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ON CONVOLUTION OF SOME TYPE OF THE NUMBERS CONNECTED WITH GENERALIZED REPUNITS

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Abstract: We will concentrate on special types of numbers

$$J_n(k) = \sum_{i=0}^{n-2} \binom{n}{i} k^{n-2-i},$$

where k is any nonnegative integer and n is any positive integer greater than 1. These numbers are a generalization of generalized repunits $R_n(b)$. In this paper some results about divisibility of $J_n(k)$ are stated. Further the generating function and a m-fold convolution formula for the numbers $J_n(k)$ is found.

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1. Introduction

The term *repunit* was coined by Beiler [2] in 1966. A repunit R_n is any integer written in decimal form as a string of 1's. The numbers 1, 11, 111, 1111, 11111, etc., are examples of repunits. Thus repunits have the form $R_n = \frac{10^n-1}{9}$. The great effort was devoted to searching of repunit primes, thus such primes which are any repunits and they are also prime numbers, see Reuschle [11], Hoppe [7], Lehmer [10] and Kraitchik [9], Williams and Dubner [16]. In recent time

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four probably prime repunits R_{49081}, R_{86453}, R_{109297}, R_{270343} have known (see Dubner [4] and [5], Baxter [1] and Voznyy and Budnyy [14]). For more facts about repunits see Yates [17]. Snyder [12] extended the notation repunit to one in which for some integer $b \geq 2$ by this way

$$R_n(b) = \frac{b^n - 1}{b - 1}. \tag{1}$$

They are called as *generalized repunits* or *repunits to base b and* consist of a string of 1's when written in base b. Some facts on the divisibility and primality of $R_n(b)$ can be found in Williams [15], Jaroma [8] and Dubner [3]. We have found m-fold convolution formula for $R_n(b)$ and some their divisibility properties in [13]. In this paper we will investigate a generalization of generalized repunits $R_n(k+1)$, which are created by subtracting the linear term in $(k+1)^n$ and dividing by the trivial divisor k^2

$$J_n(k) = \frac{(k+1)^n - nk - 1}{k^2}\ , \tag{2}$$

It is easy to realize that all the numbers $J_n(k)$ are nonnegative integers for arbitrary positive integers k and n. Hence we can also write the numbers $J_n(k)$ in the form

$$J_n(k) = \sum_{i=0}^{n-2} \binom{n}{i} k^{n-2-i}\ , \tag{3}$$

with $J_0(k) = J_1(k) = 0$.

2. The Main Results

The main results established in this paper concern some congruences for the numbers $J_n(k)$ and a m-fold convolution formula for $J_n(k)$ is derived. They are expressed in the following theorems.

Theorem 1. *Let a, b, l be any positive integers. Then*

$$J_n(al+b) \equiv \begin{cases} \binom{n}{2} \pmod 2 & \textit{iff} \quad a \mid b, \\ J_n(b) \pmod 2 & \textit{iff} \quad a \nmid b. \end{cases} \tag{4}$$

holds for $n \geqq 1$.

Corollary 2. *Let a, m, l be any positive integers. Then*

$$J_{am}(al) \equiv \begin{cases} 0 \pmod a, & a \equiv 1 \pmod 2 \ \vee \\ & (a \equiv 0 \pmod 2 \ \wedge \ m \equiv 0 \pmod 2)\ , \\ \frac{a}{2} \pmod a, & a \equiv 0 \pmod 2 \wedge \ m \equiv 1 \pmod 2\,. \end{cases}$$

Theorem 3. *Let k, $m \geq 2$, n be any positive integers. Then the following formula for the numbers $J_n(k)$ holds*

$$\sum_{\substack{n_1,n_2,\ldots n_m \\ n_1+n_2+\cdots+n_m=n}} J_{n_1}(k)\ J_{n_2}(k)\cdots J_{n_m}(k) =$$
$$= \sum_{l=0}^{n-2m} \binom{l+2m-1}{2m-1}\binom{n-m-l-1}{m-1}(k+1)^{n-2m-l}\ .$$

3. Some Lemmas and Preliminary Results

(i) Recurrence relations With respect to (2) the numbers $J_n(k)$ satisfy a non-homogeneous linear difference equation of the second order and the roots of the characteristic equation must be $\lambda_1 = k+1$ and $\lambda_2 = 1$. Hence the recurrence for the numbers $J_n(k)$ has the form

$$J_{n+2}(k) - (k+2)J_{n+1}(k) + (k+1)J_n(k) = 0 \tag{5}$$

and we obtain the sought after recurrence for the numbers $J_n(k)$ by putting (2) into the left side of (5) by the following way

$$\begin{aligned} J_{n+2}(k) &- (k+2)J_{n+1}(k) + (k+1)J_n(k) \\ &= \frac{1}{k^2}\big((k+1)^{n+2} - (n+2)\,k - 1\big) - (k+2)(k+1)^{n+1} \\ &- (n+1)\,k - 1 + (k+1)\,((k+1)^n - n\,k - 1) = 1. \end{aligned}$$

Therefore the numbers $J_n(k)$ satisfy the recurrence

$$J_{n+2}(k) - (k+2)\,J_{n+1}(k) + (k+1)\,J_n(k) = 1,$$

with the initial conditions $J_0(k) = J_1(k) = 0$. By an analogous procedure we can find for example recurrence

$$J_{n+1}(k) - (k+1)\,J_n(k) = n,\ \ J_0(k) = 0\ .$$

(ii) The generating function for the numbers $J_n(k)$. After arrangements we have for the generating function $j(x)$ of $J_n(k)$

$$\begin{aligned} j(x) &= \sum_{n=0}^{\infty} \frac{(k+1)^n - k\,n - 1}{k^2} x^n \\ &= \frac{1}{k^2}\sum_{n=0}^{\infty}(k+1)^n x^n - \frac{1}{k}\sum_{n=0}^{\infty} n x^n - \frac{1}{k^2}\sum_{n=0}^{\infty} x^n \\ &= \frac{x^2}{(1-(k+1)x)(1-x)^2}\ . \end{aligned} \tag{6}$$

4. The Proofs of the Main Theorems

Proof of Theorem 1. We will consider the following two cases. Let $b \mid a$. Then

$$\begin{aligned} J_n(al+b) = J_n(am) &= \frac{(am+1)^n - n(am) - 1}{(am)^2} = \frac{\sum_{i=2}^{n}\binom{n}{i}(am)^i}{(am)^2} \\ &= \binom{n}{2} + \sum_{i=3}^{n}\binom{n}{i}(am)^{i-2} \equiv \binom{n}{2} \pmod{a}\ . \end{aligned}$$

Let $b \nmid a$. Then $al + b \equiv b \pmod{a}$ and we have

$$\begin{aligned} J_n(al+b) &= \frac{(al+b+1)^n - n(al+b) - 1}{(al+b)^2} \\ &= \frac{\left(a\left(\sum_{i=0}^{n-1}\binom{n}{i}a^{n-i-1}l^{n-i}(b+1)^i - nl\right) + (b+1)^n - nb - 1\right)}{(al+b)^2} \\ &\equiv \frac{(b+1)^n - nb - 1}{b^2} \pmod{a}\ . \end{aligned}$$

□

Proof of Corollary 2. As the congruence $xy \equiv y \pmod{2y}$ clearly holds for any odd integer x and any integer y congruence (4) implies the assertion. □

Proof of Theorem 6. We use the following well–known fact on the generating function. If any sequence $\langle a_n \rangle$ has the generating function $A(x)$ (for example see [6], p. 355) then the m–fold convolution of the sequence $\langle a_n \rangle$ with itself has nth term equal to

$$\sum_{\substack{n_1,n_2,\dots n_m \\ n_1+n_2+\cdots+n_m=n}} a_{n_1}\, a_{n_2} \cdots a_{n_m}$$

and its generating function is $A^m(x)$. Thus, we have

$$\sum_{n=0}^{\infty} \left(\sum_{\substack{n_1,n_2,\dots n_m \\ n_1+n_2+\cdots+n_m=n}} J_{n_1}(k)\; J_{n_2}(k) \cdots J_{n_m}(k) \right) x^n = \left(\frac{x^2}{(1-x)^2(1-(k+1)x)} \right)^m .$$

On the left side we get

$$\frac{x^{2m}}{(1-x)^{2m}(1-(k+1)x)^m} = x^{2m} \frac{1}{(1-x)^{2m}} \frac{1}{(1-(k+1)x)^m}$$
$$= x^{2m} \left(\sum_{n=0}^{\infty} \binom{n+2m-1}{2m-1} x^n \right) \left(\sum_{n=0}^{\infty} \binom{n+m-1}{m-1} (k+1)^n \right) x^n$$
$$= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \binom{l+2m-1}{2m-1} \binom{n-l+m-1}{m-1} (k+1)^{n-l} \right) x^{n+2m}$$

$$= \sum_{n=2m}^{\infty} \sum_{l=0}^{n-2m} \binom{l+2m-1}{2m-1} \binom{n-2m-l+m-1}{m-1} (k+1)^{n-2m-l} x^n$$
$$= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n-2m} \binom{l+2m-1}{2m-1} \binom{n-m-l-1}{m-1} (k+1)^{n-2m-l} \right) x^n .$$

Therefore

$$\sum_{\substack{n_1,n_2,\dots n_m \\ n_1+n_2+\cdots+n_m=n}} J_{n_1}(k)\; J_{n_2}(k) \cdots J_{n_m}(k) =$$
$$= \sum_{l=0}^{n-2m} \binom{l+2m-1}{2m-1} \binom{n-m-l-1}{m-1} (k+1)^{n-2m-l} . \quad \square$$

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