International Journal of Pure and Applied Mathematics
Volume 82 No. 3 2013, 391-398
ISSN: 1311-8080 (printed version)
url: http://www.ijpam.eu

ijpam.eu

ON THE TOTAL NEGATION OF RIGIDITY

M. Sreeja
Department of Mathematics
University of Calicut
Calicut University
Malappuram (District), PIN 673 635, PO, Kerala, INDIA

Abstract: In this paper we investigate the total negation of rigidity namely anti-rigidity. A topological space X is anti-rigid if there is no rigid subspace of X with more than one point. In particular, we establish the relationships of anti-rigidity with metric spaces, scattered spaces and ordered sets with order topology etc.

AMS Subject Classification: 54A05, 54G12
Key Words: rigid, anti-rigid, scattered space, order topology

1. Introduction

In 1979, Bankston [1] introduced the notion of anti-properties in topology. He defined the total negation of a topological property P as follows. If P is any topological property, the spectrum of P, denoted by Spec(P) is the class of all cardinal numbers α such that any topological space of cardinality α has the property P. Now a topological space (X, T) is said to have the property anti-P if a subspace of it has the property P only if the cardinality of the subspace is an element of Spec(P). Authors like I. L. Reily and M. K. Vamanamurthy investigated the anti-properties arising from compactness conditions. They also characterized anti-normality in [2]. In 1990, Julie Matier and Brian M. Mcmaster in [9] showed that the anti-first countable or anti-completely separable topological spaces are precisely those that are finite, but that the anti-separable

Received: August 6, 2012

spaces exhibit several properties resembling those of the anti-compact ones. Also, they examined the minimality of anti-separable topologies. In [13], P. T. Ramachandran proved that anti-homogeneity is equivalent to hereditary rigidity. He started the investigation on anti-rigid spaces in [14]. P. T. Ramachandran and V. Kannan characterised hereditarily homogeneous spaces in [6]. The present author [15], examined several examples of anti-rigid spaces and generalized the relations between anti-rigidity with homogeneity and filters.

In this paper, we prove that any countable metric space is anti-rigid. A characterisation of an anti-rigid scattered space is also provided. Then we consider anti-rigidity of ordered spaces.

2. Scatteredness and Anti-Rigid Spaces

A topological space (X, T) is said to be *scattered* if every nonempty subset of it contains a point which is isolated with respect to the relative topology. Scattered spaces have a remarkable role in the theory of topological spaces. Some investigations on these spaces can be found in [4],[5] and [12]. Here we prove a characterisation of anti-rigidity among scattered spaces.

Theorem 1. *A scattered space is anti-rigid if and only if it is T_1.*

Proof. Consider a scattered anti-rigid space (X, T) and let $x, y \in X$. As (X, T) is anti-rigid, $\{x, y\}$ has either discrete topology or indiscrete topology. Also, as (X, T) is scattered, $\{x, y\}$ must contain an isolated point. Since (X, T) is both anti-rigid and scattered, $\{x, y\}$ must be the discrete space. Hence there exists open sets $U, V \in T$ such that $x \in U, y \in V, x \notin V, y \notin U$. This shows that (X, T) is T_1.

Conversely, let (X, T) be scattered and T_1. Let A be any subspace of (X, T) with more than one point. Then there exist $x \in A$ such that $\{x\}$ is open in A. Let $B = A - \{x\}$ and let $y \in B$ be such that $\{y\}$ is open in B. Now, take $U, V \in T$ such that $U \cap A = \{x\}$ and $V \cap B = \{y\}$. As (X, T) is T_1 there exist $G, H \in T$ such that $x \in G$, $y \notin G$, $y \in H$, $x \notin H$. Define $K = U \cap G$, $L = V \cap H$. Then both K, $L \in T$ and $x \in K, y \notin K$, $y \in L$, $x \notin L$ implies that $K \cap A = \{x\}$ and $L \cap A = \{y\}$ by construction of K and L. Thus $\{x\}$ and $\{y\}$ are open in A.

Let $h : A \to A$ by $h = (x, y)$. Then $h = h^{-1}$. Also h is one-one and onto. Let U be an open subset of A. If x, $y \in U$ or $x, y \notin U$, then $h^{-1}(U) = U$. If $x \in U, y \notin U$, then $h^{-1}(U) = (U - \{x\}) \cup \{y\}$ which is open in A as $\{x\}$ is closed in A since A is T_1. The case $x \notin U$, $y \in U$ is similar to the last discussed one. Hence h is continuous and so it is a homeomorphism on A other than identity homeomorphism. Thus A is not rigid. As $A \subset X$ is arbitrary, (X, T) is anti-rigid. □

Remark 1. There exist T_1 anti-rigid spaces which are not scattered.

Eg: The set of all rational numbers with the relative topology from the usual topology on the set of all real numbers is T_1 but not scattered. But it is anti-rigid which is shown in the next section.

3. Countable Anti-Rigid Metric Spaces

In this section we consider countable anti-rigid metric spaces.

Lemma 1. *The space of rational numbers Q is anti-rigid.*

Proof. Let $A \subset Q$ be any subspace containing more than one element of Q. If A is finite, (A, T_A) will be the discrete space and hence not rigid. Let A be infinite. As Q is T_2, any of its subspaces having more than one isolated point will not be rigid. If A contains no isolated point, then it is self dense and hence homeomorphic to Q, by [7] and hence not rigid. Now, let $a \in A$ be a unique isolated point of A. Then as A is T_2, $A - \{a\}$ does not have isolated points, and hence self dense and not rigid as discussed above. The topology $(A,\ T_A)$ is the topological sum of the two non-rigid topological spaces $\{a\}$ and A, and hence is not rigid. Thus, in any case,A is not rigid and hence Q is anti-rigid. □

Theorem 2. *Every countable metric space is anti-rigid.*

Proof. By [7], any countable metric space X is homeomorphic to a subspace of Q, the space of rational numbers and hence anti-rigid by Lemma 1. □

Remark 2. a) There exist countable non metrisable spaces which are not anti-rigid.

Eg: Consider N, the set of all natural numbers and define a topology T on N by $T = \{\emptyset, N, \{1\}, \{1, 2\}, \{1, 2, 3\}, ...\}$. Then (N, T) is countable, nonmetrisable and not anti-rigid.

b) There exist countable nonmetrisable spaces which are anti-rigid.

Eg: A countable set with co-finite topology is countable, nonmetrisable and anti-rigid.

c) There exist uncountable metrisable spaces which are not anti-rigid.

Eg: The real line with usual topology is uncountable, metrisable but not anti-rigid.

4. Anti-Rigid Order Topologies

Here we discuss the relation between anti-rigidity and order topology. Let X be a set having a simple order relation $<$. Let $\mathscr{B}$ be the collection of all subsets of X of the following types.

1. All open intervals (a, b) in X, where $(a, b) = \{x : a < x < b\}$.

2. All intervals of the form $[a_0, b)$, where a_0 is the smallest element (if any) of X, where $[a_0, b) = \{x : a_0 \leq x < b\}$.

3. All intervals of the form $(a, b_0]$ where b_0 is the largest element (if any) of X, where $(a, b] = \{x : a < x \leq b\}$.

The collection $\mathscr{B}$ is a basis for a topology on X, which is called the *order topology* on X. A linearly ordered set is said to be *well-ordered* if every non empty subset of it has a first element. A linearly ordered set is said to be *dually well-ordered* if every non empty subset of it has a last element. A linearly ordered set is said to be *semi well-ordered* if every non empty subset of it has either a first element or a last element. (see [13]).

Theorem 3. *Any countable linearly ordered set with order topology is anti-rigid.*

Proof. Every ordered space is regular and T_1. Hence, a countable ordered space is T_1, regular and second countable and hence metrisable by Urysohn Metrisation Theorem (see [3]). Thus a countable ordered space is a countable metrisable space and hence anti-rigid by Theorem 2. □

Theorem 4. *Any well-ordered set with order topology is anti-rigid.*

Proof. Let (X, T) be a well ordered set with order topology. Consider a nonempty subset A of X. Then there exist a first element $a \in A$. Also let b be the first element of $A - \{a\}$. Now, take the initial segment B of $b \in X$, which is open in X. We have $B \cap A = \{a\}$, and hence a is an isolated point of A.Thus every subspace of X contains an isolated point, ie, X is scattered. Since an order topology is T_1, (X, T) is anti-rigid by Theorem 6. □

Theorem 5. *Any dually well-ordered set with order topology is anti-rigid.*

Proof. Let (X, T) be a dually well ordered set with order topology. Consider a nonempty subset A of X. Then there exist a last element$a \in A$ of A. Also let b be the last element of $A - \{a\}$. Now, take the final segment B of $b \in X$, which is open in X. We have $B \cap A = \{a\}$, and hence a is an isolated point of A. Thus every subspace of X contains an isolated point, ie, X is scattered. Since an order topology is T_1, (X, T) is anti-rigid by Theorem 6. □

Let us combine above two theorems to get the following theorem.

Theorem 6. *Any semi well-ordered set with order topology is anti-rigid.*

Proof. Let A and B be two disjoint linearly ordered sets with the linear orders R and S respectively. Then by $A + B$ we denote the set $A \cup B$ with the linear order $R \cup S \cup \{(a, b) : a \in A, b \in B\}$ on it.

By [13] every semi well ordered set X can be written in the form $A + B$ where A is a well ordered set and B is a dually well ordered set.

Let X be any semi well ordered set with order topology. Then there exist a well ordered set A and dually well ordered set B such that $X = A + B$. By Theorem 4 and Theorem 5, both A and B are anti-rigid in the order topology. Let P be any subset of X. Then there exist well ordered set G and dually well ordered set H such that $P = G + H$, where $G = P \cap A$ and $H = P \cap B$.

If either $G = \emptyset$ or $H = \emptyset$, then P is well ordered or dually well ordered and hence contain isolated points as we have seen in the proof of Theorem 7. Now let $x \in G$ be the smallest element of G. If $|G| \geq 2$, let y be the smallest element of $G - \{x\}$ and hence $(-\infty, y) \cap G = \{y\} = (-\infty, y) \cap P$. Thus x is an isolated point of P. Now, if $G = \{x\}$, $G \cup H$ can be considered as a dually well ordered set where $x \leq y$ for all $y \in G$ and hence contain an isolated point b as we have seen in the proof of theorem 8. Thus in any case, P contains an isolated point. Since P is arbitrary, X is scattered. Since an order topology is T_1, (X, T) is anti-rigid by Theorem 6. □

Remark 3.

1. There exists an uncountable linearly ordered set with order topology which is not anti-rigid, if we assume the axiom of choice.

   Eg: The real line with usual topology is the order topology with the usual ordering, which is not anti-rigid by assuming the axiom of choice (see [8]).

2. The Cantor set, which is homeomorphic to Z_2^N is an uncountable linearly ordered set with order topology which is not anti-rigid. (see proof of Theorem 7).

3. There exists anti-rigid order topologies which are not semi well-ordered.

   Eg: Consider the set of all rational numbers with usual topology. It coincides with the set of all rational numbers with usual ordering having order topology, which is not well-ordered, dually well-ordered or semi well-ordered. We have proved in Lemma 1 that this space is anti-rigid.

5. Products and Quotients

In this section we prove that anti-rigidity is not preserved by countable products and quotients.

Definition 1. (see [3]) A subset A of a topological space (X, T) is said to be *clopen* if it is both open and closed.

Theorem 7. *In ZFC, anti-rigidity is not countably productive.*

Proof. By [8], there exists a rigid subspace K of $\mathbb{R}$. Then K is separable and second countable. Also, since K is rigid, it does not contain any interval, for otherwise we can find non-trivial homeomorphisms. Hence K is zero dimensional, since a subset of real line is zero dimensional if it does not contain any non degenerate interval. Thus we get a clopen base $\mathscr{C} = \{C_1, C_2, ..., C_n, ...\}$ for K which is countable since K is second countable and let $\mathscr{F} = \{f_k : C_k \to Z_2 : k \in N\}$ be the family of characteristic functions of $C_k, k = 1, 2, ...n, ...$, where Z_2 is $\{0, 1\}$ with discrete topology and N is the set of all natural numbers. Then each f_k is continuous as C_k is clopen. Let C be any closed subset of K and let x be any point of K which do not belong to C. Then $K - C$ is an open set containing x and hence there exists $C_k \in \mathscr{C}$ such that $x \in C_k$ and $C_k \subset K - C$. Then $f_k(x) = 1$ and $f_k(y) = 0$ for all $y \in C$. Thus $\mathscr{F}$ distinguishes points from closed sets. Also, as K is T_1, each singleton set is closed and hence $\mathscr{F}$ separates points also. Hence by embedding lemma (see [3]), K can be embedded in ${Z_2}^N$.

Thus we have proved that a rigid space can be embedded in a countable product of anti-rigid spaces. This proves the result. □

Definition 2. (see [3]) A topological property is said to be *divisible* if whenever a space has it, so does every quotient space of it.

Theorem 8. *Anti-rigidity is not divisible.*

Proof. Let $X = \{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\}$, $T = \{A \subset X : 0 \notin A\} \cup \{A \subset X : 0 \in A, X - A \text{ is finite}\}$. Then (X, T) is anti-rigid as it is of the form $P(X - A) \cup \mathscr{F}$, $\cap \mathscr{F} = \{a\}$ (see [15]).

Now, let $A = \{0, 1\}$ and define $P : X \to A$ by $P(\frac{1}{n}) = 1 \quad \forall n \in N, \quad P(0) = 0$. Then the quotient topology on A defined by the map P is $S = \{\emptyset, \{1\}, \{0, 1\}\}$, which makes A a rigid space. Thus (A, S) is a rigid space which is a quotient space of an anti-rigid space and hence the proof. □

6. Acknowledgments

I am very much thankful to my supervisor Dr. P.T. Ramachandran for his helps, advices, suggestions and criticisms during the preparation of this paper.

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