International Journal of Pure and Applied Mathematics
Volume 82 No. 4 2013, 623-629
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v82i4.11

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STABILITY ANALYSIS OF A LASER WITH A MODULATED SATURABLE ABSORBER

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Abstract: The stability analysis of the model for a laser system with a modulated saturable absorber is performed. The model is based in three equations describing the temporal evolution of the photon flux, the population inversion in the active media and the saturation coefficient of the saturable absorber. The system dynamics is discussed in order to find the best control regions of the system.

AMS Subject Classification: 34Dxx, 37Nxx, 93Dxx
Key Words: stability, Staz DeMarz equations, laser systems

1. Introduction

The dynamics of a laser system with a saturable absorber can be described by a model based on the Stats-DeMarz equations which originally were developed to describe oscillations in a Maser [1]. This model has undergone many modifications to be adopted for laser systems [2] and [3]. For a complete phenomenological description of a laser with a saturable absorber, only three equations are needed: the photon-flux equation, an equation for the population inversion

Received: October 25, 2012

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density on the active medium and the saturable population inversion equation that gives the saturation coefficient. Therefore the Stats DeMars equations for a three level laser system with a saturable absorber without modulation are written as follows.

$$
\begin{aligned}
\frac{dS}{dt} &= \Gamma\nu\sigma NS - \Gamma\nu\frac{L_\alpha}{L_m}k_\alpha S - \frac{1}{T}S,\\
\frac{dN}{dt} &= -\beta\frac{\sigma}{\hbar\omega}NS + \frac{N_0 - N}{\tau},\\
\frac{dk_\alpha}{dt} &= -\frac{2\sigma_\alpha k_\alpha S}{\hbar\omega} + \frac{k_{0\alpha} - k_\alpha}{\tau_\alpha}.
\end{aligned}
\tag{1}
$$

where S is the emitted photon density, N is the population inversion of the active medium, k_α is the resonant absorption of the saturable absorber, σ_α is the saturable absorber cross-section and N_α the population inversion of the saturable absorver ($k_\alpha = -\sigma_\alpha N_\alpha$). Γ, ν, σ, and T stand, respectively, for cavity filling coefficient, optical frequency, active medium cross-section, and photon lifetime in the cavity, β is the coefficient which accounts for the difference in population inversion coursed by lasing, L_m and L_α are, respectively, the active medium and the saturable absorber lengths, $k_{0\alpha}$ is the linear resonant saturable absorber absorption coefficient without lasing, N_0 is the population inversion in the active medium without radiation, τ and τ_α stand for relaxation time in the active medium and in the saturable absorber, respectively, and finally $\hbar\omega$ is the photon energy.

Defining the next adimensional parameters and variables: $t' = t/\tau$, $G = \tau/T$, $\delta = \tau/\tau_\alpha$, $\rho = 2\sigma_\alpha/\beta\sigma$, $\alpha = \Gamma\nu\sigma TN$ and $\alpha_\alpha = -\Gamma\nu T k_{0\alpha} L_\alpha/L_m = -\Gamma\nu T\sigma_\alpha n_\alpha/L_m$; $n(t') = \Gamma\nu\sigma TN(t')$, $n_\alpha(t') = -\Gamma\nu T k_\alpha(t') L_\alpha/L_m$ and $m(t') = \beta B\tau S(t')/\nu = \beta\sigma\tau S(t')/\hbar\omega$. The above system can be rewritten as:

$$
\begin{aligned}
\frac{dm}{dt'} &= Gm(n + n_\alpha - 1),\\
\frac{dn}{dt'} &= \alpha - n(m+1),\\
\frac{dn_\alpha}{dt'} &= \delta\alpha_\alpha - n_\alpha(\rho m + \delta).
\end{aligned}
\tag{2}
$$

All the parameters used to define the saturable absorber are fixed, except for α_α, which includes a measure of the active centers absorbent density, for this reason, α_α is used as the saturable absorbers identifying parameter. Adding an external linear sinusoidal modulation (e.g. using an Electro Optic Modulator (EOM)) directly into the saturable absorber through its main parameter (i.e.

α_α) the above equations may be transform into:

$$\begin{aligned} \frac{dm}{dt'} &= Gm(n + n_\alpha - 1), \\ \frac{dn}{dt'} &= \alpha - n(m+1), \\ \frac{dn_\alpha}{dt'} &= \delta\alpha_\alpha \left[\frac{1+\cos(\omega_c t)}{2}\right] - n_\alpha(\rho m + \delta). \end{aligned} \tag{3}$$

where ω_c stands for the external modulation frequency applied to the EOM.

These three differential equations compose the working system, it must be noted that, in absence of modulation frequency applied to the EOM, ω_c, the system returns to Eqs. (2), i.e. rate equations for a laser with a passive saturable absorber [4].

2. Linear Stability Analysis

Linear Stability Analysis is used to understand the system dynamics [5] and [6]. The analysis is based on the linear disturbance equations, these equations are derived from the original equations. As is well known the method consist in linearize the describing equations, obtain the initial state condition (i.e. when the derivatives are zero), expand the system about the initial state condition, construct the Jacobian matrix and find the eigenvectors and eigenvalues with the determinant equal to zero. This gives as a result the fixed points of the equations system, which must be analyzed in order to know what type of points are (i.e. fixed, source, saddle, etc.). The equations of interest are Eqs. 3, these equations are non-autonomous due to the explicit time-dependence found in the cosine of the third equation. To be able to perform the Linear Stability Analysis that dependence must be eliminated, to do that, a variable change must be applied, giving as a result, the next equations system:

$$\begin{aligned} \frac{dm}{dt'} &= Gm(n + n_\alpha - 1), \\ \frac{dn}{dt'} &= \alpha - n(m+1), \\ \frac{dn_\alpha}{dt'} &= \delta\alpha_\alpha \left[\frac{1+\cos(x)}{2}\right] - n_\alpha(\rho m + \delta), \\ \frac{dx}{dt} &= \omega_c. \end{aligned} \tag{4}$$

Naming the Eqs. 4 as:

$$
\begin{aligned}
f(m,n,n_\alpha,x) &= \frac{dm}{dt^{\square}} = Gm(n+n_\alpha-1),\\
g(m,n,n_\alpha,x) &= \frac{dn}{dt^{\square}} = \alpha - n(m+1),\\
h(m,n,n_\alpha,x) &= \frac{dn_\alpha}{dt^{\square}} = \delta\alpha_\alpha\left[\frac{1+\cos(x)}{2}\right] - n_\alpha(\rho m+\delta),\\
j(m,n,n_\alpha,x) &= \frac{dx}{dt} = \omega_c.
\end{aligned}
\tag{5}
$$

Assuming that $(m^{\square},n^{\square},n_\alpha^{\square},x^{\square})$ is the steady state, that is $f(m^{\square},n^{\square},n_\alpha^{\square},x^{\square}) = 0$, $g(m^{\square},n^{\square},n_\alpha^{\square},x^{\square}) = 0$, $h(m^{\square},n^{\square},n_\alpha^{\square},x^{\square}) = 0$ and $j(m^{\square},n^{\square},n_\alpha^{\square},x^{\square}) = 0$. In order to find if the steady state is stable or unstable, a small perturbation (represented by the subscript "p") must be added to it,

$$
\begin{aligned}
f &= f^{\square} + f_p,\\
g &= g^{\square} + g_p,\\
h &= h^{\square} + h_p,\\
j &= j^{\square} + j_p.
\end{aligned}
\tag{6}
$$

with $(f_p, g_p, h_p, j_p) \ll 1$.

Now, the main question for practical purposes is: Will the perturbations grow (steady state unstable) or decay (steady state stable)? To be able to observe if the perturbations grow or decay, the perturbations derivatives must be found.

$$
\begin{aligned}
\frac{df_p}{dt} &= \frac{dm}{dt} = f(m,n,n_\alpha,x) = f(m^{\square}+f_p, n^{\square}+g_p, n_\alpha^{\square}+h_p, x^{\square}+j_p)\\
&= f(m^{\square},n^{\square},n_\alpha^{\square},x^{\square}) + \frac{\delta}{\delta m} f(m^{\square},n^{\square},n_\alpha^{\square},x^{\square}) f_p\\
&\quad + \frac{\delta}{\delta n} f(m^{\square},n^{\square},n_\alpha^{\square},x^{\square}) g_p + \frac{\delta}{\delta n_\alpha} f(m^{\square},n^{\square},n_\alpha^{\square},x^{\square}) h_p\\
&\quad + \frac{\delta}{\delta x} f(m^{\square},n^{\square},n_\alpha^{\square},x^{\square}) j_p + \text{high order terms}
\end{aligned}
\tag{7}
$$

Following the Taylor series expansion shown in Eqs. (7) the derivatives for each perturbation are:

$$
\begin{aligned}
\frac{df_p}{dt} &= \frac{\delta}{\delta m} f \cdot f_p + \frac{\delta}{\delta n} f \cdot g_p + \frac{\delta}{\delta n_\alpha} f \cdot h_p + \frac{\delta}{\delta x} f \cdot j_p,\\
\frac{dg_p}{dt} &= \frac{\delta}{\delta m} g \cdot f_p + \frac{\delta}{\delta n} g \cdot g_p + \frac{\delta}{\delta n_\alpha} g \cdot h_p + \frac{\delta}{\delta x} g \cdot j_p,\\
\frac{dh_p}{dt} &= \frac{\delta}{\delta m} h \cdot f_p + \frac{\delta}{\delta n} h \cdot g_p + \frac{\delta}{\delta n_\alpha} h \cdot h_p + \frac{\delta}{\delta x} h \cdot j_p,\\
\frac{dj_p}{dt} &= \frac{\delta}{\delta m} j \cdot f_p + \frac{\delta}{\delta n} j \cdot g_p + \frac{\delta}{\delta n_\alpha} j \cdot h_p + \frac{\delta}{\delta x} j \cdot j_p.
\end{aligned} \tag{8}
$$

The system presented in Eqs. (8) can be rewritten in the matrix form:

$$
\begin{pmatrix} f_p' \\ g_p' \\ h_p' \\ j_p' \end{pmatrix} = \begin{pmatrix} \frac{\delta}{\delta m} f & \frac{\delta}{\delta n} f & \frac{\delta}{\delta n_\alpha} f & \frac{\delta}{\delta x} f \\ \frac{\delta}{\delta m} g & \frac{\delta}{\delta n} g & \frac{\delta}{\delta n_\alpha} g & \frac{\delta}{\delta x} g \\ \frac{\delta}{\delta m} h & \frac{\delta}{\delta n} h & \frac{\delta}{\delta n_\alpha} h & \frac{\delta}{\delta x} h \\ \frac{\delta}{\delta m} j & \frac{\delta}{\delta n} j & \frac{\delta}{\delta n_\alpha} j & \frac{\delta}{\delta x} j \end{pmatrix} \begin{pmatrix} f_p \\ g_p \\ h_p \\ j_p \end{pmatrix} = \mathcal{J} \begin{pmatrix} f_p \\ g_p \\ h_p \\ j_p \end{pmatrix}, \tag{9}
$$

where $\mathcal{J}$ denotes the Jacobian matrix of the original system at the steady state. Substituting the values in the Jacobian function and the next matrix is obtained:

$$
\begin{pmatrix} G(n+n_\alpha-1) & Gm & Gm & 0 \\ -n & -m-1 & 0 & 0 \\ -\rho n_\alpha & 0 & -\rho m-\delta & -\frac{\delta\alpha_\alpha}{2}\sin(x) \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{10}
$$

The next step is to find the eigenvectors and eigenvalues of the system $(\mathcal{J} - \lambda I)\partial_p = 0$, so the determinant would be:

$$
|\mathcal{J} - \lambda I| = 0. \tag{11}
$$

The matrix for the former equation is:

$$
\det \begin{pmatrix} G(n+n_\alpha-1)-\lambda & Gm & Gm & 0 \\ -n & -m-1-\lambda & 0 & 0 \\ -\rho n_\alpha & 0 & -\rho m-\delta-\lambda & -\frac{\delta\alpha_\alpha}{2}\sin(x) \\ 0 & 0 & 0 & -\lambda \end{pmatrix} = 0, \tag{12}
$$

The solutions for the perturbed steady-state of the original system are:

$$
\begin{aligned}
m^s &= 0,\\
n^s &= \alpha,\\
n_\alpha^s &= \alpha_\alpha,\\
x^s &= 0.
\end{aligned} \tag{13}
$$

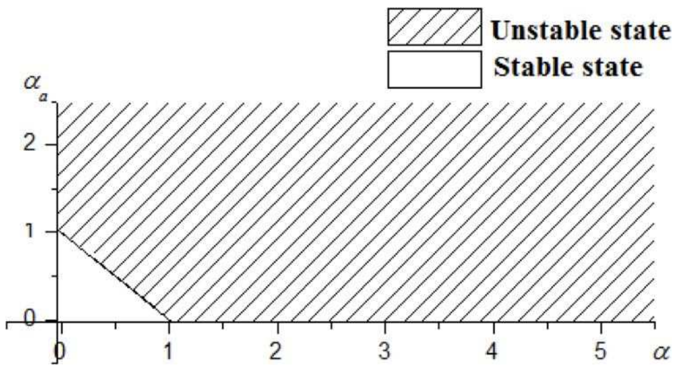

Figure 1: Stability condition given by the relation between α and α_α

Substituting the former solutions in Eqs. (12), the matrix transforms into:

$$\begin{pmatrix} G(\alpha+\alpha_\alpha-1)-\lambda & 0 & 0 & 0 \\ -\alpha & -1-\lambda & 0 & 0 \\ -\rho\alpha_\alpha & 0 & -\delta-\lambda & 0 \\ 0 & 0 & 0 & -\lambda \end{pmatrix} = 0, \tag{14}$$

Matrix (14) has the following characteristic equation:

$$[G(\alpha+\alpha_\alpha-1)-\lambda]\,[-1-\lambda][-\delta-\lambda][-\lambda] = 0, \tag{15}$$

where $\lambda_1 = G(\alpha+\alpha_\alpha-1)$, $\lambda_2 = -1$, $\lambda_3 = -\delta$ and $\lambda_4 = 0$ are eigenvalues which are all real; being λ_2 and λ_3 always negative (i.e. the perturbation will decay) and λ_4 is critically stable. Therefore, the stability condition is defined only by the sign of λ_1, i.e. the fixed point is a source when $\alpha_\alpha + \alpha > 1$ as shown in Fig. 1.

3. Conclusions

The stability analysis of the model for a laser system with a modulated saturable absorber is performed. The model is based in three equations describing the temporal evolution of the photon flux, the population inversion in the active media and the saturation coefficient of the saturable absorber. The stability condition of the systems is found to depend only on the sign of the first eigenvalue of the system λ. As is graphically shown, the stability conditions are given by the relation between α and α_α.

References

[1] H. Statz, G. De Mars, Transients and oscillation pulses in masers, In: *Quantum Electronics* (Ed. C. Townes), Columbia U. Press, New York (1960), 530-537.

[2] L. Tarassov, *Physique des Processus dans les Générateurs de Rayonnement Optique Coherent*, MIR, Moscow (1981).

[3] C.L. Tang, H. Staz, G. De Mars, Spectral output and spiking behavior of solid-state lasers, *J. Appl. Phys.*, **34** (1963), 2289-2295.

[4] M. Wilson, V. Aboites, A. Pisarchik, V. Pinto, Y. Barmenkov, Controlling a laser output through an active saturable absorber, *Rev. Mex. Fís.*, **54**, No. 3 (2011), 250-254.

[5] M. Tabor, Linear stability analysis, In: *Chaos and Integrability in Nonlinear Dynamics: An Introduction*, New York, Wiley (1989), 20-31.

[6] M. Braun, *Differential Equations and their Applications: An Introduction to Applied Mathematics*, Springer (1992).