+1))$

is commutative, where the upper horizontal arrow is

$$\mathrm{id}_M \otimes \mathcal{W}_{(d_1,\ldots,d_{n+1},e)\in D^{n+1}\times D_m \mapsto (d_1,\ldots,d_{i-1},ed_i,d_{i+1},\ldots d_{n+1})\in D^{n+1}},$$

the middle horizontal arrow is the mapping

$$\begin{aligned}
&(E \otimes \mathcal{W}_{D^{n+1}}) \underset{E\otimes \mathcal{W}_{D\{n+1\}_n}}{\times} (E \otimes \mathcal{W}_{D^{n+1}}) \\
&\xrightarrow{\frac{\left(\mathrm{id}_E \otimes \mathcal{W}_{(d_1,\ldots,d_{n+1},e)\in D^{n+1}\times D_m \mapsto (d_1,\ldots,d_{i-1},ed_i,d_{i+1},\ldots d_{n+1})\in D^{n+1}}\right) \times}{\left(\mathrm{id}_E \otimes \mathcal{W}_{(d_1,\ldots,d_{n+1},e)\in D^{n+1}\times D_m \mapsto (d_1,\ldots,d_{i-1},ed_i,d_{i+1},\ldots d_{n+1})\in D^{n+1}}\right)}} \\
&\left(E \otimes \mathcal{W}_{D^{n+1}\times D_m}\right) \underset{E\otimes \mathcal{W}_{D\{n+1\}_n}}{\times} \left(E \otimes \mathcal{W}_{D^{n+1}\times D_m}\right) \\
&= \left((E \otimes \mathcal{W}_{D^{n+1}}) \underset{E\otimes \mathcal{W}_{D\{n+1\}_n}}{\times} E \otimes \mathcal{W}_{D^{n+1}} \right) \otimes \mathcal{W}_{D_m},
\end{aligned}$$

the lower horizontal arrow is

$$\mathrm{id}_E \otimes \mathcal{W}_{\times_{D\times D_m \to D}},$$

the upper left vertical arrow is

$$\begin{gathered}
\gamma \in (M \otimes \mathcal{W}_{D^{n+1}})_{\pi(x)} \mapsto \\
\left(\nabla_x^+(\gamma), \nabla_x^-(\gamma)\right) \in (E \otimes \mathcal{W}_{D^{n+1}})_x \underset{E\otimes \mathcal{W}_{D\{n+1\}_n}}{\times} (E \otimes \mathcal{W}_{D^{n+1}})_x ,
\end{gathered}$$

the lower left vertical arrow is

$$\left(\gamma^+, \gamma^-\right) \in (E \otimes \mathcal{W}_{D^{n+1}})_x \underset{E\otimes \mathcal{W}_{D\{n+1\}_n}}{\times} (E \otimes \mathcal{W}_{D^{n+1}})_x \mapsto \gamma^+ \dot{-} \gamma^- \in (E \otimes \mathcal{W}_D)_x ,$$

the upper right vertical arrow is obtained from the upper left vertical arrow by multiplication of $\otimes \mathrm{id}_{\mathcal{W}_{D_m}}$ from the right, and the lower right vertical arrow is obtained from the lower left vertical arrow by multiplication of $\otimes \mathrm{id}_{\mathcal{W}_{D_m}}$ from the right. The upper square is commutative by the third condition in Definition 1, while the lower square is commutaive by Proposition 15, so that the outer square is also commutative, which is no other than the second condition in Definition 18.

4. For any $\sigma \in \mathbf{S}_{n+1}$, we have

$$\begin{aligned}&\nabla_x^+(\gamma^\sigma)\dot{-}\nabla_x^-(\gamma^\sigma)\\&=\left(\nabla_x^+(\gamma)\right)^\sigma \dot{-} \left(\nabla_x^-(\gamma)\right)^\sigma\\&=\nabla_x^+(\gamma)\dot{-}\nabla_x^-(\gamma),\end{aligned}$$

which implies that the assignment abides by the third condition in Definition 18.

5. It remains to show that the assignment abides by the fourth condition in Definition 18, which follows directly from the second condition in Definition 2 and the assumption that $\underline{\hat{\pi}}_{n+1,n}(\nabla_x^+) = \underline{\hat{\pi}}_{n+1,n}(\nabla_x^-)$.

□

Proposition 23. *Let $\nabla_x \in \mathbb{J}_x^{D^{n+1}}(\pi)$ and $\omega \in \mathbb{S}_x^{D^{n+1}}(\pi)$. Then the assignment $\gamma \in (M \otimes \mathcal{W}_{D^{n+1}})_{\pi(x)} \longmapsto \omega(\gamma)\dot{+}\nabla_x(\gamma)$ belongs to $\mathbb{J}_x^{D^{n+1}}(\pi)$.*

Proof. 1. Since

$$\begin{aligned}&\left(\pi \otimes \mathrm{id}_{\mathcal{W}_{D^{n+1}}}\right)\left(\omega(\gamma)\dot{+}\nabla_x(\gamma)\right)\\&=\left(\pi \otimes \mathrm{id}_{\mathcal{W}_D}\right)(\omega(\gamma)) \dot{+} \left(\pi \otimes \mathrm{id}_{\mathcal{W}_{D^{n+1}}}\right)(\nabla_x(\gamma))\\&\text{[By Proposition 12]}\\&=\gamma,\end{aligned}$$

the assignment

$$\gamma \in (M \otimes \mathcal{W}_{D^{n+1}})_{\pi(x)} \longmapsto \omega(\gamma)\dot{+}\nabla_x(\gamma)$$

stands to the first condition in Definition 1.

2. For any $\alpha \in \mathbb{R}$ and any natural number i with $1 \leq i \leq n+1$, we have

$$\begin{aligned}&\omega(\alpha \underset{i}{\cdot} \gamma)\dot{+}\nabla_x(\alpha \underset{i}{\cdot} \gamma)\\&=\alpha\omega(\gamma)\dot{+}\alpha \underset{i}{\cdot} \nabla_x(\gamma)\\&=\alpha \underset{i}{\cdot} (\omega(\gamma)\dot{+}\nabla_x(\gamma)),\end{aligned}$$

so that the assignment stands to the second condition in Definition 1.

3. To see that the assignment acquiesces in the third condition in Definition 1, it suffices to note that the diagram

$$\begin{array}{ccc}
(M\otimes\mathcal{W}_{D^{n+1}})_{\pi(x)} & \rightarrow_i & (M\otimes\mathcal{W}_{D^{n+1}})_{\pi(x)}\otimes\mathcal{W}_{D_m} \\
\downarrow & & \downarrow \\
(E\otimes\mathcal{W}_D)_x\underset{E}{\times}(E\otimes\mathcal{W}_{D^{n+1}})_x & \rightarrow_i & \left((E\otimes\mathcal{W}_D)\underset{E}{\times}(E\otimes\mathcal{W}_{D^{n+1}})_x\right)\otimes\mathcal{W}_{D_m} \\
\downarrow & & \downarrow \\
(E\otimes\mathcal{W}_{D^{n+1}})_x & \rightarrow_i & (E\otimes\mathcal{W}_{D^{n+1}})_x\otimes\mathcal{W}_{D_m}
\end{array}$$

$(1\leq i\leq n+1))$

is commutative, where the upper horizontal arrow is

$$\mathrm{id}_M\otimes\mathcal{W}_{\left(\begin{smallmatrix}\cdot\\ i\end{smallmatrix}\right)_{D^{n+1}\times D_m}},$$

the middle horizontal arrow is the composition of mappings

$$\begin{aligned}
&(E\otimes\mathcal{W}_D)\underset{M}{\times}(E\otimes\mathcal{W}_{D^{n+1}}) \\
&\xrightarrow{\left(\mathrm{id}_M\otimes\mathcal{W}_{\times_{D\times D_m\to D}}\right)\times\left(\mathrm{id}_M\otimes\mathcal{W}_{\left(\begin{smallmatrix}\cdot\\ i\end{smallmatrix}\right)_{D^{n+1}\times D_m}}\right)} \\
&(E\otimes\mathcal{W}_{D\times D_m})\underset{E\otimes\mathcal{W}_{D_m}}{\times}(E\otimes\mathcal{W}_{D^{n+1}\times D_m}) \\
&=\left((E\otimes\mathcal{W}_D)\underset{M}{\times}(E\otimes\mathcal{W}_{D^{n+1}})\right)\otimes\mathcal{W}_{D_m},
\end{aligned}$$

the lower horizontal arrow is

$$\mathrm{id}_E\otimes\mathcal{W}_{\left(\begin{smallmatrix}\cdot\\ i\end{smallmatrix}\right)_{D^{n+1}\times D_m}},$$

the upper left vertical arrow is

$$\gamma\in M\otimes\mathcal{W}_{D^{n+1}}\mapsto(\omega_x(\gamma),\nabla_x(\gamma))\in(E\otimes\mathcal{W}_D)\underset{E}{\times}(E\otimes\mathcal{W}_{D^{n+1}}),$$

he lower left vertical arrow is

$$(t,\gamma)\in(E\otimes\mathcal{W}_D)\underset{E}{\times}(E\otimes\mathcal{W}_{D^{n+1}})\mapsto t\dot{+}\gamma\in E\otimes\mathcal{W}_{D^{n+1}},$$

the upper right vertical arrow is the upper left vertical arrow multiplied by $\otimes\mathrm{id}_{\mathcal{W}_{D_m}}$ from the right, and the lower right vertical arrow is the lower left vertical arrow multiplied by $\otimes \mathrm{id}_{\mathcal{W}_{D_m}}$ from the right. The upper square is commutative by the third condition in Definition 1 and the second condition in Definition 18, while the lower square is commutaive by Proposition 15, so that the outer square is also commutative, which is no other than the third condition in Definition 1.

4. For any $\sigma \in \mathbf{S}_{n+1}$, we have

$$\begin{aligned} &\omega(\gamma^\sigma)\dot{+}\nabla_x(\gamma^\sigma) \\ &= \omega(\gamma)\dot{+}(\nabla_x(\gamma))^\sigma \\ &= (\omega(\gamma)\dot{+}\nabla_x(\gamma))^\sigma, \end{aligned}$$

so that the assignment stands to the fourth condition in Definition 1.

5. That the assignment $\gamma \in (M \otimes \mathcal{W}_{D^{n+1}})_{\pi(x)} \longmapsto \omega(\gamma)\dot{+}\nabla_x(\gamma)$ stands to the first condition of Definition 2 follows from the simple fact that the image of the assignment under $\underline{\hat{\pi}}_{n+1,n}$ coincides with $\underline{\hat{\pi}}_{n+1,n}(\nabla_x)$, which is consequent upon Proposition 20.

6. It remains to show that the assignment abides by the second condition in Definition 2, which follows directly from fourth condition in Definition 18 and the second condition in Definition 2.

□

Now we are in a position to give a definition.

Definition 24. 1. For any $\nabla_x^+, \nabla_x^- \in \mathbb{J}^{n+1}(\pi)$ with

$$\pi_{n+1,n}(\nabla_x^+) = \pi_{n+1,n}(\nabla_x^-),$$

we define $\nabla_x^+ \dot{-} \nabla_x^- \in \mathbb{S}_x^{D_{n+1}}(\pi)$ to be

$$(\nabla_x^+ \dot{-} \nabla_x^-)(\gamma) = \nabla_x^+(\gamma) \dot{-} \nabla_x^-(\gamma)$$

for any $\gamma \in (M \otimes \mathcal{W}_{D^{n+1}})_{\pi(x)}$.

2. For any $\omega \in \mathbb{S}_x^{D^{n+1}}(\pi)$ and any $\nabla_x \in \mathbb{J}^{n+1}(\pi)$, we define $\omega \dot{+} \nabla_x \in \mathbb{J}_x^{n+1}(\pi)$ to be

$$(\omega \dot{+} \nabla_x)(\gamma) = \omega(\gamma) \dot{+} \nabla_x(\gamma)$$

for any $\gamma \in (M \otimes \mathcal{W}_{D^{n+1}})_{\pi(x)}$.

With these two operations depicted in the above definition, it is easy to see that

Theorem 25. *(cf. Theorem 6.2.9 of [5]). The bundle* $\pi_{n+1,n} : \mathbb{J}^{n+1}(\pi) \to \mathbb{J}^n(\pi)$ *is an affine bundle over the vector bundle* $\mathbb{J}^n(\pi) \underset{E}{\times} \mathbb{S}^{D^{n+1}}(\pi) \to \mathbb{J}^n(\pi)$.

Proof. This follows simply from Theorem 13. □

3.2. The Theorem in the Third Approach

3.2.1. Affine Bundles

Now we turn to another kind of affine bundles, for which we can proceed in the same way as in Subsubsection 3.1.1.

Lemma 26. *The diagram*

$$\begin{array}{ccc} D_n & \xrightarrow{i_{D_n \to D_{n+1}}} & D_{n+1} \\ i_{D_n \to D_{n+1}} \downarrow & & \downarrow \Psi_{D_{n+1}} \\ D_{n+1} & \xrightarrow{\Phi_{D_{n+1}}} & D_{n+1} \oplus D \end{array}$$

is a quasi-colimit diagram, where $i : D_n \to D_{n+1}$ *is the canonical injection,* $\Phi_{D_{n+1}}(d) = (d, 0)$ *and* $\Psi_{D_{n+1}}(d) = (d, d^{n+1})$.

This implies directly that

Proposition 27. *Given* $\gamma_+, \gamma_- \in M \otimes \mathcal{W}_{D_{n+1}}$ *with*

$$\left(\mathrm{id}_M \otimes \mathcal{W}_{i_{D_n \to D_{n+1}}}\right)(\gamma_+) = \left(\mathrm{id}_M \otimes \mathcal{W}_{i_{D_n \to D_{n+1}}}\right)(\gamma_-),$$

there exists unique $\gamma \in M \otimes \mathcal{W}_{D_{n+1} \oplus D}$ *with*

$$\left(\mathrm{id}_M \otimes \mathcal{W}_{\Psi_{D_{n+1}}}\right)(\gamma) = \gamma_+ \text{ and}$$
$$\left(\mathrm{id}_M \otimes \mathcal{W}_{\Phi_{D_{n+1}}}\right)(\gamma) = \gamma_-$$

Notation 28. Under the same notation as in the above proposition, we denote

$$\left(\mathrm{id}_M \otimes \mathcal{W}_{\Xi_{D_{n+1}}}\right)(\gamma)$$

by $\gamma_+ \dot{-} \gamma_-$, where $\Xi_{D_{n+1}} : D \rightarrow D^{n+1} \oplus D$ is the mapping

$$d \in D \mapsto (0, ..., 0, d) \in D^{n+1} \oplus D$$

From the very definition of $\dot{-}$, we have

Proposition 29. *Let φ be a mapping of M into M'. Given $\gamma_+, \gamma_- \in M \otimes \mathcal{W}_{D_{n+1}}$ with*

$$\left(\mathrm{id}_M \otimes \mathcal{W}_{i_{D_n \rightarrow D_{n+1}}}\right)(\gamma_+) = \left(\mathrm{id}_M \otimes \mathcal{W}_{i_{D_n \rightarrow D_{n+1}}}\right)(\gamma_-),$$

we have

$$\begin{aligned} &\left(\mathrm{id}_{M'} \otimes \mathcal{W}_{i_{D_n \rightarrow D_{n+1}}}\right)\left(\left(\varphi \otimes \mathrm{id}_{\mathcal{W}_{D_{n+1}}}\right)(\gamma_+)\right) \\ &= \left(\mathrm{id}_{M'} \otimes \mathcal{W}_{i_{D_n \rightarrow D_{n+1}}}\right)\left(\left(\varphi \otimes \mathrm{id}_{\mathcal{W}_{D_{n+1}}}\right)(\gamma_-)\right) \end{aligned}$$

and

$$\begin{aligned} &(\varphi \otimes \mathrm{id}_{\mathcal{W}_D})\left(\gamma_+ \dot{-} \gamma_-\right) \\ &= \left(\varphi \otimes \mathrm{id}_{\mathcal{W}_{D_{n+1}}}\right)(\gamma_+) \dot{-} \left(\varphi \otimes \mathrm{id}_{\mathcal{W}_{D_{n+1}}}\right)(\gamma_-) \end{aligned}$$

It is easy to see that

Proposition 30. *1. We have*

$$\alpha\gamma_+ \dot{-} \alpha\gamma_- = \alpha^{n+1}(\gamma_+ \dot{-} \gamma_-) \quad (1 \leq i \leq n+1)$$

for any $\alpha \in \mathbb{R}$ and any $\gamma_\pm \in M \otimes \mathcal{W}_{D_{n+1}}$ with

$$\left(\mathrm{id}_M \otimes \mathcal{W}_{i_{D_n \rightarrow D_{n+1}}}\right)(\gamma_+) = \left(\mathrm{id}_M \otimes \mathcal{W}_{i_{D_n \rightarrow D_{n+1}}}\right)(\gamma_-).$$

2. The diagram

$$\begin{array}{ccc} (M \otimes \mathcal{W}_{D_{n+1}}) \underset{M \otimes \mathcal{W}_{D_n}}{\times} (M \otimes \mathcal{W}_{D_{n+1}}) & \rightarrow & \left((M \otimes \mathcal{W}_{D_{n+1}}) \underset{M \otimes \mathcal{W}_{D_n}}{\times} (M \otimes \mathcal{W}_{D_{n+1}})\right) \\ & & \otimes \mathcal{W}_{D_m} \\ \downarrow & & \downarrow \\ M \otimes \mathcal{W}_D & \rightarrow & M \otimes \mathcal{W}_{D \times D_m} \end{array}$$

commutes, where the upper horizontal arrow is the composition of mappings

$$\left(M \otimes \mathcal{W}_{D_{n+1}}\right) \underset{M\otimes\mathcal{W}_{D_n}}{\times} \left(M \otimes \mathcal{W}_{D_{n+1}}\right)$$

$$\xrightarrow{\left(\mathrm{id}_M \otimes \mathcal{W}_{\times_{D_{n+1}\times D_m \to D_{n+1}}}\right) \times \left(\mathrm{id}_M \otimes \mathcal{W}_{\times_{D_{n+1}\times D_m \to D_{n+1}}}\right)}$$

$$\left(M \otimes \mathcal{W}_{D_{n+1}\times D_m}\right) \underset{M\otimes\mathcal{W}_{D_n\times D_m}}{\times} \left(M \otimes \mathcal{W}_{D_{n+1}\times D_m}\right)$$

$$= \left(\left(M \otimes \mathcal{W}_{D_{n+1}}\right) \underset{M\otimes\mathcal{W}_{D_n}}{\times} \left(M \otimes \mathcal{W}_{D_{n+1}}\right)\right) \otimes \mathcal{W}_{D_m},$$

the lower horizontal arrow is

$$\mathrm{id}_{M\otimes\mathcal{W}_D} \otimes \mathcal{W}_{d\in D_m \mapsto d^n \in D_m},$$

the left vertical arrow is

$$(\gamma_+, \gamma_-) \in \left(M \otimes \mathcal{W}_{D_{n+1}}\right) \underset{M\otimes\mathcal{W}_{D_n}}{\times} \left(M \otimes \mathcal{W}_{D_{n+1}}\right) \mapsto \gamma_+ \dot{-} \gamma_- \in M \otimes \mathcal{W}_D,$$

and the right vertical arrow is

$$\left((\gamma_+, \gamma_-) \in \left(M \otimes \mathcal{W}_{D_{n+1}}\right) \underset{M\otimes\mathcal{W}_{D_n}}{\times} \left(M \otimes \mathcal{W}_{D_{n+1}}\right) \mapsto \gamma_+ \dot{-} \gamma_- \in M \otimes \mathcal{W}_D\right)$$
$$\otimes\, \mathrm{id}_{\mathcal{W}_{D_m}}.$$

Lemma 31. *The diagram*

$$\begin{array}{ccc} 1 & \xrightarrow{i_{1\to D}} & D \\ i_{1\to D_{n+1}} \downarrow & & \downarrow \Xi_{D_{n+1}} \\ D_{n+1} & \xrightarrow{\Phi_{D_{n+1}}} & D_{n+1} \oplus D \end{array}$$

is a quasi-colimit diagram, where $i_{1\to D_{n+1}}$ is the canonical injection.

This implies at once that

Proposition 32. *Given $t \in M \otimes \mathcal{W}_D$ and $\gamma \in M \otimes \mathcal{W}_{D_{n+1}}$ with*

$$\left(\mathrm{id}_M \otimes \mathcal{W}_{i_{1\to D_{n+1}}}\right)(\gamma) = \left(\mathrm{id}_M \otimes \mathcal{W}_{i_{1\to D}}\right)(t),$$

there exists a unique function $\widetilde{\gamma} : D_{n+1} \oplus D \to M$ *with*

$$\left(\mathrm{id}_M \otimes \mathcal{W}_{\Phi_{D_{n+1}}}\right)(\widetilde{\gamma}) = \gamma$$

and

$$\left(\mathrm{id}_M \otimes \mathcal{W}_{\Xi_{D_{n+1}}}\right)(\widetilde{\gamma}) = t.$$

Notation 33. Under the same notation as in the above proposition, we denote

$$\left(\mathrm{id}_M \otimes \mathcal{W}_{\Psi_{D_{n+1}}}\right)(\widetilde{\gamma})$$

by $t \dot{+} \gamma$.

From the very definition of $\dot{+}$ we have

Proposition 34. *Let* φ *be a mapping of* M *into* M'. *Given* $t \in M \otimes \mathcal{W}_D$ *and* $\gamma \in M \otimes \mathcal{W}_{D_{n+1}}$ *with*

$$\left(\mathrm{id}_M \otimes \mathcal{W}_{i_{1 \to D_{n+1}}}\right)(\gamma) = \left(\mathrm{id}_M \otimes \mathcal{W}_{i_{1 \to D}}\right)(t),$$

we have

$$\left(\mathrm{id}_{M'} \otimes \mathcal{W}_{i_{1 \to D}}\right)\left(\left(\varphi \otimes \mathrm{id}_{\mathcal{W}_D}\right)(t)\right) = \left(\mathrm{id}_{M'} \otimes \mathcal{W}_{i_{1 \to D_{n+1}}}\right)\left(\left(\varphi \otimes \mathrm{id}_{\mathcal{W}_{D_{n+1}}}\right)(\gamma)\right)$$

and

$$\left(\varphi \otimes \mathrm{id}_{\mathcal{W}_{D_{n+1}}}\right)(t \dot{+} \gamma) = \left(\varphi \otimes \mathrm{id}_{\mathcal{W}_D}\right)(t) \dot{+} \left(\varphi \otimes \mathrm{id}_{\mathcal{W}_{D_{n+1}}}\right)(\gamma)$$

Now we have the following affine bundle theorem.

Theorem 35. *The canonical projection*

$$\mathrm{id}_M \otimes \mathcal{W}_{i_{D_n \to D_{n+1}}} : M \otimes \mathcal{W}_{D_{n+1}} \to M \otimes \mathcal{W}_{D_n}$$

is an affine bundle over the vector bundle $(M \otimes \mathcal{W}_D) \underset{M}{\times} (M \otimes \mathcal{W}_{D_n}) \to M \otimes \mathcal{W}_{D_n}$.

3.2.2. Symmetric Forms

Definition 36. A *symmetric* D_n*-form at* $x \in E$ is a mapping $\omega_x : (M \otimes \mathcal{W}_{D_n})_{\pi(x)} \to (E \otimes \mathcal{W}_D)_x^{\perp}$ subject to the following conditions:

1. For any $\gamma \in (M \otimes \mathcal{W}_{D_n})_{\pi(x)}$ and any $\alpha \in \mathbb{R}$, we have
$$\omega_x(\alpha\gamma) = \alpha^n \omega_x(\gamma)$$
2. The diagram
$$\begin{array}{ccc} (M \otimes \mathcal{W}_{D_n})_{\pi(x)} & \xrightarrow{\mathrm{id}_M \otimes \mathcal{W}_{\times_{D_n \times D_m \to D_n}}} & (M \otimes \mathcal{W}_{D_n})_{\pi(x)} \otimes \mathcal{W}_{D_m} \\ \omega_x \downarrow & & \downarrow \omega_x \otimes \mathrm{id}_{\mathcal{W}_{D_m}} \\ (E \otimes \mathcal{W}_D)_x & \xrightarrow[\mathrm{id}_E \otimes \mathcal{W}_{(d,e) \in D \times D_m \mapsto de^n \in D}]{} & (E \otimes \mathcal{W}_D)_x \otimes \mathcal{W}_{D_m} \end{array}$$
is commutative.
3. For any simple polynomial ρ of $d \in D_n$ and any $\gamma \in (M \otimes \mathcal{W}_{D_l})_{\pi(x)}$ with $\dim_n \rho = l < n$, we have
$$\omega((\mathrm{id}_M \otimes \mathcal{W}_\rho)(\gamma)) = 0$$

Notation 37. We denote by $\mathbb{S}_x^{D_n}(\pi)$ the totality of symmetric D_n-*forms* at $x \in E$. We denote by $\mathbb{S}^{D_n}(\pi)$ the set-theoretic union of $\mathbb{S}_x^{D_n}(\pi)$'s for all $x \in E$. The canonical projection $\mathbb{S}^{D_n}(\pi) \to E$ is obviously a vector bundle.

3.2.3. The Theorem

Now we turn to a variant of Theorem 25, for which we can proceed as in 3.1.3, so that proofs of the following results are omitted or merely indicated.

Proposition 38. *Let $\nabla^+, \nabla^- \in \mathbb{J}_x^{D_{n+1}}(\pi)$ with $\pi_{n+1,n}(\nabla^+) = \pi_{n+1,n}(\nabla^-)$. Then the assignment $\gamma \in \left(M \otimes \mathcal{W}_{D_{n+1}}\right)_{\pi(x)} \longmapsto \nabla^+(\gamma) \dot{-} \nabla^-(\gamma) \in (E \otimes \mathcal{W}_D)_x$ belongs to $\mathbb{S}_x^{D_{n+1}}(\pi)$.*

Proposition 39. *Let $\nabla \in \mathbb{J}_x^{D_{n+1}}(\pi)$ and $\omega \in \mathbb{S}_x^{D_{n+1}}(\pi)$. Then the assignment $\gamma \in \left(M \otimes \mathcal{W}_{D_{n+1}}\right)_{\pi(x)} \longmapsto \omega(\gamma) \dot{+} \nabla(\gamma) \in \left(E \otimes \mathcal{W}_{D_{n+1}}\right)_x$ belongs to $\mathbb{J}_x^{D_{n+1}}(\pi)$.*

Notation 40. 1. For any $\nabla^+, \nabla^- \in \mathbb{J}^{D_{n+1}}(\pi)$ with
$$\hat{\pi}_{n+1,n}(\nabla^+) = \hat{\pi}_{n+1,n}(\nabla^-),$$
we define $\nabla^+ \dot{-} \nabla^- \in \mathbb{S}^{D_{n+1}}(\pi)$ to be
$$(\nabla^+ \dot{-} \nabla^-)(\gamma) = \nabla^+(\gamma) \dot{-} \nabla^-(\gamma)$$
for any $\gamma \in \left(M \otimes \mathcal{W}_{D_{n+1}}\right)_{\pi(x)}$.

2. For any $\nabla \in \mathbb{J}_x^{D_{n+1}}(\pi)$ and any $\omega \in \mathbb{S}_x^{D_{n+1}}(\pi)$ we define $\omega \dot{+} \nabla \in \mathbb{J}_x^{D_{n+1}}(\pi)$ to be
$$(\omega \dot{+} \nabla)(\gamma) = \omega(\gamma) \dot{+} \nabla(\gamma)$$
for any $\gamma \in \left(M \otimes \mathcal{W}_{D_{n+1}}\right)_{\pi(x)}$.

With these two operations, we have

Theorem 41. *The bundle* $\pi_{n+1,n} : \mathbb{J}^{D_{n+1}}(\pi) \to \mathbb{J}^{D_n}(\pi)$ *is an affine bundle over the vector bundle* $\mathbb{S}^{D_{n+1}}(\pi) \underset{E}{\times} \mathbb{J}^{D_n}(\pi) \to \mathbb{J}^{D_n}(\pi)$.

Proof. This follows simply from Theorem 35. □

3.3. The Comparison between the Second and Third Approaches

Now we are in a position to investigate the relationship between the affine bundles discussed in Theorems 25 and 41. Let us begin with

Lemma 42. *Let* $\gamma^{\pm} \in \left(E \otimes \mathcal{W}_{D_{n+1}}\right)_x$ *with*
$$\left(\mathrm{id}_E \otimes \mathcal{W}_{i_{D_n \to D_{n+1}}}\right)(\gamma_+) = \left(\mathrm{id}_E \otimes \mathcal{W}_{i_{D_n \to D_{n+1}}}\right)(\gamma_-).$$
Then
$$\begin{aligned}&\left(\mathrm{id}_E \otimes \mathcal{W}_{i_{D\{n+1\}_n \to D^{n+1}}}\right)\left(\left(\mathrm{id}_E \otimes \mathcal{W}_{+_{D^{n+1} \to D_{n+1}}}\right)(\gamma_+)\right)\\ &= \left(\mathrm{id}_E \otimes \mathcal{W}_{i_{D\{n+1\}_n \to D^{n+1}}}\right)\left(\left(\mathrm{id}_E \otimes \mathcal{W}_{+_{D^{n+1} \to D_{n+1}}}\right)(\gamma_-)\right)\end{aligned}$$
obtains, and we have
$$\begin{aligned}&\gamma^+ \dot{-} \gamma^-\\ &= \left(\mathrm{id}_E \otimes \mathcal{W}_{+_{D^{n+1} \to D_{n+1}}}\right)(\gamma^+) \dot{-} \left(\mathrm{id}_E \otimes \mathcal{W}_{+_{D^{n+1} \to D_{n+1}}}\right)(\gamma^-).\end{aligned}$$

Proof. Since the diagram
$$\begin{array}{ccc} D\{n+1\}_n & \xrightarrow{i_{D\{n+1\}_n \to D^{n+1}}} & D^{n+1} \\ +_{D\{n+1\}_n \to D_n} \downarrow & & \downarrow +_{D^{n+1} \to D_{n+1}} \\ D_n & \xrightarrow[i_{D_n \to D_{n+1}}]{} & D_{n+1} \end{array} \tag{1}$$

is commutative, we have

$$\begin{aligned}
&\left(\mathrm{id}_E \otimes \mathcal{W}_{i_{D\{n+1\}_n \to D^{n+1}}}\right)\left(\left(\mathrm{id}_E \otimes \mathcal{W}_{+_{D^{n+1}\to D_{n+1}}}\right)(\gamma^+)\right)\\
&= \left(\mathrm{id}_E \otimes \mathcal{W}_{+_{D\{n+1\}_n \to D_n}}\right)\left(\left(\mathrm{id}_E \otimes \mathcal{W}_{i_{D_n\to D_{n+1}}}\right)(\gamma^+)\right)\\
&= \left(\mathrm{id}_E \otimes \mathcal{W}_{+_{D\{n+1\}_n \to D_n}}\right)\left(\left(\mathrm{id}_E \otimes \mathcal{W}_{i_{D_n\to D_{n+1}}}\right)(\gamma^-)\right)\\
&= \left(\mathrm{id}_E \otimes \mathcal{W}_{i_{D\{n+1\}_n \to D^{n+1}}}\right)\left(\left(\mathrm{id}_E \otimes \mathcal{W}_{+_{D^{n+1}\to D_{n+1}}}\right)(\gamma^-)\right),
\end{aligned}$$

which establishes the coveted first statement. The second statement follows simply from a commutative cubical diagram, which is depicted here separately as the upper square (1), the lower square and the rounding four side squares:

$$\begin{array}{ccc}
D^{n+1} & \xrightarrow{\Phi_{D^{n+1}}} & D^{n+1}\oplus D\\
{\scriptstyle +_{D^{n+1}\to D_{n+1}}}\downarrow & & \downarrow {\scriptstyle +_{D^{n+1}\to D_{n+1}}\oplus \mathrm{id}_D}\\
D_{n+1} & \xrightarrow[\Phi_{D_{n+1}}]{} & D_{n+1}\oplus D
\end{array} \tag{2}$$

$$\begin{array}{ccccc}
D\{n+1\}_n & \xrightarrow{i_{D\{n+1\}_n\to D^{n+1}}} & D^{n+1} & \xrightarrow{+_{D^{n+1}\to D_{n+1}}} & D_{n+1}\\
{\scriptstyle i_{D\{n+1\}_n\to D^{n+1}}}\downarrow & & {\scriptstyle \Psi_{D^{n+1}}}\downarrow & & {\scriptstyle \Psi_{D_{n+1}}}\downarrow\\
D^{n+1} & \xrightarrow[\Phi_{D^{n+1}}]{} & D^{n+1}\oplus D & \xrightarrow[+_{D^{n+1}\to D_{n+1}}\oplus \mathrm{id}_D]{} & D_{n+1}\oplus D
\end{array} \tag{3}$$

$$\begin{array}{ccccc}
D\{n+1\}_n & \xrightarrow{+_{D\{n+1\}_n\to D_n}} & D_n & \xrightarrow{i_{D_n\to D_{n+1}}} & D_{n+1}\\
{\scriptstyle i_{D\{n+1\}_n\to D^{n+1}}}\downarrow & & {\scriptstyle i_{D_n\to D_{n+1}}}\downarrow & & {\scriptstyle \Psi_{D_{n+1}}}\downarrow\\
D^{n+1} & \xrightarrow[+_{D^{n+1}\to D_{n+1}}]{} & D_{n+1} & \xrightarrow[\Phi_{D_{n+1}}]{} & D_{n+1}\oplus D
\end{array} \tag{4}$$

□

Lemma 43. *Let* $t \in E \otimes \mathcal{W}_D$ *and* $\gamma \in E \otimes \mathcal{W}_{D_{n+1}}$ *with*

$$\begin{aligned}
&(\mathrm{id}_E \otimes \mathcal{W}_{i_{1\to D}})(t)\\
&= \left(\mathrm{id}_E \otimes \mathcal{W}_{i_{1\to D_{n+1}}}\right)(\gamma).
\end{aligned}$$

Then we have

$$\left(\mathrm{id}_E \otimes \mathcal{W}_{+_{D^{n+1}\to D_{n+1}}}\right)(t \dot{+} \gamma) = t \dot{+} \left(\mathrm{id}_E \otimes \mathcal{W}_{+_{D^{n+1}\to D_{n+1}}}\right)(\gamma)$$

Proof. This follows simply from a commutative cubical diagram, which is depicted here separately as the upper square, the lower square (2) and the rounding four side squares:

$$\begin{array}{ccc} 1 & \underrightarrow{i_{1\to D}} & D \\ \mathrm{id}_1 \downarrow & & \downarrow \mathrm{id}_D \\ 1 & \overrightarrow{i_{1\to D}} & D \end{array}$$

$$\begin{array}{ccccc} 1 & \underrightarrow{i_{1\to D}} & D & \underrightarrow{\mathrm{id}_D} & D \\ i_{1\to D^{n+1}} \downarrow & & \Xi_{D^{n+1}} \downarrow & & \Xi_{D_{n+1}} \downarrow \\ D^{n+1} & \overrightarrow{\Phi_{D^{n+1}}} & D^{n+1} \oplus D & \overrightarrow{+_{D^{n+1}\to D_{n+1}} \oplus \mathrm{id}_D} & D_{n+1} \oplus D \end{array}$$

$$\begin{array}{ccccc} 1 & \underrightarrow{\mathrm{id}_1} & 1 & \underrightarrow{i_{1\to D}} & D \\ i_{1\to D^{n+1}} \downarrow & & i_{1\to D_{n+1}} \downarrow & & \Xi_{D_{n+1}} \downarrow \\ D^{n+1} & \overrightarrow{+_{D^{n+1}\to D_{n+1}}} & D_{n+1} & \overrightarrow{\Phi_{D_{n+1}}} & D_{n+1} \oplus D \end{array}$$

□

Now we are ready to state the main result of this subsection.

Theorem 44. *We have the following:*

1. *For any* $\nabla^+, \nabla^- \in \mathbb{J}_x^{D^{n+1}}(\pi)$ *and any* $\gamma \in \left(M \otimes \mathcal{W}_{D_{n+1}}\right)_{\pi(x)}$ *with*

$$\pi_{n+1,n}\left(\nabla^+\right) = \pi_{n+1,n}\left(\nabla^-\right),$$

we have

$$\begin{aligned} &\psi_{n+1}(\nabla^+)(\gamma)\dot{-}\psi_{n+1}(\nabla^-)(\gamma) \\ &= \nabla^+(\left(\mathrm{id}_E \otimes \mathcal{W}_{+_{D^{n+1}\to D_{n+1}}}\right)(\gamma))\dot{-}\nabla^-(\left(\mathrm{id}_E \otimes \mathcal{W}_{+_{D^{n+1}\to D_{n+1}}}\right)(\gamma)) \end{aligned}$$

2. *For any*

$$\nabla \in \mathbb{J}_x^{D^{n+1}}(\pi),$$

any

$$t \in (M \otimes \mathcal{W}_D)_{\pi(x)}$$

and any $\gamma \in \left(M \otimes \mathcal{W}_{D_{n+1}}\right)_{\pi(x)}$, *we have*

$$\begin{aligned} &\left(\mathrm{id}_E \otimes \mathcal{W}_{+_{D^{n+1}\to D_{n+1}}}\right)\left(\left(\pi_{n+1,1}\left(\psi_{n+1}(\nabla)\right)\right)(t)\,\dot{+}\psi_{n+1}(\nabla)(\gamma)\right) \\ &= \left(\pi_{n+1,1}\left(\nabla\right)\right)(t)\,\dot{+}\nabla(\left(\mathrm{id}_M \otimes \mathcal{W}_{+_{D^{n+1}\to D_{n+1}}}\right)(\gamma)) \end{aligned}$$

Proof. We deal with the two statements separately.

1. Since

$$\nabla^{\pm}(\left(\mathrm{id}_M \otimes \mathcal{W}_{+_{D^{n+1}\to D_{n+1}}}\right)(\gamma))$$
$$= \left(\mathrm{id}_E \otimes \mathcal{W}_{+_{D^{n+1}\to D_{n+1}}}\right)\left(\left(\psi_{n+1}(\nabla^{\pm})\right)(\gamma)\right)$$

by the very definition of $\psi_{n+1}(\nabla^{\pm})$, we have

$$\nabla^{+}(\left(\mathrm{id}_M \otimes \mathcal{W}_{+_{D^{n+1}\to D_{n+1}}}\right)(\gamma))\dot{-}\nabla^{-}(\left(\mathrm{id}_M \otimes \mathcal{W}_{+_{D^{n+1}\to D_{n+1}}}\right)(\gamma))$$
$$= \left(\mathrm{id}_E \otimes \mathcal{W}_{+_{D^{n+1}\to D_{n+1}}}\right)\left(\left(\psi_{n+1}(\nabla^{+})\right)(\gamma)\right)$$
$$\dot{-}\left(\mathrm{id}_E \otimes \mathcal{W}_{+_{D^{n+1}\to D_{n+1}}}\right)\left(\left(\psi_{n+1}(\nabla^{-})\right)(\gamma)\right)$$
$$= \psi_{n+1}(\nabla^{+})(\gamma)\dot{-}\psi_{n+1}(\nabla^{-})(\gamma) \quad \text{[by Lemma 42]}$$

2. Since

$$\nabla(\left(\mathrm{id}_M \otimes \mathcal{W}_{+_{D^{n+1}\to D_{n+1}}}\right)(\gamma))$$
$$= \left(\mathrm{id}_E \otimes \mathcal{W}_{+_{D^{n+1}\to D_{n+1}}}\right)\left((\psi_{n+1}(\nabla))(\gamma)\right)$$

by the very definition of $\psi_{n+1}(\nabla)$ and

$$(\pi_{n+1,1}(\nabla))(t) = (\pi_{n+1,1}(\psi_{n+1}(\nabla)))(t)$$

by dint of Proposition 4, we have

$$(\pi_{n+1,1}(\nabla))(t)\dot{+}\nabla(\left(\mathrm{id}_M \otimes \mathcal{W}_{+_{D^{n+1}\to D_{n+1}}}\right)(\gamma))$$
$$= (\pi_{n+1,1}(\nabla))(t)\dot{+}\left(\mathrm{id}_E \otimes \mathcal{W}_{+_{D^{n+1}\to D_{n+1}}}\right)\left((\psi_{n+1}(\nabla))(\gamma)\right)$$
$$= \left(\mathrm{id}_E \otimes \mathcal{W}_{+_{D^{n+1}\to D_{n+1}}}\right)\left((\pi_{n+1,1}(\psi_{n+1}(\nabla)))(t)\dot{+}\psi_{n+1}(\nabla)(\gamma)\right)$$
[by Lemma 43]

□

Now we would like to discuss the relationship between $\mathbb{S}^{D^{n+1}}(\pi)$ and $\mathbb{S}^{D_{n+1}}(\pi)$.

Proposition 45. *For any* $\omega \in \mathbb{S}_x^{D^{n+1}}(\pi)$*, the mapping*

$$\gamma \in \left(M \otimes \mathcal{W}_{D_{n+1}}\right)_{\pi(x)} \mapsto \omega\left(\left(\mathrm{id}_M \otimes \mathcal{W}_{+_{D^{n+1}\to D_{n+1}}}\right)(\gamma)\right),$$

denoted by $\phi_{n+1}(\omega)$*, belongs to* $\mathbb{S}_x^{D_{n+1}}(\pi)$*, thereby giving rise to a function* $\phi_{n+1} : \mathbb{S}^{D^{n+1}}(\pi) \to \mathbb{S}^{D_{n+1}}(\pi)$.

Proof. For $n = 0$, the statement is trivial. For any $\omega \in \mathbb{S}_x^{D^{n+1}}(\pi)$, there exist $\nabla^+, \nabla^- \in \mathbb{J}_x^{D^{n+1}}(\pi)$, by dint of Theorem 25, such that

$$\pi_{n+1,n}(\nabla^+) = \pi_{n+1,n}(\nabla^-)$$

and

$$\omega = \nabla^+ \dot{-} \nabla^-.$$

Then we have the following:

1. Let $\alpha \in \mathbb{R}$ and $\gamma \in \left(M \otimes \mathcal{W}_{D_{n+1}}\right)_{\pi(x)}$. Then we have

$$\begin{aligned}
&\omega\left(\left(\mathrm{id}_M \otimes \mathcal{W}_{+_{D^{n+1}\to D_{n+1}}}\right)(\alpha\gamma)\right)\\
&= \nabla^+\left(\left(\mathrm{id}_M \otimes \mathcal{W}_{+_{D^{n+1}\to D_{n+1}}}\right)(\alpha\gamma)\right) \dot{-} \nabla^-\left(\left(\mathrm{id}_M \otimes \mathcal{W}_{+_{D^{n+1}\to D_{n+1}}}\right)(\alpha\gamma)\right)\\
&= \psi_{n+1}(\nabla^+)(\alpha\gamma) \dot{-} \psi_{n+1}(\nabla^-)(\alpha\gamma) \quad \text{[by Theorem 44]}\\
&= \alpha(\psi_{n+1}(\nabla^+)(\gamma)) \dot{-} \alpha(\psi_{n+1}(\nabla^-)(\gamma))\\
&= \alpha^{n+1}(\psi_{n+1}(\nabla^+)(\gamma) \dot{-} \psi_{n+1}(\nabla^-)(\gamma))\\
&= \alpha^{n+1}(\nabla^+(\gamma_{D^{n+1}}) \dot{-} \nabla^-(\gamma_{D^{n+1}})) \quad \text{[by Theorem 44]}\\
&= \alpha^{n+1}\omega\left(\left(\mathrm{id}_M \otimes \mathcal{W}_{+_{D^{n+1}\to D_{n+1}}}\right)(\gamma)\right)
\end{aligned}$$

   so that $\phi_{n+1}(\omega)$ abides by the first condition in Definition 36.

2. The proof that the mapping $\phi_{n+1}(\omega)$ abides by the second condition in Definition 36, which is similar to the above, is safely left to the reader.

3. Let ρ be a simple polynomial of $d \in D_{n+1}$ and $\gamma \in (M \otimes \mathcal{W}_{D_l})_{\pi(x)}$ with $\dim_{n+1}\rho = l < n+1$, we have

$$\begin{aligned}
&\omega\left(\left(\mathrm{id}_M \otimes \mathcal{W}_{+_{D^{n+1}\to D_{n+1}}}\right)((\mathrm{id}_M \otimes \mathcal{W}_\rho)(\gamma))\right)\\
&= \nabla^+\left(\left(\mathrm{id}_M \otimes \mathcal{W}_{+_{D^{n+1}\to D_{n+1}}}\right)((\mathrm{id}_M \otimes \mathcal{W}_\rho)(\gamma))\right) \dot{-}
\end{aligned}$$

$$\nabla^{-}\left(\left(\mathrm{id}_M\otimes\mathcal{W}_{+_{D^{n+1}\to D_{n+1}}}\right)\left(\left(\mathrm{id}_M\otimes\mathcal{W}_\rho\right)(\gamma)\right)\right)$$
$$=\left(\psi_{n+1}(\nabla^{+})\right)\left(\left(\mathrm{id}_M\otimes\mathcal{W}_\rho\right)(\gamma)\right)\dot{-}\left(\psi_{n+1}(\nabla^{-})\right)\left(\left(\mathrm{id}_M\otimes\mathcal{W}_\rho\right)(\gamma)\right)$$
[by Theorem 44]
$$=\left(\mathrm{id}_E\otimes\mathcal{W}_\rho\right)\left(\left(\pi_{n+1,l}(\psi_{n+1}(\nabla^{+}))\right)(\gamma)\right)$$
$$\dot{-}\left(\mathrm{id}_E\otimes\mathcal{W}_\rho\right)\left(\left(\pi_{n+1,l}(\psi_{n+1}(\nabla^{-}))\right)(\gamma)\right)$$
$$=\left(\mathrm{id}_E\otimes\mathcal{W}_\rho\right)\left(\left(\psi_l(\pi_{n+1,l}(\nabla^{+}))\right)(\gamma)\right)\dot{-}\left(\mathrm{id}_E\otimes\mathcal{W}_\rho\right)\left(\left(\psi_l(\pi_{n+1,l}(\nabla^{-}))\right)(\gamma)\right)$$
[by Proposition 4]
$$=0,$$

so that $\phi_{n+1}(\omega)$ abides by the third condition in Definition 36.

□

Let us fix our terminology.

Definition 46. Given an affine bundle $\pi_1 : E_1 \to M_1$ over a vector bundle $\xi_1 : P_1 \to M_1$ and another affine bundle $\pi_2 : E_2 \to M_2$ over another vector bundle $\xi_2 : P_2 \to M_2$, a triple (f, g, h) of mappings $f : M_1 \to M_2$, $g : E_1 \to E_2$ and $h : P_1 \to P_2$ is called a *morphism of affine bundles* from the affine bundle $\pi_1 : E_1 \to M_1$ over the vector bundle $\xi_1 : P_1 \to M_1$ to the affine bundle $\pi_2 : E_2 \to E_2$ over the vector bundle $\xi_2 : P_2 \to M_2$ provided that they abide by the following three conditions:

1. (f, g) is a morphism of bundles from π_1 to π_2. In other words, the following diagram is commutative:

$$\begin{array}{ccc} E_1 & \underset{\rightarrow}{g} & E_2 \\ \pi_1\downarrow & & \downarrow\pi_2 \\ M_1 & \vec{f} & E_2 \end{array}$$

2. (f, h) is a morphism of bundles from ξ_1 to ξ_2. In other words, the following diagram is commutative:

$$\begin{array}{ccc} P_1 & \underset{\rightarrow}{h} & P_2 \\ \xi_1\downarrow & & \downarrow\xi_2 \\ M_1 & \vec{f} & E_2 \end{array}$$

3. For any $x \in M_1$, $(g\mid_{E_{1,x}}, h\mid_{P_{1,x}})$ is a morphism of affine spaces from $(E_{1,x}, P_{1,x})$ to $(E_{2,x}, P_{2,x})$.

Using this terminology, we can summarize Theorem 44 succinctly as follows:

Theorem 47. *The triple* $(\psi_n, \psi_{n+1}, \phi_{n+1} \times \psi_n)$ *of mappings is a morphism of affine bundles from the affine bundle* $\pi_{n+1,n} : \mathbb{J}^{D^{n+1}}(\pi) \to \mathbb{J}^{D^n}(\pi)$ *over the vector bundle* $\mathbb{S}^{D^{n+1}}(\pi) \underset{E}{\times} \mathbb{J}^{D^n}(\pi) \to \mathbb{J}^{D^n}(\pi)$ *in Theorem 25 to the affine bundle* $\pi_{n+1,n} : \mathbb{J}^{D_{n+1}}(\pi) \to \mathbb{J}^{D_n}(\pi)$ *over the vector bundle* $\mathbb{S}^{D_{n+1}}(\pi) \underset{E}{\times} \mathbb{J}^{D_n}(\pi) \to \mathbb{J}^{D_n}(\pi)$ *in Theorem 41.*

4. Taylor Representations

The following results are merely special cases of the general Taylor-type theorem such as seen in [2] (Part III, Proposition 5.2).

Proposition 48. *Any* $f \in \mathbb{R}^p \otimes \mathcal{W}_{D_n}$ *is of a unique Taylor representation of the form*

$$\delta \in \mathbb{R} \longmapsto (x^i) + \delta(y_1^i) + \delta^2(y_2^i) + ... + \delta^n(y_n^i) \in \mathbb{R}^p$$

with $(x^i), (y_1^i), (y_2^i), ..., (y_n^i) \in \mathbb{R}^p$.

Proposition 49. *Any* $f \in \mathbb{R}^p \otimes \mathcal{W}_{D^n}$ *is of a unique Taylor representation of the form*

$$(\delta_1, ..., \delta_n) \in \mathbb{R}^n \longmapsto (x^i) + \sum_{r=1}^{n} \sum_{1 \leq k_1 < ... < k_r \leq n} \delta_{k_1} ... \delta_{k_r} (y^i_{k_1,...,k_r}) \in \mathbb{R}^p$$

with $(x^i), (y^i_{k_1,...,k_r}) \in \mathbb{R}^p$.

Proposition 50. *Any* $f \in \mathbb{R}^p \otimes \mathcal{W}_{D(n)_m}$ *is of a unique Taylor representation of the form*

$$(\delta_1, ..., \delta_n) \in \mathbb{R}^n \longmapsto (x^i) + \sum_{r=1}^{m} \sum_{1 \leq k_1 \leq ... \leq k_r \leq n} \delta_{k_1} ... \delta_{k_r} (y^i_{k_1,...,k_r}) \in \mathbb{R}^p$$

with $(x^i), (y^i_{k_1,...,k_r}) \in \mathbb{R}^p$.

5. The Basic Framework with Coordinates

This section is inspired much by [1].

5.1. The Basic Framework

Notation 51. We denote by $\mathcal{J}^n(\pi)$ the totality of

$$\gamma \in E \otimes \mathcal{W}_{D(p)_n}$$

such that

$$\left(\pi \otimes \mathrm{id}_{\mathcal{W}_{D(p)_n}}\right)(\gamma) \in M \otimes \mathcal{W}_{D(p)_n}$$

is degenerate in the sense that

$$\begin{aligned}&\left(\pi \otimes \mathrm{id}_{\mathcal{W}_{D(p)_n}}\right)(\gamma)\\ &= \left(\mathrm{id}_M \otimes \mathcal{W}_{D(p)_n \to 1}\right)(\gamma')\end{aligned}$$

for some $\gamma' \in M \otimes \mathcal{W}_1 = M$.

Notation 52. We denote by $\mathcal{S}^n(\pi)$ the totality of

$$t \in E \otimes \mathcal{W}_{D({}_{p+n}C_{n+1})}$$

such that

$$\left(\pi \otimes \mathrm{id}_{\mathcal{W}_{D({}_{p+n}C_{n+1})}}\right)(t) \in M \otimes \mathcal{W}_{D({}_{p+n}C_{n+1})}$$

is degenerate in the sense that

$$\begin{aligned}&\left(\pi \otimes \mathrm{id}_{\mathcal{W}_{D({}_{p+n}C_{n+1})}}\right)(t)\\ &= \left(\mathrm{id}_M \otimes \mathcal{W}_{D({}_{p+n}C_{n+1}) \to 1}\right)(t')\end{aligned}$$

for some $t' \in M \otimes \mathcal{W}_1 = M$.

Remark 53. 1. Each $\gamma \in E \otimes \mathcal{W}_{D(p)_n}$ can be identified unquely with a sequence

$$\left(x^1, ..., x^p, u^1, ..., u^q, x^i_{i_1}, x^i_{i_1,i_2}, ..., x^i_{i_1,i_2,...,i_n}, u^j_{i_1}, u^j_{i_1,i_2}, ..., u^j_{i_1,i_2,...,i_n}\right)_{1 \leq i_1 \leq i_2 \leq ... \leq i_n \leq p}$$

of real numbers at length

$$p + q + (p+q)\, p + ... + (p+q)_{p+n-1}\, C_n$$

in the sense that the Taylor representation of γ is

$$(\delta_1, ..., \delta_p) \in \mathbb{R}^p \mapsto \left(x^1, ..., x^p, u^1, ..., u^q\right) + \sum_{1 \leq i_1 \leq p} \left(x^1_{i_1}, ..., x^p_{i_1}, u^1_{i_1}, ..., u^q_{i_1}\right) \delta_{i_1}$$
$$+ \sum_{1 \leq i_1 \leq i_2 \leq p} \left(x^1_{i_1,i_2}, ..., x^p_{i_1,i_2}, u^1_{i_1,i_2}, ..., u^q_{i_1,i_2}\right) \delta_{i_1} \delta_{i_2} + ...$$
$$+ \sum_{1 \leq i_1 \leq i_2 \leq ... \leq i_n \leq p} \left(x^1_{i_1,i_2,...,i_n}, ..., x^p_{i_1,i_2,...,i_n}, u^1_{i_1,i_2,...,i_n}, ..., u_{i_1,i_2,}\right.$$
$$\left. ..., {i_n}^q\right) \delta_{i_1} \delta_{i_2} ... \delta_{i_n}$$
$$\in \mathbb{R}^{p+q}$$

2. Each $\nabla \in \mathcal{J}^n(\pi)$ can be identified uniquely with a sequence

$$\left(x^1, ..., x^p, u^1, ..., u^q, u^j_{i_1}, u^j_{i_1,i_2}, ..., u^j_{i_1,i_2,...,i_n}\right)_{1 \leq i_1 \leq i_2 \leq ... \leq i_n \leq p}$$

of real numbers at length

$$p + q + qp + ... + q_{p+n-1}C_n$$

in the sense that the Taylor representation of ∇ is

$$(\delta_1, ..., \delta_p) \in \mathbb{R}^p \mapsto \left(x^1, ..., x^p, u^1, ..., u^q\right) + \sum_{1 \leq i_1 \leq p} \left(0, ...0, u^1_{i_1}, ..., u^q_{i_1}\right) \delta_{i_1}$$
$$+ \sum_{1 \leq i_1 \leq i_2 \leq p} \left(0, ...0, u^1_{i_1,i_2}, ..., u^q_{i_1,i_2}\right) \delta_{i_1} \delta_{i_2} + ...$$
$$+ \sum_{1 \leq i_1 \leq i_2 \leq ... \leq i_n \leq p} \left(0, ...0, u^1_{i_1,i_2,...,i_n}, ..., u^q_{i_1,i_2,...,i_n}\right) \delta_{i_1} \delta_{i_2} ... \delta_{i_n}$$
$$\in \mathbb{R}^{p+q}$$

3. Each $t \in E \otimes \mathcal{W}_{D(_{p+n}C_{n+1})}$ can be identified unquely with a sequence

$$\left(x^1, ..., x^p, u^1, ..., u^q, x^1_{i_1,i_2,...,i_{n+1}}, ..., x^p_{i_1,i_2,...,i_{n+1}}, u^1_{i_1,i_2,...,i_{n+1}}, ...,\right.$$
$$\left. u^q_{i_1,i_2,...,i_{n+1}}\right)_{1 \leq i_1 \leq i_2 \leq ... \leq i_{n+1} \leq p}$$

of real numbers at length

$$p + q + (p+q)_{p+n} C_{n+1}$$

in the sense that the Taylor representation of t is

$$\begin{aligned}
&\left(\delta_{i_1,i_2,...,i_{n+1}}\right)_{1\leq i_1\leq i_2\leq ...\leq i_{n+1}\leq p} \\
&\in \mathbb{R}^{(_{p+n}C_{n+1})} \mapsto \left(x^1,...,x^p,u^1,...,u^q\right) \\
&+ \sum_{1\leq i_1\leq i_2\leq ...\leq i_{n+1}\leq p} \left(x^1_{i_1,i_2,...,i_{n+1}},...,x^p_{i_1,i_2,...,i_{n+1}},u^1_{i_1,i_2,...,i_{n+1}},..., \right. \\
&\quad \left. u^q_{i_1,i_2,...,i_{n+1}}\right)\delta_{i_1,i_2,...,i_{n+1}} \\
&\in \mathbb{R}^{p+q}
\end{aligned}$$

4. Each $\omega \in \mathcal{S}^{n+1}(\pi)$ can be identified unquely with a sequence

$$\left(x^1,...,x^p,u^1,...,u^q,u^1_{i_1,i_2,...,i_{n+1}},...,u^q_{i_1,i_2,...,i_{n+1}}\right)_{1\leq i_1\leq i_2\leq ...\leq i_{n+1}\leq p}$$

of real numbers at length

$$p+q+q_{p+n}C_{n+1}$$

in the sense that the Taylor representation of ω is

$$\begin{aligned}
&\left(\delta_{i_1,i_2,...,i_{n+1}}\right)_{1\leq i_1\leq i_2\leq ...\leq i_{n+1}\leq p} \\
&\in \mathbb{R}^{(_{p+n}C_{n+1})} \mapsto \left(x^1,...,x^p,u^1,...,u^q\right) + \\
&\sum_{1\leq i_1\leq i_2\leq ...\leq i_{n+1}\leq p} \left(0,...0,u^1_{i_1,i_2,...,i_{n+1}},...,u^q_{i_1,i_2,...,i_{n+1}}\right)\delta_{i_1,i_2,...,i_{n+1}} \in \mathbb{R}^{p+q}
\end{aligned}$$

Notation 54. Since $D(p)_n \subset D(p)_{n+1}$, there is a canonical projection $E\otimes \mathcal{W}_{D(p)_{n+1}} \to E\otimes \mathcal{W}_{D(p)_n}$, which restricts itself naturally to $\mathcal{J}^{n+1}(\pi) \to \mathcal{J}^n(\pi)$. Both of them are denoted by $\pi_{n+1,n}$. We have

$$\begin{aligned}
&\pi_{n+1,n}\left(x^1,...,x^p,u^1,...,u^q,x^i_{i_1},...,x^j_{i_1,i_2,...,i_n},x^i_{i_1,i_2,...,i_{n+1}},u^j_{i_1},..., \right. \\
&\quad \left. u^j_{i_1,i_2,...,i_n},u^j_{i_1,i_2,...,i_{n+1}}\right) \\
&= \left(x^1,...,x^p,u^1,...,u^q,x^i_{i_1},x^i_{i_1,i_2},...,x^i_{i_1,i_2,...,i_n},u^j_{i_1},u^j_{i_1,i_2},...,u^j_{i_1,i_2,...,i_n}\right)
\end{aligned}$$

and

$$\begin{aligned}
&\pi_{n+1,n}\left(x^1,...,x^p,u^1,...,u^q,u^j_{i_1},...,u^j_{i_1,i_2,...,i_n},u^j_{i_1,i_2,...,i_{n+1}}\right) \\
&= \left(x^1,...,x^p,u^1,...,u^q,u^j_{i_1},...,u^j_{i_1,i_2,...,i_n}\right)
\end{aligned}$$

5.2. The Affine Bundle Theorem within the Basic Framework

It is easy to see that

Lemma 55. *The diagram*

$$\begin{array}{ccc} D(p)_n & \underrightarrow{i_{D(p)_n \to D(p)_{n+1}}} & D(p)_{n+1} \\ i_{D(p)_n \to D(p)_{n+1}} \downarrow & & \downarrow \Psi_{D(p)_{n+1}} \\ D(p)_{n+1} & \overrightarrow{\Phi_{D(p)_{n+1}}} & D(p)_{n+1} \oplus D\left({}_{p+n}C_{n+1}\right) \end{array}$$

is a quasi-colimit diagram, where $\Phi_{D(p)_{n+1}}$ is the canonical injection of $D(p)_{n+1}$ into $D(p)_{n+1} \oplus D\left({}_{p+n}C_{n+1}\right)$, and $\Psi_{D(p)_{n+1}}$ is the mapping

$$(d_1, ..., d_p) \in D(p)_{n+1} \mapsto \left(d_1, ..., d_p, d_1^{n+1}, d_1^n d_2, ...\right) \in D(p)_{n+1} \oplus D\left({}_{p+n}C_{n+1}\right)$$

with the sequence $d_1^{n+1}, d_1^n d_2, ...$ being that of $d_{k_1} d_{k_2} ... d_{k_{n+1}}$'s $(1 \leq k_1 \leq k_2 \leq ... \leq k_{n+1} \leq n+1)$ in lexicographic order.

This implies at once that

Proposition 56. *Given $\gamma_+, \gamma_- \in E \otimes \mathcal{W}_{D(p)_{n+1}}$ with*

$$\left(\mathrm{id}_E \otimes \mathcal{W}_{i_{D(p)_n \to D(p)_{n+1}}}\right)(\gamma_+) = \left(\mathrm{id}_E \otimes \mathcal{W}_{i_{D(p)_n \to D(p)_{n+1}}}\right)(\gamma_-),$$

there exists unique $\gamma \in E \otimes \mathcal{W}_{D(p)_{n+1} \oplus D\left({}_{p+n}C_{n+1}\right)}$ with

$$\left(\mathrm{id}_E \otimes \mathcal{W}_{\Psi_{D(p)_{n+1}}}\right)(\gamma) = \gamma_+ \text{ and}$$
$$\left(\mathrm{id}_E \otimes \mathcal{W}_{\Phi_{D(p)_{n+1}}}\right)(\gamma) = \gamma_-$$

Notation 57. Under the same notation as in the above proposition, we denote

$$\left(\mathrm{id}_E \otimes \mathcal{W}_{\Xi_{D(p)_{n+1}}}\right)(\gamma)$$

by

$$\gamma_+ \dot{-} \gamma_-,$$

where $\Xi_{D^n} : D\left({}_{p+n}C_{n+1}\right) \to D(p)_{n+1} \oplus D\left({}_{p+n}C_{n+1}\right)$ is the canonical injection.

Remark 58. Given

$$\gamma_+ = \left(x^1,...,x^p,u^1,...,u^q,x^i_{i_1},x^i_{i_1,i_2},...,x^i_{i_1,i_2,...,i_{n+1}},u^j_{i_1},u^j_{i_1,i_2},...,u^j_{i_1,i_2,...,i_{n+1}}\right),$$
$$\gamma_- = \left(y^1,...,y^p,v^1,...,v^q,y^i_{i_1},y^i_{i_1,i_2},...,y^i_{i_1,i_2,...,i_{n+1}},v^j_{i_1},v^j_{i_1,i_2},...,v^j_{i_1,i_2,...,i_{n+1}}\right)$$
$$\in E\otimes\mathcal{W}_{D(p)_{n+1}},$$

we have

$$\left(\mathrm{id}_E\otimes\mathcal{W}_{i_{D(p)_n\to D(p)_{n+1}}}\right)(\gamma_+) = \left(\mathrm{id}_E\otimes\mathcal{W}_{i_{D(p)_n\to D(p)_{n+1}}}\right)(\gamma_-)$$

iff

$$\left(x^1,...,x^p,u^1,...,u^q,x^i_{i_1},x^i_{i_1,i_2},...,x^i_{i_1,i_2,...,i_n},u^j_{i_1},u^j_{i_1,i_2},...,u^j_{i_1,i_2,...,i_n}\right)$$
$$= \left(y^1,...,y^p,v^1,...,v^q,y^i_{i_1},y^i_{i_1,i_2},...,y^i_{i_1,i_2,...,i_n},v^j_{i_1},v^j_{i_1,i_2},...,v^j_{i_1,i_2,...,i_n}\right),$$

in which we get

$$\gamma_+\dot{-}\gamma_-$$
$$= \left(x^1,...,x^p,u^1,...,u^q,u^1_{i_1,i_2,...,i_{n+1}}-v^1_{i_1,i_2,...,i_{n+1}},...,u^q_{i_1,i_2,...,i_{n+1}}-v^q_{i_1,i_2,...,i_{n+1}}\right)$$

From the very definition of $\dot{-}$, we have

Proposition 59. *Let F be a mapping of E into E'. Given $\gamma_+,\gamma_-\in E\otimes\mathcal{W}_{D(p)_{n+1}}$ with*

$$\left(\mathrm{id}_E\otimes\mathcal{W}_{i_{D(p)_n\to D(p)_{n+1}}}\right)(\gamma_+) = \left(\mathrm{id}_E\otimes\mathcal{W}_{i_{D(p)_n\to D(p)_{n+1}}}\right)(\gamma_-),$$

we have

$$\left(\mathrm{id}_{E'}\otimes\mathcal{W}_{i_{D(p)_n\to D(p)_{n+1}}}\right)\left(\left(F\otimes\mathrm{id}_{\mathcal{W}_{D(p)_{n+1}}}\right)(\gamma_+)\right)$$
$$= \left(\mathrm{id}_{E'}\otimes\mathcal{W}_{i_{D(p)_n\to D(p)_{n+1}}}\right)\left(\left(F\otimes\mathrm{id}_{\mathcal{W}_{D(p)_{n+1}}}\right)(\gamma_-)\right)$$

and

$$\left(F\otimes\mathrm{id}_{\mathcal{W}_{D\left(p+{}_nC_{n+1}\right)}}\right)(\gamma_+\dot{-}\gamma_-)$$
$$= \left(F\otimes\mathrm{id}_{\mathcal{W}_{D(p)_{n+1}}}\right)(\gamma_+)\dot{-}\left(F\otimes\mathrm{id}_{\mathcal{W}_{D(p)_{n+1}}}\right)(\gamma_-)$$

Lemma 60. *The diagram*

$$\begin{array}{ccc} 1 & \overrightarrow{i_{1\to D({}_{p+n}C_{n+1})}} & D\left({}_{p+n}C_{n+1}\right) \\ i_{1\to D(p)_{n+1}} \downarrow & & \downarrow \Xi_{D(p)_{n+1}} \\ D(p)_{n+1} & \underrightarrow{\Phi_{D(p)_{n+1}}} & D(p)_{n+1} \oplus D\left({}_{p+n}C_{n+1}\right) \end{array}$$

is a quasi-colimit diagram.

This implies at once that

Proposition 61. *Given $t \in E \otimes \mathcal{W}_{D({}_{p+n}C_{n+1})}$ and $\gamma \in E \otimes \mathcal{W}_{D(p)_{n+1}}$ with*

$$\left(\mathrm{id}_E \otimes \mathcal{W}_{i_{1\to D({}_{p+n}C_{n+1})}}\right)(t) = \left(\mathrm{id}_E \otimes \mathcal{W}_{i_{1\to D(p)_{n+1}}}\right)(\gamma),$$

there exists unique $\gamma' \in E \otimes \mathcal{W}_{D(p)_{n+1}\oplus D({}_{p+n}C_{n+1})}$ with

$$\left(\mathrm{id}_E \otimes \mathcal{W}_{\Xi_{D(p)_{n+1}}}\right)(\gamma') = t \text{ and}$$
$$\left(\mathrm{id}_E \otimes \mathcal{W}_{\Phi_{D(p)_{n+1}}}\right)(\gamma') = \gamma.$$

Notation 62. Under the same notation as in the above proposition, we denote

$$\left(\mathrm{id}_E \otimes \mathcal{W}_{\Psi_{D(p)_{n+1}}}\right)(\gamma')$$

by $t \dot{+} \gamma$.

Remark 63. Given

$$\gamma = \left(x^1, ..., x^p, u^1, ..., u^q, x^i_{i_1}, ..., x^i_{i_1,i_2,...,i_{n+1}}, u^j_{i_1}, ..., u^j_{i_1,i_2,...,i_{n+1}}\right)$$
$$\in E \otimes \mathcal{W}_{D(p)_{n+1}}$$

and

$$t = \left(y^1, ..., y^p, v^1, ..., v^q, y^1_{i_1,i_2,...,i_{n+1}}, ..., y^p_{i_1,i_2,...,i_{n+1}}, v^1_{i_1,i_2,...,i_{n+1}}, ..., v^q_{i_1,i_2,...,i_{n+1}}\right)$$
$$\in E \otimes \mathcal{W}_{D({}_{p+n}C_{n+1})},$$

we have

$$\left(\mathrm{id}_E \otimes \mathcal{W}_{i_{1\to D({}_{p+n}C_{n+1})}}\right)(t) = \left(\mathrm{id}_E \otimes \mathcal{W}_{i_{1\to D(p)_{n+1}}}\right)(\gamma)$$

iff

$$\left(x^1,...,x^p,u^1,...,u^q\right)=\left(y^1,...,y^p,v^1,...,v^q\right),$$

in which we get

$$t\dot{+}\gamma=\Big(x^i,u^j,x^i_{i_1},...,x^i_{i_1,i_2,...,i_n},x^i_{i_1,i_2,...,i_{n+1}}\\ +y^i_{i_1,i_2,...,i_{n+1}},u^j_{i_1},...,u^j_{i_1,i_2,...,i_n},u^j_{i_1,i_2,...,i_{n+1}}+v^j_{i_1,i_2,...,i_{n+1}}\Big)$$

From the very definition of $\dot{+}$, we have

Proposition 64. *Let F be a mapping of E into E'. Given $t\in E\otimes\mathcal{W}_{D({}_{p+n}C_{n+1})}$ and $\gamma\in E\otimes\mathcal{W}_{D(p)_{n+1}}$ with*

$$\left(\mathrm{id}_E\otimes\mathcal{W}_{i_{1\to D({}_{p+n}C_{n+1})}}\right)(t)=\left(\mathrm{id}_E\otimes\mathcal{W}_{i_{1\to D(p)_{n+1}}}\right)(\gamma),$$

we have

$$\left(\mathrm{id}_{E'}\otimes\mathcal{W}_{i_{1\to D({}_{p+n}C_{n+1})}}\right)\left(\left(F\otimes\mathrm{id}_{\mathcal{W}_{D({}_{p+n}C_{n+1})}}\right)(t)\right)\\ =\left(\mathrm{id}_{E'}\otimes\mathcal{W}_{i_{1\to D(p)_{n+1}}}\right)\left(\left(F\otimes\mathrm{id}_{\mathcal{W}_{D(p)_{n+1}}}\right)(\gamma)\right)$$

and

$$\left(F\otimes\mathrm{id}_{\mathcal{W}_{D(p)_{n+1}}}\right)(t\dot{+}\gamma)=\left(F\otimes\mathrm{id}_{\mathcal{W}_{D({}_{p+n}C_{n+1})}}\right)(t)\dot{+}\left(F\otimes\mathrm{id}_{\mathcal{W}_{D(p)_{n+1}}}\right)(\gamma),$$

Now we have

Theorem 65. *The canonical projection*

$$\mathrm{id}_E\otimes\mathcal{W}_{i_{D(p)_n\to D(p)_{n+1}}}:E\otimes\mathcal{W}_{D(p)_{n+1}}\to E\otimes\mathcal{W}_{D(p)_n}$$

is an affine bundle over the vector bundle

$$\left(E\otimes\mathcal{W}_{D({}_{p+n}C_{n+1})}\right)^{\perp}\underset{E}{\times}\left(E\otimes\mathcal{W}_{D(p)_n}\right)\to E\otimes\mathcal{W}_{D(p)_n}.$$

Theorem 66. *The mapping $\pi_{n+1,n}:\mathcal{J}^{n+1}(\pi)\to\mathcal{J}^n(\pi)$ is an affine bundle over the vector bundle $\mathcal{S}^{n+1}(\pi)\underset{E}{\times}\mathcal{J}^n(\pi)\to\mathcal{J}^n(\pi)$.*

6. The First Approach with Coordinates

Definition 67. We define $\theta_{\mathbf{J}^1(\pi)}^{\mathcal{J}^1(\pi)} : \mathcal{J}^1(\pi) \to \mathbf{J}^1(\pi)$ to be

$$\begin{aligned}
&\theta_{\mathbf{J}^1(\pi)}^{\mathcal{J}^1(\pi)} \left(x^1, ..., x^p, u^1, ..., u^q, u_i^j\right) \\
&= \left[\delta \in \mathbb{R} \mapsto \left(x^1, ..., x^p\right) + \left(y^1, ..., y^p\right) \delta \in \mathbb{R}^p\right] \in (M \otimes \mathcal{W}_D)_{(x^1,...,x^p)} \mapsto \\
&\left[\delta \in \mathbb{R} \mapsto \left(x^1, ..., x^p, u^1, ..., u^q\right) + \left(y^1, ..., y^p, \sum_{i=1}^{p} u_i^j y^i\right) \delta \in \mathbb{R}^{p+q}\right] \\
&\in (E \otimes \mathcal{W}_D)_{(x^1,...,x^p,u^1,...,u^q)}
\end{aligned}$$

Remark 68. It is easy to see that the right-hand side of the above formula belongs to $\mathbf{J}^1(\pi)$.

Theorem 69. *The mapping* $\theta_{\mathbf{J}^1(\pi)}^{\mathcal{J}^1(\pi)} : \mathcal{J}^1(\pi) \to \mathbf{J}^1(\pi)$ *is bijective.*

Remark 70. This gives a coordinate description of $\mathbf{J}^1(\pi)$.

Now we are going to consider $\widetilde{\mathbf{J}}^2(\pi) = \mathbf{J}^1(\pi_1)$, which has a coordinate description as follows:

$$\begin{aligned}
&\theta_{\mathbf{J}^1(\pi_1)}^{\mathcal{J}^1(\pi_1)} \left(x^i, u^j, u_{i_1}^j, u_{;i_2}^j, u_{i_1;i_2}^j\right) \\
&= \left[\delta \in \mathbb{R} \mapsto \left(x^i\right) + \left(y^i\right) \delta \in \mathbb{R}^p\right] \in (M \otimes \mathcal{W}_D)_{(x^i)} \mapsto \\
&\left[\delta \in \mathbb{R} \mapsto \left(x^i, u^j, u_{i_1}^j\right) + \left(y^i, \sum_{i_2=1}^{p} u_{;i_2}^j y^{i_2}, \sum_{i_2=1}^{p} u_{i_1;i_2}^j y^{i_2}\right) \delta \in \mathbb{R}^{p+q+pq}\right] \\
&\in (E \otimes \mathcal{W}_D)_{\left(x^i,u^j,u_{i_1}^j\right)},
\end{aligned}$$

for which we get

Proposition 71. *We have*

$$(x^i, u^j; u_{i_1}^j; u_{;i_2}^j, u_{i_1;i_2}^j) \in \hat{\mathbf{J}}^2(\pi)$$

iff $u_i^j = u_{;i}^j$ *for all* $1 \leq i \leq p$ *and all* $1 \leq j \leq q$.

Proof. It is easy to see that $\left(x^i, u^j, u^j_{i_1}, u^j_{;i_2}, u^j_{i_1;i_2}\right) \in \mathbf{J}^1(\pi_1)$ is $\pi_{1,0}$-related to $\left(x^i, u^j, u^j_{i_1}\right) \in \mathbf{J}^1(\pi)$ iff

$$u^j + \delta\Sigma_{i_2=1}^n y^{i_2} u^j_{;i_2} = u^j + \delta\Sigma_{i_1=1}^n y^{i_1} u^j_{i_1} \qquad (1 \leq j \leq q)$$

for all $(y^1, ..., y^p) \in \mathbb{R}^p$ and all $\delta \in \mathbb{R}$, which is tantamount to saying that

$$u^j_{;i} = u^j_i$$

for all $1 \leq i \leq p$ and all $1 \leq j \leq q$. This completes the proof. □

Notation 72. Thus the coordinate $(x^i, u^j; u^j_{i_1}; u^j_{;i_2}, u^j_{i_1;i_2}) \in \hat{\mathbf{J}}^2(\pi)$ can be simplified to $(x^i, u^j, u^j_{i_1}, u^j_{i_1;i_2})$.

Now we take a step forward.

Proposition 73. *Let* $(x^i, u^j, u^j_{i_1}, u^j_{i_1;i_2}) \in \hat{\mathbf{J}}^2(\pi)$*. Then* $(x^i, u^j, u^j_{i_1}, u^j_{i_1;i_2}) \in \mathbf{J}^2(\pi)$ *iff*

$$u^j_{i_1;i_2} = u^j_{i_2;i_1}$$

for all $1 \leq i_1, i_2 \leq p$ *and all* $1 \leq j \leq q$.

Proof. Let $\gamma \in (M \otimes \mathcal{W}_{D^2})_{(x^i)}$. Then γ is of the Taylor representation

$$\begin{aligned}(\delta_1, \delta_2) \in \mathbb{R}^2 &\longmapsto (x^1, ..., x^p) + \delta_1(y^1_1, ..., y^p_1) + \delta_2(y^1_2, ..., y^p_2) + \delta_1\delta_2(y^1_{12}, ..., y^p_{12}) \\ &= (x^1 + \delta_1 y^1_1, ..., x^p + \delta_1 y^p_1) + \delta_2(y^1_2 + \delta_1 y^1_{12}, ..., y^p_2 + \delta_1 y^p_{12}) \\ &= (x^1 + \delta_2 y^1_2, ..., x^p + \delta_2 y^p_2) + \delta_1(y^1_1 + \delta_2 y^1_{12}, ..., y^p_1 + \delta_2 y^p_{12}) \\ &\in \mathbb{R}^p\end{aligned}$$

Let $\nabla_{(x^i, u^j, u^j_{i_1})} = \left(x^i, u^j, u^j_{i_1}, u^j_{i_1;i_2}\right)$ and $\nabla_{(x^i, u^j)} = (x^i, u^j, u^j_{i_1})$. Then we have

$$\begin{aligned}&\delta_1 \in \mathbb{R} \mapsto \nabla_{(x^i, u^j)}(\gamma(\cdot, 0))(\delta_1) \\ &= \delta_1 \in \mathbb{R} \mapsto (x^i + \delta_1 y^i_1, u^j + \delta_1 \Sigma_{i_1=1}^p y^{i_1}_1 u^j_{i_1}) \in \mathbb{R}^{p+q} \\ &\delta_1 \in \mathbb{R} \mapsto \nabla_{(x^i, u^j, u^j_{i_1})}(\gamma(\cdot, 0))(\delta_1) \in \mathbb{R}^{p+q+pq} \\ &= \delta_1 \in \mathbb{R} \mapsto (x^i + \delta_1 y^i_1, u^j + \delta_1 \sum_{i_1=1}^p y^{i_1}_1 u^j_{i_1}, u^j_{i_1} + \delta_1 \sum_{i_2=1}^p y^{i_2}_1 u^j_{i_1;i_2}) \in \mathbb{R}^{p+q+pq}\end{aligned}$$

while we have

$$(\delta_1,\delta_2)\in\mathbb{R}^2\mapsto\nabla_{\nabla_{(x^i,u^j)}(\gamma(\cdot,0))(\delta_1)}(\gamma(\delta_1,\cdot))(\delta_2)\in\mathbb{R}^{p+q}$$
$$=(\delta_1,\delta_2)\in\mathbb{R}^2\mapsto\left(\begin{array}{c}x^i+\delta_1y_1^i+\delta_2y_2^i+\delta_1\delta_2y_{12}^i,u^j+\delta_1\sum_{i_1=1}^p y_1^{i_1}u_{i_1}^j+\\ \delta_2\sum_{i_1=1}^p\left(y_2^{i_1}+\delta_1y_{12}^{i_1}\right)\left(u_{i_1}^p+\delta_1\sum_{i_2=1}^p y_1^{i_2}u_{i_1;i_2}^j\right)\end{array}\right)\in\mathbb{R}^{p+q}$$
$$=(\delta_1,\delta_2)\in\mathbb{R}^2\mapsto\left(\begin{array}{c}x^i+\delta_1y_1^i+\delta_2y_2^i+\delta_1\delta_2y_{12}^i,u^j+\\ \delta_1\sum_{i_1=1}^p y_1^{i_1}u_{i_1}^j+\delta_2\sum_{i_1=1}^p y_2^{i_1}u_{i_1}^j+\\ \delta_1\delta_2\sum_{i_1=1}^p y_{12}^{i_1}u_{i_1}^j+\delta_1\delta_2\sum_{i_2=1}^p\sum_{i_1=1}^p y_1^{i_2}y_2^{i_1}u_{i_1;i_2}^j\end{array}\right)\in\mathbb{R}^{p+q} \quad (5)$$

On the other hand, we have

$$\delta_2\in\mathbb{R}\mapsto\nabla_{(x^i,u^j)}(\gamma(0,\cdot))(\delta_2)\in\mathbb{R}^{p+q}$$
$$=\delta_2\in\mathbb{R}\mapsto\left(x^i+\delta_2y_2^i,u^j+\delta_2\sum_{i_1=1}^p y_2^{i_1}u_{i_1}^j\right)\in\mathbb{R}^{p+q}$$
$$\delta_2\in\mathbb{R}\mapsto\nabla_{(x^i,u^j,u_{i_1}^j)}(\gamma(0,\cdot))(\delta_2)\in\mathbb{R}^{p+q+pq}$$
$$=\delta_2\in\mathbb{R}\mapsto\left(x^i+\delta_2y_2^i,u^j+\delta_2\sum_{i_1=1}^p y_2^{i_1}u_{i_1}^j,u_{i_1}^j+\delta_2\sum_{i_2=1}^p y_2^{i_2}u_{i_1;i_2}^j\right)\in\mathbb{R}^{p+q+pq}$$

while we have

$$(\delta_1,\delta_2)\in\mathbb{R}^2\mapsto\nabla_{\nabla_{(x^i,u^j)}(\gamma(0,\cdot))(\delta_2)}(\gamma(\cdot,\delta_2))(\delta_1)\in\mathbb{R}^{p+q}$$
$$=(\delta_1,\delta_2)\in\mathbb{R}^2\mapsto\left(\begin{array}{c}x^i+\delta_1y_1^i+\delta_2y_2^i+\delta_1\delta_2y_3^i,u^j+\delta_2\sum_{i_1=1}^p y_2^{i_1}u_{i_1}^j+\\ \delta_1\sum_{i_1=1}^p\left(y_1^{i_1}+\delta_2y_3^{i_1}\right)\left(u_{i_1}^j+\delta_2\sum_{i_2=1}^p y_2^{i_2}u_{i_1;i_2}^j\right)\end{array}\right)\in\mathbb{R}^{p+q}$$
$$=(\delta_1,\delta_2)\in\mathbb{R}^2\mapsto\left(\begin{array}{c}x^i+\delta_1y_1^i+\delta_2y_2^i+\delta_1\delta_2y_3^i,u^j+\\ \delta_1\sum_{i_1=1}^p y_1^{i_1}u_{i_1}^j+\delta_2\sum_{i_1=1}^p y_2^{i_1}u_{i_1}^j+\\ \delta_1\delta_2\sum_{i_1=1}^p y_3^{i_1}u_{i_1}^j+\delta_1\delta_2\sum_{i_2=1}^p\sum_{i_1=1}^p y_1^{i_1}y_2^{i_2}u_{i_1;i_2}^j\end{array}\right)\in\mathbb{R}^{p+q}$$
$$=(\delta_1,\delta_2)\in\mathbb{R}^2\mapsto\left(\begin{array}{c}x^i+\delta_1y_1^i+\delta_2y_2^i+\delta_1\delta_2y_3^i,u^j+\\ \delta_1\sum_{i_1=1}^p y_1^{i_1}u_{i_1}^j+\delta_2\sum_{i_1=1}^p y_2^{i_1}u_{i_1}^j+\\ \delta_1\delta_2\sum_{i_2=1}^p y_3^{i_2}u_{i_2}^j+\delta_1\delta_2\sum_{i_1=1}^p\sum_{i_2=1}^p y_1^{i_1}y_2^{i_2}u_{i_1;i_2}^j\end{array}\right)\in\mathbb{R}^{p+q} \quad (6)$$

Therefore it follows from (5) and (6) that

$$(\delta_1,\delta_2)\in\mathbb{R}^2\mapsto\nabla_{\nabla_{(x^i,u^j)}(\gamma(\cdot,0))(\delta_1)}(\gamma(\delta_1,\cdot))(\delta_2)\in\mathbb{R}^{p+q}$$

$$= (\delta_1, \delta_2) \in \mathbb{R}^2 \mapsto \nabla_{\nabla_{(x^i,u^j)}(\gamma(0,\cdot))(\delta_2)}(\gamma(\cdot, \delta_2))(\delta_1) \in \mathbb{R}^{p+q}$$

for all $\gamma \in (M \otimes \mathcal{W}_{D^2})_{(x^i)}$ iff

$$u^j_{i_1;i_2} = u^j_{i_2;i_1}$$

for all $1 \leq i_1, i_2 \leq p$ and all $1 \leq j \leq q$. This completes the proof. □

Definition 74. Thus we have defined a bijection $\theta^{\mathcal{J}^2(\pi)}_{\mathbf{J}^2(\pi)} : \mathcal{J}^2(\pi) \to \mathbf{J}^2(\pi)$, which goes formally as follows:

$$\begin{aligned}
&\theta^{\mathcal{J}^2(\pi)}_{\mathbf{J}^2(\pi)} \left(x^i, u^j, u^j_{i_1}, u^j_{i_1,i_2}\right) \\
&= \left[\delta \in \mathbb{R} \mapsto \left(x^i\right) + \left(y^i\right)\delta \in \mathbb{R}^p\right] \in (M \otimes \mathcal{W}_D)_{(x^i)} \mapsto \\
&\left[\delta \in \mathbb{R} \mapsto \theta^{\mathcal{J}^1(\pi)}_{\mathbf{J}^1(\pi)} \left(\left(x^i, u^j, u^j_{i_1}\right) + \left(y^i, \sum_{i_1=1}^p u^j_{i_1} y^{i_1}, \sum_{i_2=1}^p u^j_{i_1,i_2} y^{i_2}\right)\delta\right) \in \mathbf{J}^1(\pi)\right] \\
&\in \left(\mathbf{J}^1(\pi) \otimes \mathcal{W}_D\right)_{\theta^{\mathcal{J}^1(\pi)}_{\mathbf{J}^1(\pi)}\left(x^i,u^j,u^j_{i_1}\right)}
\end{aligned}$$

We can go on by induction on n.

Theorem 75. *The mapping $\theta^{\mathcal{J}^{n+1}(\pi)}_{\mathbf{J}^{n+1}(\pi)} : \mathcal{J}^{n+1}(\pi) \to \mathbf{J}^{n+1}(\pi)$, which is defined to be*

$$\begin{aligned}
&\theta^{\mathcal{J}^{n+1}(\pi)}_{\mathbf{J}^{n+1}(\pi)} \left(x^i, u^j, u^j_{i_1}, u^j_{i_1,i_2}, ..., u^j_{i_1,i_2,...,i_{n+1}}\right) \\
&= \left[\delta \in \mathbb{R} \mapsto \left(x^i\right) + \left(y^i\right)\delta \in \mathbb{R}^p\right] \in (M \otimes \mathcal{W}_D)_{(x^i)} \mapsto \\
&\left[\begin{array}{c} \delta \in \mathbb{R} \mapsto \\ \theta^{\mathcal{J}^n(\pi)}_{\mathbf{J}^n(\pi)} \left(\begin{array}{c} \left(x^i, u^j, u^j_{i_1}, ..., u^j_{i_1,i_2,...,i_n}\right) + \\ \left(y^i, \sum_{i_1=1}^p u^j_{i_1} y^{i_1}, ..., \sum_{i_{n+1}=1}^p u^j_{i_1,i_2,...,i_{n+1}} y^{i_{n+1}}\right)\delta \end{array}\right) \in \mathbf{J}^n(\pi) \end{array}\right] \\
&\in (\mathbf{J}^n(\pi) \otimes \mathcal{W}_D)_{\theta^{\mathcal{J}^n(\pi)}_{\mathbf{J}^n(\pi)}\left(x^i,u^j,u^j_{i_1},...,u^j_{i_1,i_2,...,i_n}\right)}
\end{aligned}$$

by induction on n, is bijective.

7. The Second Approach with Coordinates

Definition 76. We define mappings $\theta_{\mathbb{J}^{D^n}(\pi)}^{\mathcal{J}^n(\pi)} : \mathcal{J}^n(\pi) \to \mathbb{J}^{D^n}(\pi)$ as $\varphi_n \circ \theta_{\mathbf{J}^n(\pi)}^{\mathcal{J}^n(\pi)}$.

Remark 77. Since $\mathbb{J}^D(\pi) = \mathbf{J}^1(\pi)$ and φ_1 is the identity transformation, we have

$$\begin{aligned}
&\theta_{\mathbb{J}^{D^1}(\pi)}^{\mathcal{J}^1(\pi)}\left(x^i, u^j, u^j_{i_1}\right) \\
&= \left[\delta \in \mathbb{R} \longmapsto (x^i) + \delta(y^i) \in \mathbb{R}^p\right] \in (M \otimes \mathcal{W}_D)_{(x^i)} \mapsto \\
&\left[\delta \in \mathbb{R} \longmapsto (x^i, u^j) + \delta(y^i, \sum_{i_1=1}^{p} y^{i_1} u^j_{i_1}) \in \mathbb{R}^{p+q}\right] \\
&\in (E \otimes \mathcal{W}_D)_{(x^i, u^j)}
\end{aligned}$$

With due regard to Theorem 75, it is easy to see that

Lemma 78. *Given* $\left(x^i, u^j, u^j_{i_1}, u^j_{i_1,i_2}, ..., u^j_{i_1,i_2,...,i_{n+1}}\right) \in \mathcal{J}^{n+1}(\pi)$*, we have*

$$\begin{aligned}
&\theta_{\mathbb{J}^{D^{n+1}}(\pi)}^{\mathcal{J}^{n+1}(\pi)}\left(x^i, u^j, u^j_{i_1}, u^j_{i_1,i_2}, ..., u^j_{i_1,i_2,...,i_{n+1}}\right) \\
&= \left[(\delta_1, ..., \delta_{n+1}) \in \mathbb{R}^{n+1} \longmapsto (x^i) + \sum_{r=1}^{n+1} \sum_{1 \le k_1 < ... < k_r \le n+1} \delta_{k_1} ... \delta_{k_r} (y^i_{k_1,...,k_r}) \in \mathbb{R}^p\right] \\
&\in (M \otimes \mathcal{W}_{D^{n+1}})_{(x^i)} \mapsto \\
&\left[\begin{array}{c}
\theta_{\mathbb{J}^{D^n}(\pi)}^{\mathcal{J}^n(\pi)}\left(\begin{array}{c}
(\delta_1, ..., \delta_{n+1}) \in \mathbb{R}^{n+1} \longmapsto \\
\left(x^i, u^j, u^j_{i_1}, ..., u^j_{i_1,i_2,...,i_n}\right) + \\
\left(y^i_{n+1}, \sum_{i_1=1}^p u^j_{i_1} y^{i_1}_{n+1}, ..., \sum_{i_{n+1}=1}^p u^j_{i_1,i_2,...,i_{n+1}} y^{i_{n+1}}_{n+1}\right) \delta_{n+1}
\end{array}\right) \\
\left(\begin{array}{c}
(\delta_1, ..., \delta_n) \in \mathbb{R}^n \longmapsto (x^i) + \\
\sum_{r=1}^n \sum_{1 \le k_1 < ... < k_r \le n} \delta_{k_1} ... \delta_{k_r} (y^i_{k_1,...,k_r} + y^i_{k_1,...,k_r,n+1} \delta_{n+1}) \in \mathbb{R}^p
\end{array}\right)
\end{array}\right] \\
&\in (E \otimes \mathcal{W}_{D^{n+1}})_{(x^i, u^j)}
\end{aligned}$$

Now we are going to determine $\theta_{\mathbb{J}^{D^2}(\pi)}^{\mathcal{J}^2(\pi)}$.

Theorem 79. *Given* $\left(x^{i},u^{j},u_{i_{1}}^{j},u_{i_{1},i_{2}}^{j}\right)\in\mathcal{J}^{2}(\pi)$, *we have*

$$\theta_{\mathbb{J}^{D^{2}}(\pi)}^{\mathcal{J}^{2}(\pi)}\left(x^{i},u^{j},u_{i_{1}}^{j},u_{i_{1},i_{2}}^{j}\right)$$
$$=\left[(\delta_{1},\delta_{2})\in\mathbb{R}^{2}\mapsto\left(x^{i}\right)+\left(y_{1}^{i}\right)\delta_{1}+\left(y_{2}^{i}\right)\delta_{2}+\left(y_{12}^{i}\right)\delta_{1}\delta_{2}\in\mathbb{R}^{p}\right]$$
$$\in\left(M\otimes\mathcal{W}_{D^{2}}\right)_{(x^{i})}\mapsto$$
$$\left[\begin{array}{c}(\delta_{1},\delta_{2})\in\mathbb{R}^{2}\mapsto(x^{i},u^{j})+(y_{1}^{i},\sum_{i_{1}=1}^{p}y_{1}^{i_{1}}u_{i_{1}}^{j})\delta_{1}+(y_{2}^{i},\sum_{i_{1}=1}^{p}y_{2}^{i_{1}}u_{i_{1}}^{j})\delta_{2}+\\(y_{12}^{i},\sum_{i_{1}=1}^{p}\sum_{i_{2}=1}^{p}y_{1}^{i_{1}}y_{2}^{i_{2}}u_{i_{1},i_{2}}^{j}+\sum_{i_{1}=1}^{p}y_{12}^{i_{1}}u_{i_{1}}^{j})\delta_{1}\delta_{2}\in\mathbb{R}^{p+q}\end{array}\right]$$
$$\in\left(E\otimes\mathcal{W}_{D^{2}}\right)_{(x^{i},u^{j})}$$

Proof. The Taylor representation of

$$\theta_{\mathbb{J}^{D^{2}}(\pi)}^{\mathcal{J}^{2}(\pi)}\left(x^{i},u^{j},u_{i_{1}}^{j},u_{i_{1},i_{2}}^{j}\right)$$
$$\left((\delta_{1},\delta_{2})\in\mathbb{R}^{2}\mapsto\left(x^{i}\right)+\left(y_{1}^{i}\right)\delta_{1}+\left(y_{2}^{i}\right)\delta_{2}+\left(y_{12}^{i}\right)\delta_{1}\delta_{2}\in\mathbb{R}^{p}\right)$$
$$=(\delta_{1},\delta_{2})\in\mathbb{R}^{2}\mapsto$$
$$\theta_{\mathbb{J}^{D}(\pi)}^{\mathcal{J}^{1}(\pi)}\left(\left(x^{i},u^{j},u_{i_{1}}^{j}\right)+\left(y_{2}^{i},\sum_{i_{1}=1}^{p}y_{2}^{i_{1}}u_{i_{1}}^{j},\sum_{i_{2}=1}^{p}y_{2}^{i_{2}}u_{i_{1},i_{2}}^{j}\right)\delta_{2}\right)$$
$$\left(\delta_{1}\in\mathbb{R}\mapsto\left(x^{i}+y_{2}^{i}\delta_{2}\right)+\left(y_{1}^{i}+y_{12}^{i}\delta_{2}\right)\delta_{1}\right)$$
$$\in\mathbb{R}^{p+q}$$

goes as follows:

$$\left(x^{i}+y_{2}^{i}\delta_{2},u^{j}+\left(\sum_{i_{1}=1}^{p}y_{2}^{i_{1}}u_{i_{1}}^{j}\right)\delta_{2}\right)+$$
$$\left(y_{1}^{i}+y_{12}^{i}\delta_{2},\sum_{i_{1}=1}^{p}\left(u_{i_{1}}^{j}+\left(\sum_{i_{2}=1}^{p}y_{2}^{i_{2}}u_{i_{1},i_{2}}^{j}\right)\delta_{2}\right)\left(y_{1}^{i_{1}}+y_{12}^{i_{1}}\delta_{2}\right)\right)\delta_{1}$$
$$=(x^{i},u^{j})+(y_{1}^{i},\sum_{i_{1}=1}^{p}y_{1}^{i_{1}}u_{i_{1}}^{j})\delta_{1}+(y_{2}^{i},\sum_{i_{1}=1}^{p}y_{2}^{i_{1}}u_{i_{1}}^{j})\delta_{2}+$$
$$(y_{12}^{i},\sum_{i_{1}=1}^{p}\sum_{i_{2}=1}^{p}y_{1}^{i_{1}}y_{2}^{i_{2}}u_{i_{1},i_{2}}^{j}+\sum_{i_{1}=1}^{p}y_{12}^{i_{1}}u_{i_{1}}^{j})\delta_{1}\delta_{2}$$

so that we have the coveted result. □

Generally, by the same token, we have

Theorem 80. *Given* $(x^i, u^j, u^j_{i_1}, u^j_{i_1,i_2}, ..., u^j_{i_1,i_2,...,i_n}) \in \mathcal{J}^n(\pi)$, *we have*

$$\theta^{\mathcal{J}^n(\pi)}_{\mathbb{J}^{D^n}(\pi)}(x^i, u^j, u^j_{i_1}, u^j_{i_1,i_2}, ..., u^j_{i_1,i_2,...,i_n})$$
$$= \left[(\delta_1, ..., \delta_n) \in \mathbb{R}^n \longmapsto (x^i) + \sum_{r=1}^{n} \sum_{1 \leq k_1 < ... < k_r \leq n} \delta_{k_1} ... \delta_{k_r} (y^i_{k_1,...,k_r}) \in \mathbb{R}^p \right]$$
$$\in (M \otimes \mathcal{W}_{D^n})_{(x^i)} \mapsto$$
$$\left[\begin{array}{c} (\delta_1, ..., \delta_n) \in \mathbb{R}^n \longmapsto (x^i, u^j) + \\ \sum_{r=1}^{n} \sum_{1 \leq k_1 < ... < k_r \leq n} \delta_{k_1} ... \delta_{k_r} (y^i_{k_1,...,k_r}, \sum \sum_{i_1=1}^{p} ... \sum_{i_s=1}^{p} y^{i_1}_{\mathbf{J}_1} ... y^{i_s}_{\mathbf{J}_s} u^j_{i_1,...,i_s}) \in \mathbb{R}^{p+q} \end{array} \right]$$
$$\in (E \otimes \mathcal{W}_{D^n})_{(x^i, u^j)}$$

where the completely undecorated $\sum$ *is taken over all partitions of the set* $\{k_1, ..., k_r\}$ *into nonempty subsets* $\{\mathbf{J}_1, ..., \mathbf{J}_s\}$, *and if* $\mathbf{J} = \{k_1, ..., k_t\}$ *is a set of natural numbers with* $k_1 < ... < k_t$, *then* $y^i_{\mathbf{J}}$ *denotes* $y^i_{k_1,\cdots,k_t}$.

Proof. By using Lemma 78, we can proceed by induction on n. The details are safely left to the reader. □

Definition 81. We define mappings $\theta^{\mathcal{S}^n(\pi)}_{\mathbb{S}^{D^n}(\pi)} : \mathcal{S}^n(\pi) \to \mathbb{S}^{D^n}(\pi)$ to be

$$\theta^{\mathcal{S}^n(\pi)}_{\mathbb{S}^{D^n}(\pi)} \left(x^i, u^j, u^j_{i_1,i_2,...,i_n} \right)$$
$$= \left[(\delta_1, ..., \delta_n) \in \mathbb{R}^n \longmapsto (x^i) + \sum_{r=1}^{n} \sum_{1 \leq k_1 < ... < k_r \leq n} \delta_{k_1} ... \delta_{k_r} (y^i_{k_1,...,k_r}) \in \mathbb{R}^p \right]$$
$$\in (M \otimes \mathcal{W}_{D^n})_{(x^i)} \mapsto$$
$$\left[\delta \in \mathbb{R} \longmapsto \left(x^i, u^j + \delta \sum_{1 \leq i_1 \leq ... \leq i_n \leq p} y^{i_1}_1 ... y^{i_n}_n u^j_{i_1,i_2,...,i_n} \right) \in \mathbb{R}^{p+q} \right]$$
$$\in (E \otimes \mathcal{W}_D)^{\perp}_{(x^i, u^j)}$$

It is easy to see that

Proposition 82. *The mappings* $\theta^{\mathcal{S}^n(\pi)}_{\mathbb{S}^{D^n}(\pi)} : \mathcal{S}^n(\pi) \to \mathbb{S}^{D^n}(\pi)$ *are bijective.*

It is also easy to see that

Proposition 83. *Given*

$$\nabla = (x^i, u^j, u^j_{i_1}, u^j_{i_1,i_2}, ..., u^j_{i_1,i_2,...,i_{n+1}}),$$

$$\nabla' = (y^i, v^j, v^j_{i_1}, v^j_{i_1,i_2}, ..., v^j_{i_1,i_2,...,i_{n+1}})\mathcal{J}^{n+1}(\pi),$$

we have

$$\pi_n^{n+1}(\nabla) = \pi_n^{n+1}(\nabla')$$

iff

$$\pi_n^{n+1}\left(\theta^{\mathcal{J}^{n+1}(\pi)}_{\mathbb{J}^{D^{n+1}}(\pi)}(\nabla)\right) = \pi_n^{n+1}\left(\theta^{\mathcal{J}^{n+1}(\pi)}_{\mathbb{J}^{D^{n+1}}(\pi)}(\nabla')\right),$$

in which we get

$$\theta^{\mathcal{S}^{n+1}(\pi)}_{\mathbb{S}^{D^{n+1}}(\pi)}\left(\nabla \dot{-} \nabla'\right) = \theta^{\mathcal{J}^{n+1}(\pi)}_{\mathbb{J}^{D^{n+1}}(\pi)}(\nabla) \dot{-} \theta^{\mathcal{J}^{n+1}(\pi)}_{\mathbb{J}^{D^{n+1}}(\pi)}(\nabla')$$

Theorem 84. *The mappings $\theta^{\mathcal{J}^{n}(\pi)}_{\mathbb{J}^{D^{n}}(\pi)} : \mathcal{J}^n(\pi) \to \mathbb{J}^{D^n}(\pi)$ are bijective.*

Proof. We proceed by induction on n. The mapping $\theta^{\mathcal{J}^{1}(\pi)}_{\mathbb{J}^{D}(\pi)} : \mathcal{J}^1(\pi) \to \mathbb{J}^D(\pi)$ is obviously bijective. By Proposition 83 and the induction hypothesis, $\left(\theta^{\mathcal{J}^{n+1}(\pi)}_{\mathbb{J}^{D^{n+1}}(\pi)}, \theta^{\mathcal{S}^{n+1}(\pi)}_{\mathbb{S}^{D^{n+1}}(\pi)} \underset{E}{\times} \theta^{\mathcal{J}^{n}(\pi)}_{\mathbb{J}^{D^{n}}(\pi)}, \theta^{\mathcal{J}^{n}(\pi)}_{\mathbb{J}^{D^{n}}(\pi)}\right)$ gives a morphism of affine bundles from the affine bundle $\pi_{n+1,n} : \mathcal{J}^{n+1}(\pi) \to \mathcal{J}^n(\pi)$ over the vector bundle $\mathcal{S}^{n+1}(\pi) \underset{E}{\times} \mathcal{J}^n(\pi) \to \mathcal{J}^n(\pi)$ to the affine bundle $\pi_{n+1,n} : \mathbb{J}^{D^{n+1}}(\pi) \to \mathbb{J}^{D^n}(\pi)$ over the vector bundle $\mathbb{S}^{D^{n+1}}(\pi) \underset{E}{\times} \mathbb{J}^{D^n}(\pi) \to \mathbb{J}^{D^n}(\pi)$ is an isomorphism of affine bundles, so that the mapping $\theta^{\mathcal{J}^{n+1}(\pi)}_{\mathbb{J}^{D^{n+1}}(\pi)} : \mathcal{J}^{n+1}(\pi) \to \mathbb{J}^{D^{n+1}}(\pi)$ is bijective. □

Corollary 85. *The mappings $\varphi_n : \mathbf{J}^n(\pi) \to \mathbb{J}^{D^n}(\pi)$ are bijective.*

Proof. This follows simply from Theorems 75 and 84 and the commutativity of the following diagram:

$$\begin{array}{ccc} & \mathcal{J}^n(\pi) & \\ \theta^{\mathcal{J}^{n}(\pi)}_{\mathbf{J}^{n}(\pi)} \swarrow & & \searrow \theta^{\mathcal{J}^{n}(\pi)}_{\mathbb{J}^{D^{n}}(\pi)} \\ \mathbf{J}^n(\pi) & \underset{\varphi_n}{\rightarrow} & \mathbb{J}^{D^n}(\pi) \end{array}$$

□

8. The Third Approach with Coordinates

Definition 86. We define mappings $\theta_{\mathbb{J}^{D_n}(\pi)}^{\mathcal{J}^n(\pi)} : \mathcal{J}^n(\pi) \to \mathbb{J}_{D_n}(\pi)$ as $\psi_n \circ \theta_{\mathbb{J}^{D^n}(\pi)}^{\mathcal{J}^n(\pi)}$.

Now we are going to determine $\theta_{\mathbb{J}^{D_2}(\pi)}^{\mathcal{J}^2(\pi)}$.

Theorem 87. *Given* $\left(x^i, u^j, u^j_{i_1}, u^j_{i_1,i_2}\right) \in \mathcal{J}^2(\pi)$, *we have*

$$\begin{aligned}
&\theta_{\mathbb{J}^{D_2}(\pi)}^{\mathcal{J}^2(\pi)}\left(x^i, u^j, u^j_{i_1}, u^j_{i_1,i_2}\right)\\
&= \left[\delta \in \mathbb{R} \longmapsto \left(x^i\right) + \left(y^i_1\right)\delta + \frac{1}{2}\left(y^i_2\right)\delta^2 \in \mathbb{R}^p\right]\\
&\in \left(M \otimes \mathcal{W}_{D_2}\right)_{(x^i)} \mapsto\\
&\left[\begin{array}{c}\delta \in \mathbb{R} \longmapsto (x^i, u^j) + (y^i_1, \sum_{i_1=1}^p y^{i_1}_1 u^j_{i_1})\delta + \\ \frac{1}{2}(y^i_2, \sum_{i_1=1}^p \sum_{i_2=1}^p y^{i_1}_1 y^{i_2}_1 u^j_{i_1,i_2} + \sum_{i_1=1}^p y^{i_1}_2 u^j_{i_1})\delta^2 \in \mathbb{R}^{p+q}\end{array}\right]\\
&\in \left(E \otimes \mathcal{W}_{D_2}\right)_{(x^i,u^j)}
\end{aligned}$$

Proof. The Taylor representation of

$$\left(\mathrm{id}_M \otimes \mathcal{W}_{(d_1,d_2)\in D^2 \mapsto d_1+d_2 \in D_2}\right)\left(\delta \in \mathbb{R} \longmapsto \left(x^i\right) + \left(y^i_1\right)\delta + \frac{1}{2}\left(y^i_2\right)\delta^2 \in \mathbb{R}^p\right)$$

is

$$\begin{aligned}
&(\delta_1,\delta_2) \in \mathbb{R}^2 \longmapsto \left(x^i\right) + \left(y^i_1\right)(\delta_1+\delta_2) + \frac{1}{2}\left(y^i_2\right)(\delta_1+\delta_2)^2 \in \mathbb{R}^p\\
&= (\delta_1,\delta_2) \in \mathbb{R}^2 \longmapsto \left(x^i\right) + \left(y^i_1\right)\delta_1 + \left(y^i_1\right)\delta_2 + \left(y^i_2\right)\delta_1\delta_2 \in \mathbb{R}^p
\end{aligned}$$

so that its transformation under the mapping $\theta_{\mathbb{J}^{D^2}(\pi)}^{\mathcal{J}^2(\pi)}\left(x^i, u^j, u^j_{i_1}, u^j_{i_1,i_2}\right)$ is

$$\begin{aligned}
&(\delta_1,\delta_2) \in \mathbb{R}^2 \mapsto (x^i, u^j) + (y^i_1, \sum_{i_1=1}^p y^{i_1}_1 u^j_{i_1})\delta_1 + (y^i_1, \sum_{i_1=1}^p y^{i_1}_1 u^j_{i_1})\delta_2 +\\
&\quad (y^i_2, \sum_{i_1=1}^p \sum_{i_2=1}^p y^{i_1}_1 y^{i_2}_1 u^j_{i_1,i_2} + \sum_{i_1=1}^p y^{i_1}_2 u^j_{i_1})\delta_1\delta_2\\
&\quad \in \mathbb{R}^{p+q}
\end{aligned}$$

$$= (\delta_1, \delta_2) \in \mathbb{R}^2 \mapsto (x^i, u^j) + (y_1^i, \sum_{i_1=1}^{p} y_1^{i_1} u_{i_1}^j)(\delta_1 + \delta_2) +$$
$$\frac{1}{2}(y_2^i, \sum_{i_1=1}^{p}\sum_{i_2=1}^{p} y_1^{i_1} y_1^{i_2} u_{i_1,i_2}^j + \sum_{i_1=1}^{p} y_2^{i_1} u_{i_1}^j)(\delta_1 + \delta_2)^2$$
$$\in \mathbb{R}^{p+q}$$

Therefore we have the coveted result. □

Generally, by the same token, we have

Theorem 88. *Given* $(x^i, u^j, u_{i_1}^j, u_{i_1,i_2}^j, ..., u_{i_1,i_2,...,i_n}^j) \in \mathcal{J}^n(\pi)$, *we have*

$$\theta_{\mathbb{J}^{D_n}(\pi)}^{\mathcal{J}^n(\pi)}(x^i, u^j, u_{i_1}^j, u_{i_1,i_2}^j, ..., u_{i_1,i_2,...,i_n}^j)$$
$$= \left[\delta \in \mathbb{R} \longmapsto (x^i) + \sum_{k=1}^{n} \frac{\delta^k}{k!}(y_k^i) \in \mathbb{R}^p\right] \in (M \otimes \mathcal{W}_{D_n})_{(x^i)} \mapsto$$
$$\left[\delta \in \mathbb{R} \longmapsto (x^i, u^j) + \sum_{k=1}^{n} \frac{\delta^k}{k!} \sum \sum_{i_1=1}^{p} ... \sum_{i_r=1}^{p} \left(y_k^i, u_{i_1,...,i_r}^j y_{k_1}^{i_1} ... y_{k_r}^{i_r}\right) \in \mathbb{R}^{p+q}\right]$$
$$\in (E \otimes \mathcal{W}_{D_n})_{(x^i,u^j)}$$

where the undecorated $\sum$ *is taken over all partitions of the positive integer* k *into positive integers* $k_1, ..., k_r$ *(so that* $k = k_1 + ... + k_r$*) with* $1 \leq k_1 \leq ... \leq k_r \leq n$*.*

Definition 89. We define mappings $\theta_{\mathbb{S}^{D_n}(\pi)}^{\mathcal{S}^n(\pi)} : \mathcal{S}^n(\pi) \to \mathbb{S}^{D_n}(\pi)$ to be

$$\theta_{\mathbb{S}^{D_n}(\pi)}^{\mathcal{S}^n(\pi)}\left(x^i, u^j, u_{i_1,i_2,...,i_n}^j\right)$$
$$= \left[\delta \in \mathbb{R} \longmapsto (x^i) + \sum_{k=1}^{n} \frac{\delta^k}{k!}(y_k^i) \in \mathbb{R}^p\right] \in (M \otimes \mathcal{W}_{D_n})_{(x^i)} \mapsto$$
$$\left[\delta \in \mathbb{R} \longmapsto \left(x^i, u^j + \frac{\delta}{n!} \sum_{1 \leq i_1 \leq ... \leq i_n \leq p} y_1^{i_1} ... y_1^{i_n} u_{i_1,i_2,...,i_n}^j\right) \in \mathbb{R}^{p+q}\right]$$
$$\in (E \otimes \mathcal{W}_D)_{(x^i,u^j)}^{\perp}$$

It is easy to see that

Proposition 90. *The mappings* $\theta^{\mathcal{S}^n(\pi)}_{\mathbb{S}^{D_n}(\pi)} : \mathcal{S}^n(\pi) \to \mathbb{S}^{D_n}(\pi)$ *are bijective.*

It is also easy to see that

Proposition 91. *Given*

$$\nabla = (x^i, u^j, u^j_{i_1}, u^j_{i_1,i_2}, ..., u^j_{i_1,i_2,...,i_{n+1}}),$$

$$\nabla' = (y^i, v^j, v^j_{i_1}, v^j_{i_1,i_2}, ..., v^j_{i_1,i_2,...,i_{n+1}})\mathcal{J}^{n+1}(\pi),$$

we have

$$\pi^{n+1}_n(\nabla) = \pi^{n+1}_n(\nabla')$$

iff

$$\pi^{n+1}_n\left(\theta^{\mathcal{J}^{n+1}(\pi)}_{\mathbb{J}^{D_{n+1}}(\pi)}(\nabla)\right) = \pi^{n+1}_n\left(\theta^{\mathcal{J}^{n+1}(\pi)}_{\mathbb{J}^{D_{n+1}}(\pi)}(\nabla')\right),$$

in which we get

$$\theta^{\mathcal{S}^{n+1}(\pi)}_{\mathbb{S}^{D_{n+1}}(\pi)}\left(\nabla \dot{-} \nabla'\right) = \theta^{\mathcal{J}^{n+1}(\pi)}_{\mathbb{J}^{D_{n+1}}(\pi)}(\nabla) \dot{-} \theta^{\mathcal{J}^{n+1}(\pi)}_{\mathbb{J}^{D_{n+1}}(\pi)}(\nabla')$$

Now we have

Theorem 92. *The mappings* $\theta^{\mathcal{J}^n(\pi)}_{\mathbb{J}^{D_n}(\pi)} : \mathcal{J}^n(\pi) \to \mathbb{J}^{D_n}(\pi)$ *are bijective.*

Proof. The mapping $\theta^{\mathcal{J}^1(\pi)}_{\mathbb{J}^D(\pi)} : \mathcal{J}^1(\pi) \to \mathbb{J}^D(\pi)$ is obviously bijective. We proceed by induction on n. By Proposition 91 and the induction hypothesis, $\left(\theta^{\mathcal{J}^{n+1}(\pi)}_{\mathbb{J}^{D_{n+1}}(\pi)}, \theta^{\mathcal{S}^{n+1}(\pi)}_{\mathbb{S}^{D_{n+1}}(\pi)} \underset{E}{\times} \theta^{\mathcal{J}^n(\pi)}_{\mathbb{J}^{D_n}(\pi)}, \theta^{\mathcal{J}^n(\pi)}_{\mathbb{J}^{D_n}(\pi)}\right)$ gives a morphism of affine bundles from the affine bundle $\pi_{n+1,n} : \mathcal{J}^{n+1}(\pi) \to \mathcal{J}^n(\pi)$ over the vector bundle $\mathcal{S}^{n+1}(\pi) \underset{E}{\times} \mathcal{J}^n(\pi) \to \mathcal{J}^n(\pi)$ to the affine bundle $\pi_{n+1,n} : \mathbb{J}^{D_{n+1}}(\pi) \to \mathbb{J}^{D_n}(\pi)$ over the vector bundle $\mathbb{S}^{D_{n+1}}(\pi) \underset{E}{\times} \mathbb{J}^{D_n}(\pi) \to \mathbb{J}^{D_n}(\pi)$ is an isomorphism of affine bundles, so that the mapping $\theta^{\mathcal{J}^{n+1}(\pi)}_{\mathbb{J}^{D_{n+1}}(\pi)} : \mathcal{J}^{n+1}(\pi) \to \mathbb{J}^{D_{n+1}}(\pi)$ is bijective. $\square$

Corollary 93. *The mappings* $\psi_n : \mathbb{J}^{D^n}(\pi) \to \mathbb{J}^{D_n}(\pi)$ *are bijective.*

Proof. This follows simply from Theorems 84 and 92 and the commutativity of the following diagram:

$$
\begin{array}{ccc}
 & \mathcal{J}^n(\pi) & \\
\theta^{\mathcal{J}^n(\pi)}_{\mathbb{J}^{D^n}(\pi)} \swarrow & & \searrow \theta^{\mathcal{J}^n(\pi)}_{\mathbb{J}^{D_n}(\pi)} \\
\mathbb{J}^{D^n}(\pi) & \underset{\psi_n}{\rightarrow} & \mathbb{J}^{D_n}(\pi)
\end{array}
$$

□

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