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ON THE COMPOUND BINOMIAL RISK MODEL WITH STOCHASTIC INCOME

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Abstract: In this paper we consider a risk model where both premiums and claims follow compound binomial processes. A difference equation and a defective renewal equation satisfied respectively by the expected discounted penalty function are derived. Then for a special penalty function and geometrically distributed claim amounts, an explicit expression of the expected discounted penalty function is obtained. Finally, we investigate the case when premiums are degenerate at a constant amount. We derive an alternative defective renewal equation in terms of the roots of the generalized Lundberg's fundamental equation.

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Key Words: compound binomial model, defective renewal equation, difference equation, expected discounted penalty function

1. Introduction

The classical compound binomial model is one of the most popular discrete time renewal risk models in actuarial literature in ruin theory. In this model, the claim number process $\{N(t), t \in \mathbb{N}\}$ is assumed to be a renewal process with independent and identically distributed (i.i.d.) interclaim times $\{W_j, j \in \mathbb{N}^+\}$ having probability function (p.f.) $k_1(l) = p_1 q_1^{l-1}$ for $l \in \mathbb{N}^+$, where $0 < p_1 <$

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$1, q_1 = 1-p_1$. The individual claim amounts $\{X_j, j \in \mathbb{N}^+\}$, independent of $N(t)$, are positive i.i.d. random variables with p.f. $f(x)$, cumulative distribution function (c.d.f.) $F(x) = 1 - \overline{F}(x)$. Then the surplus process $U(t)$ can be expressed as

$$U(t) = u + t - \sum_{i=1}^{N(t)} X_i,$$

where $u \in \mathbb{N}$ is the initial surplus. The readers are referred to Cheng *et al.* [1] and the references therein for this well known risk model.

In the the discrete time renewal risk model, the premium rate is assumed to be 1 per period. The deterministic premium income evidently fails to capture the uncertainty of the customer's arrivals and payments. It is natural to generalize the compound binomial risk process by introducing an additional compound binomial process to keep track of the income received by the insurer. More precisely, the number of the customers up to time t is denoted by $\{M(t), t \in \mathbb{N}\}$, which is a binomial process with i.i.d. inter-arrival times $\{V_j, j \in \mathbb{N}^+\}$ having p.f. $k_2(l) = p_2 q_2^{l-1}$ for $l \in \mathbb{N}^+$, where $0 < p_2 < 1, q_2 = 1-p_2$. The premium sizes are given by the sequence of i.i.d. positive random variables $\{Y_j, j \in \mathbb{N}^+\}$. From the practical point of view, the c.d.f. of Y_1 should have finite support. Without loss of generality, we assume that Y_1 is concentrated on $\{1, 2, \cdots, n\}, n \in \mathbb{N}^+$ with p.f. $g(x)$. Independence is assumed among $\{N(t)\}$, $\{M(t)\}$, $\{X_j\}$ and $\{Y_j\}$. The company's surplus is then given by

$$U(t) = u + \sum_{i=1}^{M(t)} Y_i - \sum_{i=1}^{N(t)} X_i. \tag{1}$$

Furthermore, we assume

$$p_2 E(Y_1) > p_1 E(X_1) \tag{2}$$

to guarantee a positive survival probability.

For the risk model (1), define the time of ruin $T = \min\{t \in \mathbb{N}^+; U(t) < 0\}$ with $T = \infty$ if ruin does not occur. If ruin occurs, $|U(T)|$ is the deficit at ruin and $U(T-1)$ is the surplus immediately prior to ruin. For $v \in (0, 1)$, the expected discounted penalty function is defined as

$$m_v(u) = \mathrm{E}[v^T \omega(U(T-1), |U(T)|) I_{\{T<\infty\}} \,|\, U(0) = u],$$

where $\omega(x, y) : \mathbb{N} \times \mathbb{N}^+ \to \mathbb{N}$ is a nonnegative function and $I_{\{.\}}$ is the indicator function. The quantity $\omega(U(T-1), |U(T)|)$ can be interpreted as the penalty at the time of ruin for the surplus $U(T-1)$ and deficit $|U(T)|$.

The expected discounted penalty function was introduced by Gerber and Shiu [3], which has been a unified tool to study the ruin problems in actuarial literature. With the assumption of unit premium rate, the discrete time renewal risk model has been investigated by many authors. For the classical compound binomial risk model, Cheng *et al*. [1] derived the moment generating function of the time to ruin for zero initial surplus and then they derived recursively the joint distribution of the surplus prior to ruin and the deficit at ruin. Li [8] derived a recursive formula for the discounted penalty function with claim waiting times having a discrete K_m distribution. In a subsequent paper of Li [9], the discounted penalty function was explicitly expressed in terms of a compound geometric distribution function, and the explicit expressions for the probability generating function (p.g.f.) of the time of ruin, the joint and marginal distributions of the surplus before ruin, the deficit at ruin, the claim causing ruin, as well as their moments were derived. Wu and Li [12] obtained a recursive formula satisfied by the penalty function for the discrete renewal risk model with arbitrary interclaim times. Cossette *et al*. [2] consider a discrete time renewal risk model with premium rate $c \in \mathbb{N}^+$, they investigate the aggregate claim amount process and both finite-time and infinite-time ruin probabilities. Under the framework of the compound binomial model with a general premium rate $c \in \mathbb{N}^+$, Landriault [6] proposed a generalization of the expected discounted penalty function and derive an explicit expression for this generalized analytic tool in terms of the zeros of a matrix determinant. See [7,10,14] for other various extensions of the compound binomial model.

The rest of this paper is structured as follows. In Section 2, a difference equation for the discounted penalty function is derived. In Section 3, we show that the Gerber-Shiu function satisfies a defective renewal equation. An explicit expression is obtained for the expected discounted penalty function, provided the penalty only depends on the deficit and the claims are geometrically distributed. Finally, Section 4 specifies the defective renewal equation for the Gerber-Shiu function when the premiums are degenerate at a constant amount.

Similar topics have been discussed by Labbé [5] and Yang [13].

2. Difference Equation for $m_v(u)$

Throughout the entire paper we use $\hat{\vartheta}(z)$ to denote the p.g.f. with dummy variable z for any p.f. ϑ .

This section is concerned with the derivation of the difference equation satisfied by the expected discounted penalty function $m_v(u)$. Recall that W_1

is the time until the first claim and V_1 is the time until the first premium. It is easy to see that $\Pr(W_1 < V_1) = \frac{p_1q_2}{p_1q_2+p_2}$, $\Pr(V_1 < W_1) = \frac{q_1p_2}{p_1q_2+p_2}$, $\Pr(W_1 = V_1) = \frac{p_1p_2}{p_1q_2+p_2}$.

Let $Z = \min\{W_1, V_1\}$, then Z is also a geometrical distribution with p.f. $\Pr(Z = l) = (p_1q_2 + p_2)(q_1q_2)^{l-1}$ for $l \in \mathbb{N}^+$. Denote by

$$m_v(u|Z = l) = \mathrm{E}[v^T \omega(U(T-1), |U(T)|)I_{\{T<\infty\}}\,|\,U(0) = u, Z = l],$$

by using the law of total probability, we have

$$\begin{aligned}
&m_v(u)\\
&= (p_1q_2 + p_2)\sum_{l=1}^{\infty}(q_1q_2)^{l-1}m_v(u\,|\,Z = l)\\
&= (p_1q_2 + p_2)\sum_{l=1}^{\infty}(q_1q_2)^{l-1}\Big\{m_v(u|Z = l, W_1 < V_1)\Pr(W_1 < V_1)\\
&\quad + m_v(u|Z = l, V_1 = W_1)\Pr(V_1 = W_1)\\
&\quad + m_v(u|Z = l, V_1 < W_1)\Pr(V_1 < W_1)\Big\}\\
&= \sum_{l=1}^{\infty}(q_1q_2)^{l-1}v^l\Bigg\{p_1q_2\Bigg[\sum_{j=1}^{u}m_v(u-j)f(j) + \sum_{j=u+1}^{\infty}\omega(u, j-u)f(j)\Bigg]\\
&\quad + p_1p_2\Bigg[\sum_{k=1}^{n}g(k)\Bigg(\sum_{j=1}^{u+k}m_v(u+k-j)f(j)\\
&\quad + \sum_{j=u+k+1}^{\infty}\omega(u+k, j-u-k)f(j)\Bigg)\Bigg] + q_1p_2\sum_{k=1}^{n}m_v(u+k)g(k)\Bigg\}. \qquad (3)
\end{aligned}$$

For notational convenience, we denote by $a = \frac{p_1q_2v}{1-q_1q_2v}$, $b = \frac{p_1p_2v}{1-q_1q_2v}$, $c = \frac{q_1p_2v}{1-q_1q_2v}$, and define $\xi(u) = \sum_{j=u+1}^{\infty}\omega(u, j-u)f(j)$, then equation (3) can be rewritten as

$$\begin{aligned}
m_v(u) = a&\Bigg[\sum_{j=1}^{u}m_v(u-j)f(j) + \xi(u)\Bigg]\\
&+ b\sum_{k=1}^{n}g(k)\Bigg[\sum_{j=1}^{u+k}m_v(u+k-j)f(j) + \xi(u+k)\Bigg]\\
&+ c\sum_{k=1}^{n}m_v(u+k)g(k). \qquad (4)
\end{aligned}$$

Further, let $m_v * f(u) = \sum_{j=1}^{u} m_v(u-j)f(j)$ and $\gamma(u) = a\xi(u) + b\sum_{k=1}^{n} \xi(u+k)g(k)$, then equation (4) can be re-expressed as

$$\begin{aligned} m_v(u) &= am_v * f(u) + b\sum_{k=1}^{n} m_v * f(u+k)g(k) \\ &\quad + c\sum_{k=1}^{n} m_v(u+k)g(k) + \gamma(u). \end{aligned} \tag{5}$$

For any function $h(u), u \in \mathbb{N}$, we define the difference operator $\triangle$ as $\triangle h(u) = h(u+1) - h(u)$. Using the property of difference operator $\triangle$ (see chapter 2 of [4])

$$h(u+k) = \sum_{j=0}^{k} \binom{k}{j} \triangle^j h(u),$$

equation (5) can be expressed in terms of the difference operator as

$$\begin{aligned} m_v(u) &= am_v * f(u) + b\sum_{k=1}^{n} g(k) \sum_{j=0}^{k} \binom{k}{j} \triangle^j m_v * f(u) \\ &\quad + c\sum_{k=1}^{n} g(k) \sum_{j=0}^{k} \binom{k}{j} \triangle^j m_v(u) + \gamma(u) \\ &= am_v * f(u) + b\sum_{j=0}^{n} \triangle^j m_v * f(u) \sum_{k=j}^{n} \binom{k}{j} g(k) \\ &\quad + c\sum_{j=0}^{n} \triangle^j m_v(u) \sum_{k=j}^{n} \binom{k}{j} g(k) + \gamma(u). \end{aligned} \tag{6}$$

Let $\chi_j^1 = I_{\{j=0\}} - c\sum_{k=j}^{n}\binom{k}{j}g(k)$, $\chi_j^2 = aI_{\{j=0\}} + b\sum_{k=j}^{n}\binom{k}{j}g(k)$, then equation (6) implies that

$$\sum_{j=0}^{n} \chi_j^1 \triangle^j m_v(u) = \sum_{j=0}^{n} \chi_j^2 \triangle^j m_v * f(u) + \gamma(u). \tag{7}$$

By letting $A(z) = \sum_{j=0}^{n} z^j \chi_j^1$, $B(z) = \sum_{j=0}^{n} z^j \chi_j^2$, we obtain by (7)

$$A(\triangle) m_v(u) = B(\triangle) m_v * f(u) + \gamma(u). \tag{8}$$

Since $A(z)$ and $B(z)$ are two polynomials of degree n in z, we conclude that $m_v(u)$ satisfies a non-homogeneous difference equation of order n. The solution to (8) can be obtained from the general theory on difference equations, see [4] for more details.

3. Defective Renewal Equation for $m_v(u)$

Along the similar lines as [12], we consider, for $i \in \mathbb{N}, j \in \mathbb{N}^+$

$$f_3(i,j,t \,|\, u) = \Pr\{U(T-1) = i, |U(T)| = j, T = t \,|\, U(0) = u\},$$

which is the joint p.f. of the surplus just before ruin, deficit at ruin and ruin time. And we define $f_2(i,t \,|\, u) = \sum_{j=1}^{\infty} f_3(i,j,t \,|\, u)$ as the joint p.f. of $U(T-1)$ and T. Then $f_1(i \,|\, u) = \sum_{t=1}^{\infty} v^t f_2(i,t \,|\, u)$ can be viewed as the discounted p.f. of $U(T-1)$. Further, define

$$p_i(j) = f(i+j+1)/\overline{F}(i+1), \quad j \in \mathbb{N}^+,$$

then the usual conditional probability formula gives the following relation

$$f_3(i,j,t \,|\, u) = p_i(j) f_2(i,t \,|\, u).$$

Similar to the method used for the ordinary discrete time renewal risk model in [8], conditioning on the first time when the surplus process drops below the surplus u, we have

$$\begin{aligned} m_v(u) = & \sum_{j=1}^{u} \sum_{i=0}^{\infty} \sum_{t=1}^{\infty} v^t m_v(u-j) f_3(i,j,t \,|\, 0) \\ & + \sum_{j=u+1}^{\infty} \sum_{i=0}^{\infty} \sum_{t=1}^{\infty} v^t \omega(i+u, j-u) f_3(i,j,t \,|\, 0). \end{aligned} \tag{9}$$

The technique can be used to derive equation (9) is that the premium income process $\sum_{i=1}^{M(t)} Y_i$ has stationary independent increments property. Define $\phi_v = \sum_{i=0}^{\infty} f_1(i \,|\, 0)$. Note the positive safety loading condition (2), we conclude that $0 < \phi_v < 1$ by imitating the proof of Proposition 3.1 of [5]. Now let

$$\varsigma_v(j) = \frac{1}{\phi_v} \sum_{i=0}^{\infty} p_i(j) f_1(i \,|\, 0), j \in \mathbb{N}^+,$$

then $\varsigma_v(j)$ is a proper p.f.. After some simple modifications, equation (9) yields the following defective renewal equation for the Gerber-Shiu function:

$$m_v(u) = \phi_v \sum_{j=1}^{u} m_v(u-j) \varsigma_v(j) + L_{v,\omega}(u), \tag{10}$$

where $L_{v,\omega}(u) = \sum_{j=u+1}^{\infty} \sum_{i=0}^{\infty} \omega(i+u, j-u) p_i(j) f_1(x\,|\,0)$.

In general the quantities ϕ_v, ς_v and $L_{v,\omega}$ can be calculated explicitly only in very specific situations. We will illustrate this fact in what follows. The calculations of Wu and Li [12] may be used to generate other such examples. By choosing $\omega(i,j) = s^i w_1(j)$, we denote

$$\psi_{v,s} = \mathrm{E}[v^T s^{U(T-1)} \omega_1(|U(T)|) I_{\{T<\infty\}} \,|\, U(0) = u].$$

Now we deal with the Gerber-Shiu function $\psi_{v,s}$ with assumption that claim amounts are geometrically distributed with $f(x) = (1-\rho)\rho^{x-1}, x \in \mathbb{N}^+$. From Section 4.2 of [12], we know that $\psi_{v,s}$ satisfies

$$\psi_{v,s} = \eta_v(s)\{\alpha(\rho s)^u + (1-\alpha)[\rho + \phi_v(1-\rho)]^u\}, \tag{11}$$

where $\eta_v(s) = \mathrm{E}[\omega_1(X_1)] \sum_{i=0}^{\infty} s^i f_1(i\,|\,0)$ and $\alpha = \frac{\rho(1-s)}{\rho(1-s)+\phi_v(1-\rho)}$.

With equation (11) at hand, the explicit expression for $\psi_{v,s}$ reduces to the calculation of ϕ_v and $\eta_v(s)$. We now show that the equation (4) derived in Section 2 allows us to compute the quantities directly.

Substituting (11) into (4), by simplifying and rearranging we have

$$\begin{aligned} 0 = {} & [a + b\hat{g}(\rho s)](\rho s)^u \mathrm{E}[\omega_1(X_1)] + \eta_v(s)\bigg\{ \frac{\alpha(1-\rho)\rho^u}{\rho(1-s)}[a(1-s^u) \\ & + b(\hat{g}(\rho) - s^u \hat{g}(\rho s))] + \frac{1-\alpha}{\phi_v}[(\rho + \phi_v(1-\rho))^u (a + b\hat{g}(\rho) \\ & + \phi_v(1-\rho)) - \rho^u(a + b\hat{g}(\rho))] + \alpha(\rho s)^u[c\hat{g}(\rho s) - 1] \\ & + (1-\alpha)(\rho + \phi_v(1-\rho))^u[c\hat{g}(\rho + \phi_v(1-\rho)) - 1]\bigg\}. \end{aligned} \tag{12}$$

From the definition of ϕ_v one has that $\lim_{s\to 1} \eta_v(s) = \phi_v \mathrm{E}[\omega_1(X_1)]$. Since ϕ_v does not depend on s and ω_1, by letting $\omega_1(j) = 1$ for $j \in \mathbb{N}^+$ and taking $s \to 1$ in (12), we obtain

$$\begin{aligned} 0 = {} & (a + b\hat{g}(\rho))\rho^u \\ & + [(\rho + \phi_v(1-\rho))^u(a + b\hat{g}(\rho + \phi_v(1-\rho))) - \rho^u(a + b\hat{g}(\rho))] \\ & + \phi_v(\rho + \phi_v(1-\rho))^u[c\hat{g}(\rho + \phi_v(1-\rho)) - 1], \end{aligned} \tag{13}$$

equation (13) implies that

$$\phi_v = (b + c\phi_v)\hat{g}(\rho + \phi_v(1-\rho)) + a. \tag{14}$$

It can be shown that equation (14) has a unique solution between 0 and 1 as follows. By letting $l = \rho + \phi_v(1-\rho)$, then equation (14) is equivalently written as $k(l) = \hat{g}(l)$ with $\rho < l < 1$, where $k(l) = \frac{l-[\rho+a(1-\rho)]}{cl+b(1-\rho)-c\rho}$. Note that $k(l)$ and $\hat{g}(l)$ are strictly concave and convex on $[\rho, 1]$, respectively. Further, it is easy to see that $k(\rho) < 0 < \hat{g}(\rho)$ and $k(1) = \frac{1-a}{b+c} > 1 = \hat{g}(1)$. We conclude that there is only one root to the equation $k(l) = \hat{g}(l)$ within the interval $(\rho, 1)$.

On the other hand, by noting that $\frac{\alpha(1-\rho)}{\rho(1-s)} = \frac{1-\alpha}{\phi_v}$, substitution (13) back into (12) yields

$$\eta_v(s) = \frac{\phi_v(a + b\hat{g}(\rho s))\mathrm{E}[\omega_1(X_1)]}{[b(1-\alpha) - c\phi_v\alpha]\hat{g}(\rho s) + a(1-\alpha) + \phi_v\alpha}. \tag{15}$$

It is clear that the denominator in identity (15) is nonzero.

4. The Case when Premiums are Degenerate at a Constant Amount

In this section we consider the special case that the premium income Y_1 is concentrated on $d \in \mathbb{N}^+$ with $g(x) = 1$ if $x = d$ and $g(x) = 0$ otherwise. In this case, the equation (4) reduces to

$$\begin{aligned} m_v(u) = a\left[\sum_{j=1}^{u} m_v(u-j)f(j) + \xi(u)\right] + b\Bigg[\sum_{j=1}^{u+d} m_v(u+d-j)f(j) \\ + \xi(u+d)\Bigg] + cm_v(u+d). \end{aligned} \tag{16}$$

For notational convenience of later use, we consider the discrete operator defined on any real valued function $p(x), x \in \mathbb{N}$ as follows:

$$T_r p(y) = \sum_{x=0}^{\infty} r^x p(x+y), \quad r \in \mathbb{C}, y \in \mathbb{N}.$$

Let $r_1, r_2, \cdots, r_k$ are distinct complex numbers. Now define $\pi_k(z) = \prod_{i=1}^{k}(z - r_i)$, then one has easily that

$$T_z T_{r_k} \cdots T_{r_1} p(y) = \frac{z^k T_z p(y)}{\pi_k(z)} - \sum_{i=1}^{k} \frac{r_i^k T_{r_i} p(y)}{(z-r_i)\pi_k'(r_i)}, \quad y \in \mathbb{N}, \tag{17}$$

while the p.g.f. for $T_{r_k}\cdots T_{r_1}p(x)$ is given by

$$T_zT_{r_k}\cdots T_{r_1}p(0)=\frac{z^k\hat{p}(z)}{\pi_k(z)}-\sum_{i=1}^{k}\frac{r_i^k\hat{p}(r_i)}{(z-r_i)\pi_k'(r_i)}. \tag{18}$$

The authors are referred to Li [8] for more details on the properties of the operator T_r defined on the real valued function $q(x), x\in\mathbb{N}^+$. In that case, equation (17) is still valid for $y\in\mathbb{N}^+$, and the corresponding p.g.f. for $T_{r_k}\cdots T_{r_1}p(x)$ becomes

$$zT_zT_{r_k}\cdots T_{r_1}q(1)=\frac{z^k\hat{q}(z)}{\pi_k(z)}-\sum_{i=1}^{k}\frac{zr_i^{k-1}\hat{q}(r_i)}{(z-r_i)\pi_k'(r_i)}.$$

Let $f(0)=0$, then the p.f. of the claim amount can be written as $f(x), x\in\mathbb{N}$. Thus, both equation (17) and (18) hold for $f(x)$.

Now multiplying (16) by z^{u+d} and summing over u from 0 to ∞ yield

$$\hat{m_v}(z)=\frac{az^d\hat{\xi}(z)+bz^dT_z\xi(d)-\beta(z)}{z^d-c-(az^d+b)\hat{f}(z)} \tag{19}$$

where $\beta(z)=\sum_{k=0}^{d-1}m_v(k)\big[cz^k+b\sum_{j=k+1}^{d-1}z^df(k-j)\big]$ is a polynomial of degree $d-1$ in z.

Lemma 1. *For $0<v<1$, the denominator of the term on the right-hand side of (19) has exactly d roots, say $r_1,r_2,\cdots,r_d$, within the unit circle.*

Proof. Let $\mathcal{S}$ denote the unite circle. For $z\in\mathcal{S}$, we have

$$\begin{aligned}|z^d-c|&\geq|z^d|-c=1-c\\ &>\frac{v-q_1q_2v}{1-q_1q_2v}-c=a+b\geq|(az^d+b)\hat{f}(z)|.\end{aligned} \tag{20}$$

By Rouché Theorem, equations $z^d-c=0$ and $z^d-c-(az^d+b)\hat{f}(z)=0$ have the same number of roots within the unite circle $\mathcal{S}$. Since the former has d roots, then the later also has d roots. □

In the rest of this paper, the d roots are assumed to be distinct. Now we are ready to give the main result of this section.

Theorem 2. *In the case that the premium is concentrated on $d\in\mathbb{N}^+$, the expected discounted penalty function $m_v(u)$ satisfies the following defective renewal equation:*

$$m_v(u)=\sum_{j=0}^{u}m_v(u-j)\varphi(j)+\varrho(u), \tag{21}$$

where

$$\varphi(j) = aT_{r_d} \cdots T_{r_1} f(j) + bT_{r_d} \cdots T_{r_1} f(j+d), \quad j \in \mathbb{N}$$

and

$$\varrho(u) = aT_{r_d} \cdots T_{r_1} \xi(u) + bT_{r_d} \cdots T_{r_1} \xi(u+d), \quad u \in \mathbb{N}.$$

Proof. Since $\hat{m_v}(z)$ is analytic within the unite circle, we conclude that the roots of denominator are also the zeros of the numerator in (19). Therefore

$$\beta(r_i) = ar_i^d \hat{\xi}(r_i) + br_i^d T_{r_i} \xi(d), \quad i = 1, 2, \cdots, d.$$

Since $\beta(z)$ is a polynomial of degree $d-1$ in z, by using the Lagrange interpolating polynomial yields

$$\beta(z) = \left[a \sum_{i=1}^{d} \frac{r_i^d \hat{\xi}(r_i)}{(z-r_i)\pi_d'(r_i)} + b \sum_{i=1}^{d} \frac{r_i^d T_{r_i} \xi(d)}{(z-r_i)\pi_d'(r_i)} \right] \pi_d(z). \tag{22}$$

by (17), (18) and (22) we have

$$\begin{aligned} \frac{az^d \hat{\xi}(z) + bz^d T_z \xi(d) - \beta(z)}{\pi_d(z)} &= aT_z T_{r_d} \cdots T_{r_1} \xi(0) \\ &\quad + bT_z T_{r_d} \cdots T_{r_1} \xi(d). \end{aligned} \tag{23}$$

Now denote by

$$h(z) = c + (az^d + b)\hat{f}(z) = h_1(z) + h_2(z) + h_3(z),$$

where $h_1(z) = c + b\sum_{j=1}^{d-1} z^j f(j)$, $h_2(z) = az^d \hat{f}(z)$, $h_3(z) = bz^d T_z f(d)$. By using the Lagrange interpolating polynomial again for z^d, we get

$$\begin{aligned} z^d &= \sum_{i=1}^{d} \frac{h(r_i) z}{r_i(z-r_i)} \frac{\pi_d(z)}{\pi_d'(r_i)} \\ &= \left[\sum_{i=1}^{d} \frac{h(r_i)}{r_i \pi_d'(r_i)} + \sum_{i=1}^{d} \frac{h(r_i)}{(z-r_i)\pi_d'(r_i)} \right] \pi_d(z). \end{aligned} \tag{24}$$

Division by $\pi_d(z)$ from both side of (24) and take limit $z \to \infty$ result in $\sum_{i=1}^{d} \frac{h(r_i)}{r_i \pi_d'(r_i)} = 1$. Then (24) can be rewritten as

$$\frac{z^d}{\pi_d(z)} = 1 + \sum_{i=1}^{d} \frac{h(r_i)}{(z-r_i)\pi_d'(r_i)}. \tag{25}$$

Then (17), (18) and (25) imply that

$$\frac{z^d - h(z)}{\pi_d(z)} = 1 - \sum_{j=1}^{3}\left[\frac{h_j(z)}{\pi_d(z)} - \sum_{i=1}^{d}\frac{h_j(r_i)}{(z-r_i)\pi_d'(r_i)}\right]$$
$$= 1 - aT_zT_{r_d}\cdots T_{r_1}f(0) - bT_zT_{r_d}\cdots T_{r_1}f(d). \quad (26)$$

Combining (23) and (26) yields

$$\hat{m_v}(z) = \hat{m_v}(z)[aT_zT_{r_d}\cdots T_{r_1}f(0) + bT_zT_{r_d}\cdots T_{r_1}f(d)]$$
$$+ aT_zT_{r_d}\cdots T_{r_1}\xi(0) + bT_zT_{r_d}\cdots T_{r_1}\xi(u+d). \quad (27)$$

Inverting (27) leads to (21).

Now we illustrate the renewal equation (21) is defective. In fact, from (26) we know that

$$\sum_{j=0}^{\infty}\varphi(j) = aT_1T_{r_d}\cdots T_{r_1}f(0) + bT_1T_{r_d}\cdots T_{r_1}f(d)$$
$$= 1 - \frac{1-(a+b+c)}{\pi_d(1)} < 1. \quad (28)$$

This completes the proof of Theorem 2. □

For the solution of the renewal equation, asymptotic estimation and bounds can be obtained by using the results in [11]. For the special case where $\omega(i,j) = 1$, we use the notation

$$\psi_v(u) = \mathrm{E}[v^T I_{\{T<\infty\}}|U(0) = u],$$

to denote the p.g.f. of ruin time. In this case, $\xi(u) = \sum_{j=u+1}^{\infty} f(j) = T_1 f(u+1)$. Therefore, $\varrho(u) = T_1\varphi(u+1)$, then equation (21) implies that

$$\psi_v(u) = \sum_{j=0}^{u} m_v(u-j)\varphi(j) + T_1\varphi(u+1).$$

Define $\theta = \frac{\pi_d(1)+a+b+c-1}{\pi_d(1)}$ and $\kappa(j) = \frac{\varphi(j)}{\theta}, j \in \mathbb{N}$, the equation (28) implies that $\kappa(j)$ is a proper p.f.. Therefore $\psi_v(u)$ is a discrete compound geometric tail

$$\psi_v(u) = \sum_{i=1}^{\infty}(1-\theta)\theta^i\overline{K^{*i}}(u),$$

where $K(j) = 1 - \overline{K}(j)$ is the corresponding c.d.f. of $\kappa(j)$, $\overline{K^{*i}}$ is the tail of ith convolution of K with itself. Then the p.g.f. of ruin time with zero initial surplus is

$$\psi_v(0) = \sum_{i=1}^{\infty}(1-\theta)\theta^i[1-k(0)^i] = \frac{\theta(1-k(0))}{1-\theta k(0)},$$

where $k(0) = \frac{1}{\theta}\left[aT_{r_d}\cdots T_{r_1}f(0) + bT_{r_d}\cdots T_{r_1}f(d)\right]$.

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