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AROUND COMPLETE CLASSIFICATION OF LIÉNARD EQUATION AND APPLICATION

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Abstract: In this paper, we present a *complete* symmetry classification of Liénard equation $\ddot{x} = f(x)\dot{x} + g(x)$. The transformations we consider are of the form $(t, x) \to (at + \alpha(x),\ \beta(x))$. They preserve the set of periodic solutions. The machinery used to accomplish this complete classification is Ritt-Kolchin's theory of characteristic sets. Also, we give an example presented in work of J. Goard [7], who using Lie symmetry methods in finance.

1. Introduction

In 1928, the french engineer A. Liénard introduced the differential equation

$$\ddot{x} = f(x)\dot{x} + g(x) \tag{1}$$

where f and g are two real-valued analytic functions. Today, there is no doubt,

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equation (1) is one of the most important differential equations not only in mathematics but also in electronics (where the equation (1) appears as Raylegh or Van der Pol equation), cardiology (modeling the electric heart activity), neurology (modeling neurons activity), biology, mechanic, seismology and physics.

To this end, perturbed soliton equations represent those which differ slightly from the standard soliton equations and represent physically the more realistic experimental situations, especially the effect of various forms of dissipation, dispersion etc. which are treated as perturbation. Various perturbation methods have been developed to study these perturbed soliton equations [8]. We can shall show that some of the well known perturbed soliton equations can be reduced to equation (1) by using suitable ansatz, also represent the exact solutions of these perturbed equations. The Perturbed Cubic Nonlinear Schr¨odinger Equation [8], Perturbed Modified KdV Equation and Perturbed Wadati-Segur-Ablowitz (WSA) equation [11].

Consider the local diffeomorphisms $\varphi \in Diff^{loc}(\mathbb{R}^2)$

$$(t, x) \to (at + \alpha(x),\ \beta(x)) \tag{2}$$

where α and β are two real-valued function such that $a\beta_x \neq 0$. The inequality refers to the invertibility of the transformations φ. It is not hard to see that such transformations form a Lie pseudogroup (Roughly speaking, a pseudogroup is an almost group: the main differences are (i) the composition is not always defined and (ii) there are many neutral elements depending on the open set on which they are defined. A pseudogroup is of Lie if it is defined by a system of partial differential equations (PDE) (and inequations) called the Lie defining equations) and have the important feature of preserving periodic solutions. Using the Kolchin's theory of characteristic sets we give a complete symmetry classification of Liénard equation (1) with respect of the global diffeomorphisms (2).

2. Lie's Method. An Overview

Every differential equation $y^{(n+1)} = f(x, y, \cdots, y^{(n)})$ defines a Pfaffian system [4] $\mathcal{E}_f = (M, \Delta_f)$ where $M := J^n(\mathbb{R}, \mathbb{R})$ is the n–the order jets space of real-valued functions $y(x) : \mathbb{R} \to \mathbb{R}$ and Δ_f is the one-dimensional distribution generated by the Cartan's field

$$D_x := \partial_x + y_1\partial_y + y_2\partial_{y_1} + \cdots + f(\mathbf{x})\partial_{y_n}$$

where $\mathbf{x} = (x, y, \cdots, y^{(n)}) \in M$. Now, in this context the definition of a symmetry is

Definition 1. A symmetry of the Pfaffian system $\mathcal{E}_f = (M, \Delta_f)$ is a local diffeormorphism $\varphi \in Diff^{loc}(M)$ which preserves the contact structure of $\mathcal{E}_f$ i.e.

$$\varphi_*(\Delta_f) = \Delta_f.$$

Symmetries in this definition are internal [1]. The set of all symmetries of a given Pfaffian system $\mathcal{E}_f$ is a Lie pseudogroup denoted by $Aut^{loc}(\mathcal{E}_f) \subset Diff^{loc}M$. Since the distribution Δ_f is involutive, $Aut^{loc}(\mathcal{E}_f)$ is the symmetry pseudogroup of a foliation. Such a pseudogroup is infinite dimensional. And this why in practice (in order to classify), we restrict ourselves to symmetries belonging to a certain Lie pseudogroup $\Phi \subset Diff^{loc}M$ of local diffeomorphisms of interest. Let $\mathcal{S}_f = Aut^{loc}(\mathcal{E}_f) \cap \Phi$ denotes the Lie pseudogroup of such symmetries. Its defining equations are given by the *non linear* PDE's system

$$\varphi_*(\Delta_f) = \Delta_f \text{ et } \varphi \in \Phi \tag{3}$$

where the second constraint means that φ fulfills the Lie defining equations of the Lie pseudogroup Φ.

The non linear PDE system (3) simplifies to a *linear* system if we switch to the calculation of *infinitesimal generators* of the Lie pseudogroup $\mathcal{S}_f$. Now we present briefly this technique due to S. Lie. A good reference is the book [5] but also [7] and [2].

Let G be one-dimensional Lie group (in practice G is the additive group $(\mathbb{R}, +)$). Recall that a one-parameter transformations group on manifold M is a a map $(\epsilon, p) \in G \times M \to \varphi_\epsilon(p) \in M$ satisfying $\varphi_{\epsilon+\tau}(p) = \varphi_\epsilon \circ \phi_\tau(p)$ and if e is the identity element of G, φ_e is the identity transformation. Each one-parameter transformations group φ_ϵ induces a vector field X in the following manner. For each $p \in M$, X_p is the tangent vector of the curve $\gamma(\epsilon) = \varphi_\epsilon(p)$ at the point $p = \varphi_0(p)$ i.e. $\frac{d\varphi_\epsilon(p)}{d\epsilon}|_{\epsilon=0} = X_p$. The vector field X is called *infinitesimal generator* associated to the one-parameter group φ_ϵ. Conversely, to each vector field X we can associate a "local" one-parameter transformations group. The diffeomorphism $M \ni p \to \varphi_\epsilon(p) \in M$ is called the *flow* or the dynamic generated by X. [If we can take $\epsilon = \infty$, for each p , X is said to be complete. If M is compact, every X is complete]. The operator

$$\mathcal{L}_X : \Gamma\left(\otimes^r TM \otimes^s T^*M\right) \to \Gamma\left(\otimes^r TM \otimes^s T^*M\right)$$

defined by

$$\mathcal{L}_X = \lim_{\epsilon \to 0} \frac{\varphi_\epsilon^* - Id}{\epsilon}$$

is called the *Lie derivative* in the direction X and we have

$$\varphi_\epsilon^* = Id + \epsilon \mathcal{L}_X + O(\epsilon^2). \tag{4}$$

In particular, if Y is a vector field ($Y \in \Gamma(TM)$) then we have $\mathcal{L}_X(Y) = [X, Y]$.

Let us go back to our symmetries: now we are looking for local one-parameter symmetry groups. We know, such symmetries are of the form $\varphi_\epsilon(p) = p + \epsilon X(p) + O(\epsilon^2)$ for all $p \in M$ and for a certain $X \in \Gamma(TM)$ [Of course one needs to combine this with the fact that they also of the form $(t, x) \to (at + \alpha(x),\ \beta(x))$ which is explained in Section 3]. Applying (4) to $\varphi_\epsilon^*(\Delta) = \Delta$, shows that a transformation φ_ϵ is a symmetry of the Pfaffian system $\mathcal{E}_f = (M, \Delta_f)$ if and only if

$$\mathcal{L}_X \Delta = 0 \mod \Delta. \tag{5}$$

The components of the vector field X (called the *infinitesimals*) are now solutions of a *linear* PDE's system. The fluxes (the φ_ϵ) are recovered by solving the system of ordinary differential equations $\frac{d\varphi_\epsilon(p)}{d\epsilon}|_{\epsilon=0} = X_p$ with the initial condition $p = \varphi_0(p)$.

Lie's Classical Method

A symmetry of a differential equation is a transformation mapping an arbitrary solution to another solution of the differential equation. The classical Lie groups of point invariance transformations depend on continuous parameters and act on the system's graph space that is co-ordinatised by the independent and dependent variables. As these symmetries can be determined by an explicit computational algorithm If a partial differential equation (PDE) is invariant under a point symmetry, one can often find similarity solutions or invariant solutions which are invariant under some subgroup of the full group admitted by the PDE. These solutions result from solving a reduced equation in fewer variables.

3. Ritt–Kolchin's Theory of Characteristic Set. An Overview

Reference books are [8] and [6]. A differential ring (field) is by definition a ring (field) endowed with a set of derivations. We denote by Θ, the commutative monoid generated by these derivations. Let $U = \{u_1, u_2, \ldots, u_n\}$ be a set of differential indeterminate. The monoid Θ acts freely on the set U and defines a new set ΘU called the set of derivatives. A *ranking* is a total ordering on the set of derivatives which is compatible with the action of the derivations on ΘU. An *elimination ranking* is such that

$$u \succ v \text{ implies } \theta u \succ \phi v \text{ for all } \theta, \phi \in \Theta \text{ and } u, v \in U.$$

For a fixed ranking, one can define the leader (main variable) of a differential polynomial p to be the highest (with respect to the fixed ranking) derivative appearing in p. We denote the leader of p by u_p.

Let K be a differential field and $A = K\{u_1, \dots, c_n\}$ be the differential ring of the differential polynomial with coefficients in K. If $p \in A \setminus K$ and $q \in A$ then the differential polynomial q is said to be partially reduced with respect to p if q is free of every proper derivation of the leader of p. And it is said to be reduced with respect to p if it is partially reduced with respect to p and $\deg(q, u_p) < \deg(p, u_p)$. A subset of A is said to be auto reduced if no element of the subset belongs to K and each element of the subset is reduced with respect to all the others.

Let $\Sigma \subset A$ be a set of differential polynomials. Suppose that Σ has no (non zero) element of K (i.e., the differential system Σ is consistent)

Definition 2. A subset $C \subset A$ is said to be *characteristic* set of Σ if it is auto reduced and if Σ is free of every non zero element reduced with respect to C.

Let I be a differential ideal generated by Σ. The radical ideal of I is by definition the differential ideal $\sqrt{I} = \{a \in A : a^r \in I \text{ for some } r \in N\}$.

Proposition 3. *Fix a ranking. The radical ideal $\sqrt{I}$ can be decomposed as*

$$\sqrt{I} = I_1 \cap \cdots \cap I_s,$$

in terms of the differential ideals $I_1, \dots, I_s$ represented by the characteristic set $C_1, \cdots, C_s$.

Proof. See for instance [3] □

4. Symmetry Classification of the Liénard Equation

As announced, we consider the pseudogroup of transformations $\varphi \in Diff^{loc}(\mathbb{R}^2)$ of the form

$$\begin{cases} \bar{t} = at + \alpha(x), & a \neq 0 \\ \bar{x} = \beta(x), & \beta_x \neq 0 \end{cases} \tag{6}$$

where $a \in \mathbb{R}$ is an arbitrary constant and α, β are two arbitrary functions.

Proposition 4. *Any transformation of the form (6) maps a periodic function $x(t)$ of period T to another periodic solution of period equals to aT.*

Proof. Let us prove that there exists $\bar{T}$ such that $\bar{x}(\bar{t}+\bar{T}) = \bar{x}(\bar{t})$.

The graph of $x(t)$ is invariant under the translation $\lambda_T : (x,t) \to (x, t+T)$.

$$\begin{tikzcd}
(x,\, t) \arrow[r, "\lambda_T"] \arrow[d, "\varphi"'] & (x,\, t+T) \arrow[d, "\varphi"] \\
(\beta(x),\, \alpha(x)+at) \arrow[r, "\lambda_{aT}"] & (\beta(x),\, \alpha(x)+a(t+T))
\end{tikzcd}$$

We conclude that the graph of $\bar{x}(\bar{t})$ is invariant under $\varphi \circ \lambda_T \circ \varphi^{-1}$. $\square$

4.1. Generation of Lie Equations

Let us first determine the infinitesimal generators X with fluxes of the form (6). Let make the substitution

$$\bar{t} = t + \epsilon A(x,t) + O(\epsilon^2),\ \bar{x} = x + \epsilon B(x,t) + O(\epsilon^2),$$

in the defining equations of the Lie pseudogroup (6):

$$\frac{\partial^2 \bar{t}}{\partial t^2} = 0,\ \frac{\partial^2 \bar{t}}{\partial x \partial t} = 0,\ \frac{\partial \bar{x}}{\partial t} = 0,\ \frac{\partial \bar{t}}{\partial t}\frac{\partial \bar{x}}{\partial x} \neq 0.$$

We obtain

$$\frac{\partial^2 A(x,t)}{\partial t^2} = 0,\ \frac{\partial^2 A(x,t)}{\partial x \partial t} = 0,\ \frac{\partial B(x,t)}{\partial t} = 0.$$

This allows one to deduce that the infinitesimal generators X must be of the form

$$X = (\lambda t + A(x))\frac{\partial}{\partial t} + B(x)\frac{\partial}{\partial x}.$$

Now Lie equations are obtained by writing that the Lie derivative $[X, D_t]$ is zero modulo $D_t = \frac{\partial}{\partial t} + p\frac{\partial}{\partial x} + (f(x)p + g(x))\frac{\partial}{\partial p}$ where $p = \dot{x}$. We obtain the ODE system

$$\begin{cases} Bg_x + B_x g - 2\lambda g = 0, \\ B_{x,x} - 2A_x f = 0, \\ A_{x,x} = 0, \\ Bf_x + 3A_x g + \lambda f = 0, \\ \lambda_x = 0. \end{cases} \tag{7}$$

The system (7) depends on two arbitrary functions f and g and linear in the differential unknowns A, B andλ.

Theorem 5. *The classification below is complete.*

Proof. Let $I \subset (\mathbb{Q}\{f, g, A, B, \lambda\}, \partial_x)$ denotes the differential ideal generated by the system (7). For the elimination ranking $(A, B, \lambda) \succ (f, g)$ which eliminates the indeterminate A, B and λ in the system (7) i.e., characterizes the ideal $\sqrt{I} \cap \mathbb{Q}\{f,\, g\}$, we have

$$\sqrt{I} \cap \mathbb{Q}\{f,\, g\} = \sqrt{I_1} \cap \mathbb{Q}\{f,\, g\} \cap \cdots \sqrt{I_8} \cap \mathbb{Q}\{f,\, g\}$$

where the differential ideals $\sqrt{I_i} \cap \mathbb{Q}\{f,\, g\}$ represent all different cases (constraints) below. $\square$

In the following paragraphs, we give the characteristic representations of the ideals $\sqrt{I_i}$.

The Generic Case

The first characteristic set is

$$A_x = 0,\ \ B = 0, \lambda = 0.$$

This is the generic case, there is no constraint on the functions f and g. The dimension of the corresponding Lie algebra is equals to the number of points under the three stairs associated to the unknowns A, B and λ. Hence equals to one. The integration shows that the infinitesimal generator is $X_1 = \frac{\partial}{\partial t}$ and the corresponding fluxes form the one-parameter Lie group of temporal translations. Van der Pol equation $\ddot{x} - \epsilon(1 - x^2)\dot{x} + x = 0$, belongs to this class.

Case II

It has four subcases where the number of points under the stairs corresponding to the unknowns $A,\ B$ and λ (i.e. the dimension of the symmetry Lie algebra) is equal to two. If $X_2 = (\lambda t + A(x))\frac{\partial}{\partial t} + B(x)\frac{\partial}{\partial x}$ is another vector field (different from $X_1 = \frac{\partial}{\partial t}$) then $[X_2, X_1] = \lambda X_1$. The symmetry Lie algebra is consequently the affine algebra $\mathfrak{a}(1, \mathbb{R})$ if $\lambda \neq 0$ or the abelian algebra otherwise. In both situations, it solvable and Liénard equation can be reduced to a quadrature ([6], [2]).

Subcase II-1. $3g_{xx}+2ff_x \neq 0$ **and** $g \neq 0$ The first of the four characteristic sets is

$$\begin{cases} \lambda_x = 0, \\ A_x = -\dfrac{\lambda\,(f g_{xx} - 2f_x g_x)}{g\,(3g_{xx} + 2ff_x)}, \\ B = -2\dfrac{\lambda\left(f^2 + 3g_x\right)}{3g_{xx} + 2ff_x}, \\ g_{xxx} = \dfrac{5g_{xx}gff_x + 6g{g_{xx}}^2 - 2g{f_x}^2 g_x - 3g_{xx}{g_x}^2 - 2ff_x{g_x}^2}{g\,(f^2 + 3g_x)} \\ f_{xx} = \dfrac{9g_{xx}gf_x - 3g_{xx}g_x f + 6f{f_x}^2 g - 2f^2 f_x g_x}{2g\,(f^2 + 3g_x)} \end{cases}$$

The two last equations (in addition to the inequalities) constrain the function f and g. They characterize the differential ideal $\sqrt{I_2} \cap \{f, g\}$. The other equations give the functions A, B and λ. In particular, one sees that λ furnishes a non zero structure constant. This proves:

Proposition 6. *The symmetry Lie algebra in this case is isomorphic to* $\mathfrak{a}(1, \mathbb{R})$.

To give an example of such equations, one can consider the equation $\ddot{x} = x\dot{x} + x^3$. The infinitesimal generators are $X_1 = \frac{\partial}{\partial t}$, $X_2 = t\frac{\partial}{\partial t} - x\frac{\partial}{\partial x}$ and the fluxes generated by X_1 and X_2 form the two-dimensional Lie group if special affine transformations $(t, x) \to (\lambda t + \mu, \frac{x}{\lambda})$. The Jacobian $\begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix}$ of such transformation has determinant equals to 1.

Subcase II.2. $3g_{xx} = -2ff_x, f_{xx} = 0$ **and** $g \neq 0$ We have the characteristic set

$$\begin{cases} \lambda = 0, \\ A_x = -\dfrac{Bf_x}{3g}, \\ B_x = \dfrac{Bg_x}{g}, \\ f_{xx} = 0, \\ g_{xx} = -\dfrac{2}{3}ff_x. \end{cases}$$

The first equation immediately proves

Proposition 7. *The symmetry Lie algebra is the two-dimensional abelian algebra.*

The integration of the two last equations, which characterize $\sqrt{I_3} \cap \{f, g\}$, yields (the a_i are arbitrary constants)

$$\begin{cases} f(x) = a_1 x + a_2, \\ g(x) = -\dfrac{1}{9} a_1^2 x^3 - \dfrac{1}{3} a_1 a_2 x^2 + a_3 x + a_4. \end{cases}$$

And this in turn yields

$$\begin{cases} X_1 = \dfrac{\partial}{\partial t}, \\ X_2 = -x \dfrac{a_1}{3a_4} \dfrac{\partial}{\partial t} + \left(1 + x \dfrac{a_3}{a_4} - x^2 \dfrac{a_2 a_1}{3a_4} - x^3 \dfrac{{a_1}^2}{9a_4}\right) \dfrac{\partial}{\partial x}. \end{cases}$$

Subcase II.3. $3g_{xx} - 2g_x^2 \neq 0, f_{xx} \neq 0$ **and** $g \neq 0$ Here the function A, B and λ satisfy

$$\begin{cases} \lambda_x = 0, \\ A_x = \dfrac{\lambda\left(-3\, g f f_{x,x} + 9\, {f_x}^2 g + f^3 f_x\right)}{9 g^2 f_{x,x}}, \\ B = -\dfrac{\lambda\left(9\, g f_x + f^3\right)}{3 g f_{x,x}}. \end{cases}$$

The first equation proves

Proposition 8. *The symmetry Lie algebra is isomorphic to* $\mathfrak{a}(1, \mathbb{R})$.

The functions f and g satisfy the ODE system $(\sqrt{I_4} \cap \{f, g\})$

$$\begin{cases} f^2 = -3 g_x, \\ g_{xxxx} = -\dfrac{1}{2{g_x}^2 g\left(3 g g_{xx} - 2{g_x}^2\right)} (2 g^2 {g_{xx}}^4 + g_x {g_{xx}}^2 g^2 g_{xxx} - 10 {g_x}^2 g^2 {g_{xxx}}^2 \\ -9 {g_x}^2 {g_{xx}}^3 g + 18 {g_x}^3 g_{xx} g g_{xxx} + 4 {g_x}^4 {g_{xx}}^2 - 8 g_{xxx} {g_x}^5). \end{cases}$$

As example of equation lying in this class, we can take $\ddot{x} = x^2 \dot{x} - \frac{1}{15} x^5$ for which we have $X_1 = \frac{\partial}{\partial t}$ and $X_2 = t \frac{\partial}{\partial t} - \frac{x}{2} \frac{\partial}{\partial x}$. The corresponding fluxes form the two-parameter group of affine transformations $(t, x) \to (\lambda t + \mu, \frac{x}{\sqrt{\lambda}})$.

Subcase II.4. $g = 0,\ f f_x \neq 0$ The last of the four subcases is

$$\begin{cases} \lambda_x = 0, \\ A_x = \dfrac{\lambda \left({f_x}^2 f_{xx} + f f_x f_{xxx} - 2 f {f_{xx}}^2 \right)}{2 {f_x}^3 f}, \\ B = -\dfrac{\lambda f}{f_x}, \\ f_{xxxx} = \dfrac{-f {f_x}^3 f_{xxx} - 6 {f_{xx}}^3 f^2 + 6 f^2 f_x f_{xx} f_{xxx} + {f_{xx}}^2 f {f_x}^2 + {f_x}^4 f_{xx}}{f^2 {f_x}^2} \\ g = 0. \end{cases}$$

We deduce that the symmetry Lie algebra can not be abelian and hence:

Proposition 9. *The symmetry Lie algebra is isomorphic to* $\mathfrak{a}(1, \mathbb{R})$.

Clearly, a Liénard equation such that $g = 0$ has the first integral $t - \int^x \frac{1}{\int f(s) + C}$. Hence, such equations can not have limit cycles.

4.1.1. Third Case $g \neq 0$

The characteristic presentation is

$$\begin{cases} \lambda_x = 0, \\ A_x = -\dfrac{f \left(9 \lambda g - B f^2 \right)}{27 g^2}, \\ B_x = \dfrac{6 \lambda g - B f^2}{3g}, \\ f_x = -\dfrac{f^3}{9g}, \\ g_x = \dfrac{f^2}{3} \end{cases}$$

The number of points under the stairs of λ, A and B shows that the Lie algebra is three-dimensional.

Proposition 10. *In this case the Lie algebra is 3-dimensional and generated by*

$$\begin{cases} X_1 = \frac{\partial}{\partial t}, \\ X_2 = (x \frac{f(0)}{3g(0)} + t) \frac{\partial}{\partial t} + \left(2\,x - x^2 \frac{f(0)^2}{3g(0)} + \frac{1}{81}\, x^3 \frac{f(0)^4}{g(0)^2} \right) \frac{\partial}{\partial x} \\ X_3 = x \frac{f(0)^3}{27 g(0)^2} \frac{\partial}{\partial t} + \left(1 - x \frac{f(0)^2}{3g(0)} + x^2 \frac{f(0)^4}{27 g(0)^2} - x^3 \frac{1}{729} \frac{f(0)^6}{g(0)^3} \right) \frac{\partial}{\partial x}, \end{cases}$$

where$f(0), g(0)$ *denote the values of* f *et* g *at* $x = 0$.

Proof. The integration of the last two equations constraining f and g, yields $\{g(x) = a_1,\ f(x) = 0\}$ or $\left\{f(x) = a_1 x + a_2,\ g(x) = -\frac{(a_1 x + a_2)^3}{9 a_1}\right\}$. The infinitesimal generators are obtained by computing the Taylor series of the functions λ, A and B at zero. $\square$

4.1.2. Fourth Case

This case corresponds to the characteristic set

$$\begin{cases} \lambda = 0, \\ A_{x,x} = 0, \\ B_{x,x} = 2\, A_x f, \\ f_x = 0, \\ g = 0. \end{cases}$$

We deduce:

Proposition 11. *The symmetry algebra is four-dimensional. Moreover, Liénard equation is necessarily of the form*

$$\ddot{x} = a\dot{x},$$

where $a \in \mathbb{R}$. The infinitesimal generators are

$$X_1 = \frac{\partial}{\partial t},\ X_2 = \frac{\partial}{\partial x},\ X_3 = x\frac{\partial}{\partial x},\ X_4 = x\frac{\partial}{\partial t} + x^2\frac{\partial}{\partial x},$$

generating $gl(2, \mathbb{R})$. The corresponding fluxes are

$$(t, x) \to (t - \ln(1 - \varepsilon x) + \mu,\ \sigma\frac{x}{1 - \varepsilon x} + \nu).$$

where ϵ, μ, σ and ν are the group parameters

4.1.3. Fifth Case

This case completes the classification. We have the characteristic set

$$\begin{cases} \lambda_x = 0, \\ A_{xx} = 0, \\ B_{xx} = 0, \\ f = 0, \\ g = 0. \end{cases}$$

The last two equations implies that the last differential ideal $\sqrt{I_8} \cap \mathbb{Q}\{f, g\}$ is generated by $\{f, g\}$. We have immediately

Proposition 12. *Liénard equation is reduced to*

$$\ddot{x} = 0.$$

The symmetry Lie algebra is generated by the vector fields

$$X_1 = \frac{\partial}{\partial t},\ X_2 = \frac{\partial}{\partial x},\ X_3 = x\frac{\partial}{\partial x},\ X_4 = x\frac{\partial}{\partial t},\ X_5 = t\frac{\partial}{\partial t}.$$

The corresponding fluxes form a five-parameter transformations group

$$(t, x) \to (\lambda t + \mu + \epsilon x, \rho x + \sigma).$$

$(\lambda, \mu, \epsilon, \rho, \sigma)$ *are the group parameters.*

5. Application

In this section we will describe an example presented in work of *Joanna Goard* who using Lie symmetry methods in finance.

5.1. Bond Pricing Problem

The value of interest rate derivatives, such as bonds and swaps, naturally depends on the interest rates. We can proof that when the short-term interest rate, follows a stochastic differential equation of the form

$$dr = u(r, t)dt + w(r, t)dX, \tag{ES}$$

where X is an increment in a Wiener process, the price of a zero-coupon bond $V(r, t; T)$, if $t = T$ we have $V(r, T) = 1$, will satisfy the partial differential equation (PDE)

$$\frac{\partial V}{\partial t} + \frac{w^2}{2}\frac{\partial^2 V}{\partial r^2} + (u - \lambda w)\frac{\partial V}{\partial r} - rV = 0 \tag{PDE}$$

where $\lambda(r; t)$ is the market price of risk.

Several models describing the dynamics of the short rate can be put in the form

$$dr = (\alpha + \beta r)dt + \sigma r^{\gamma} dX,$$

where α, β, γ and σ are constants.

5.2. Classical Lie Symmetry Classification of Equation (ES)

In this section we assume that the short-term risk-neutral interest rate r follows the stochastic process

$$dr = b(r,t)w(r,t)^2 dt + w(r,t)dX,$$

where $w = cr^n, b(r,t) = a(t)r^p - qr^m$, with $c \neq 0, q, n \neq 0, m \neq 0$ and p are constants . Also, we can write the short-term interest rate in the form

$$dr = \left[b(r,t)w^2 + \lambda(r;t)w\right] dt + wdX$$

moreover we write the PDE

$$\frac{\partial^2 V}{\partial r^2} + \frac{2}{w^2}\frac{\partial V}{\partial t} + 2b(r,t)\frac{\partial V}{\partial r} - \frac{2r}{w^2}V = 0$$

we find that for arbitrary $a(t), n, p, c, q$ and m,PDF has a one dimensional symmetry generated by the vector-field

$$V\frac{\partial}{\partial V}$$

and the infinite-dimensional symmetry generated by

$$g(r,t)\frac{\partial}{\partial V}$$

where g is any solution to (PDE).

5.3. Bond Price Solutions

In this section we give an example so that the final condition $V(r,T) = 1$ is satisfied (using the symmetries- the bond pricing equation- to construct a invariant solution).

Example. We have the short-rate r

$$dr = \left[c^2 r(a(t) - qr) + \lambda(r;t)cr^{\frac{3}{2}}\right] dt + cr^{\frac{3}{2}}dX$$

we can check $w = cr^{\frac{3}{2}}, b = \frac{a(t)}{r^2} - \frac{q}{r}$

Also, the risk-neutral process is

$$dr = \left[c^2 r(a(t) - qr)\right] dt + cr^{\frac{3}{2}}dX$$

For more detail see (Joanna Goard 2008).

Remark. A Lie symmetry classification can significantly expand the class of analytically solvable models in mathematical finance.

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