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SOME NEW STEFFENSEN LIKE THREE-STEP METHODS FOR SOLVING NONLINEAR EQUATIONS

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Abstract: Some well-known methods involving first derivative create numerical difficulties and fail to converge in the neighborhood of the required root, so that in this paper, we have constructed two new sixth order and one seventh order convergence derivative free methods for solving nonlinear equations. The proposed methods are based on composition of Newton's method with known methods. Each derivative free method requires four functional evaluations per iteration. The new methods attain the efficiency indexes 1.56 and 1.63 respectively, which makes them competitive. Convergence analysis and numerical examples are also given to illustrate and compare the accuracy and efficiency of the proposed methods with existing methods.

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Key Words: nonlinear equations, efficiency index, order of convergence, Steffensen like methods, computational order of convergence

1. Introduction

Real world phenomenon are often concerned with nonlinear problems, such as solving the problems related to finance, space expeditions, engineering, biotechnology, traffic controlling, and all the other branches of science and social sci-

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ences, all these problems required mathematical modeling. The mathematical models of such problems give rise to the equations of the form

$$f(x) = 0, \tag{1}$$

where f is a real valued nonlinear function. Such type of equations are called nonlinear equations. One of the oldest and most basic problems in mathematics is that of solving nonlinear equation (1). This problem has motivated many theoretical developments including the fact that solution formulas do not exist in general. Thus, the development of algorithms for finding solutions has historically been an important enterprize. Locating zeroes of such functions has been given much attention from several decades due to its importance in applied sciences.

Newton's method is the best known iterative method for solving nonlinear equations [1], given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2... \tag{2}$$

which converges quadratically. But it has a major weakness, one has to calculate the derivative of $f(x)$ at each approximation. Frequently, $f'(x)$ is far more difficult to evaluate and needs more arithmetic operations to calculate than $f(x)$.

Steffensen overcomes the problem of the derivative evaluation in Newton's method, replacing $f'(x)$ by forward difference approximation, known as Steffensen's method [2]

$$x_{n+1} = x_n - \frac{f(x_n)^2}{f(x_n + f(x_n)) - f(x_n)}, \quad n = 0, 1, 2... \tag{3}$$

which also converges quadratically. Based on this method, many two-step and three-step derivative free iterative schemes have been proposed [3],[4].

In [3] a cubically convergent derivative free iterative method has been proposed by Dehghan and Hajarian as follows

$$y_n = x_n - \frac{f(x_n)^2}{f(x_n + f(x_n)) - f(x_n)},$$

$$x_{n+1} = x_n - \frac{f(x_n)\big(f(y_n) + f(x_n)\big)}{f(x_n + f(x_n)) - f(x_n)}, \tag{4}$$

where its efficiency index is 1.442.

In [4] a fourth order iterative algorithm has been developed by Cordero et al. and iterative scheme is given by

$$y_n = x_n - \frac{2f(x_n)^2}{f(x_n + f(x_n)) - f(x_n - f(x_n))},$$

$$x_{n+1} = x_n - \frac{2f(x_n)^2}{f(x_n + f(x_n)) - f(x_n - f(x_n))} \frac{f(y_n) - f(x_n)}{2f(y_n) - f(x_n)}, \tag{5}$$

which has efficiency index 1.414.
In [4] a sixth order convergence derivative free algorithm also has been derived by Cordero et al., which is written as

$$y_n = x_n - \frac{2f(x_n)^2}{f(x_n + f(x_n)) - f(x_n - f(x_n))},$$

$$z_n = y_n - \frac{y_n - x_n}{2f(y_n) - f(x_n)} f(y_n),$$

$$x_{n+1} = z_n - \frac{y_n - x_n}{2f(y_n) - f(x_n)} f(z_n), \tag{6}$$

which has efficiency index 1.430.

The processes of removing the derivatives usually increase the number of function evaluation per iteration. In our methods we used the technique of composition of Newton's method with the known methods, which not only increase the order of the method as high as possible but also reduce the number of function evaluations and improve the efficiency index of the composed method. In view of this fact, the new steffensen like methods are significantly better when compared with the established methods described above.

The paper is organized as follows: In Section 2, two sixth order convergence methods and one seventh order convergence method with their convergence analysis are given. In Section 3, the methods are described as the form of algorithms. In Section 4, a comparison of different derivative free methods with the proposed methods is given. Finally, the conclusions are given Section 5.

2. Two Sixth Order and One Seventh Order Method

In this section, based on Steffensen's technique, we have given two sixth order and one seventh order method for solving nonlinear equations. In the proposed methods, the divided difference is used to get the better approximation to the derivative of the function.

2.1. First Sixth Order Method (M1)

In [5], a method of fourth order has been presented by the following scheme

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$x_{n+1} = y_n + \frac{f(y_n)}{f'(x_n)} - \frac{2f(y_n)f(x_n)}{f'(x_n)\big(f(x_n) - f(y_n)\big)}, \tag{7}$$

by combining the method (7) with Newton's method, we obtain a new three-step iterative algorithm without memory as follows

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$zn = y_n + \frac{f(y_n)}{f'(x_n)} - \frac{2f(y_n)f(x_n)}{f'(x_n)\big(f(x_n) - f(y_n)\big)},$$

$$x_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}, \tag{8}$$

which required of two evaluations of the first derivative of the function. To remedy these derivatives firstly, we approximate $f'(x_n)$ by the divided difference of order one

$$f'(x_n) \approx f[w_n, x_n] = \frac{f(w_n) - f(x_n)}{w_n - x_n}, \tag{9}$$

where $w_n = x_n + f(x_n)$.
Secondly, approximating $f'(z_n)$ by the linear combination of divided differences

$$f'(z_n) \approx f[y_n, z_n] - f[x_n, y_n] + f[x_n, z_n]. \tag{10}$$

Thus, our new three-step derivative free iterative algorithm without memory is given as

$$y_n = x_n - \frac{f(x_n)}{f[w_n, x_n]},$$

$$z_n = y_n + \frac{f(y_n)}{f[w_n, x_n]} - \frac{2f(y_n)f(x_n)}{f[w_n, x_n]\big(f(x_n) - f(y_n)\big)},$$

$$x_{n+1} = z_n - \frac{f(z_n)}{f[y_n, z_n] - f[x_n, y_n] + f[x_n, z_n]}. \tag{11}$$

Theorem 1 demonstrates its convergence analysis.

Theorem 1. *If $\alpha \in I$ be a simple root of a sufficiently differentiable function $f : I \subseteq \mathbf{R} \to \mathbf{R}$ in an open interval I. If the initial approximation x_0 is sufficiently close to α, then the derivative free method defined by (11) has order of convergence six.*

Proof. Let α be the simple root of $f(x)$, i.e. $f(\alpha) = 0$, $f'(\alpha) \neq 0$, and the error equation is $e_n = x_n - \alpha$.
By Taylor‘s expansion of $f(x_n)$ about $x = \alpha$ and putting $f(\alpha) = 0$, we have

$$f(x_n) = c_1 e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + c_5 e_n^5 + c_6 e_n^6 + O(e_n^7), \tag{12}$$

where

$$c_k = \frac{f^k(\alpha)}{k!}, \qquad k = 1, 2, 3, 4, ...$$

Expanding the Taylor series of $f(w_n)$ about the solution α, we have

$$f(w_n) = \sum_{i=1}^{\infty} c_i (e_n - f(x_n))^i,$$

substituting $f(x_n)$ given by (12) gives us

$$\begin{aligned} f(w_n) &= c_1(1 + c_1)e_n + (c_1 c_2 + (1 + c_1)^2 c_2)e_n^2 \\ &\quad + (2(1 + c_1)c_2^2 + c_1 c_3 + (1 + c_1)^3 c_3)e_n^3 + ... + O(e_n^7). \end{aligned} \tag{13}$$

Substituting (12) and (13) in the first step of (11) gives us

$$\begin{aligned} y_n - \alpha &= x_n - \alpha - \frac{f(x_n)}{f[w_n, x_n]} \\ &= \frac{c_2(c_1 + 1)}{c_1} e_n^2 + \frac{c_1 c_3(c_1^2 + 3c_1 + 2) - c_2^2(c_1^2 + 2c_1 + 2)}{c_1^2} e_n^3 + ... + O(e_n^7). \end{aligned} \tag{14}$$

Using the Taylor expansion of $f(y_n)$ about the solution α, we have

$$f(y_n) = \sum_{i=1}^{\infty} c_i (y_n - \alpha)^i,$$

substituting equation (14) into the preceding equation

$$f(y_n) = c_2(c_1 + 1)e_n^2 + \frac{c_1 c_3(c_1^2 + 3c_1 + 2) - c_2^2(c_1^2 + 2c_1 + 2)}{c_1} e_n^3 + ... + O(e_n^7). \tag{15}$$

$$\frac{f(y_n)}{f[w_n,x_n]} - \frac{2f(y_n)f(x_n)}{f[w_n,x_n]\big(f(x_n)-f(y_n)\big)} = -\frac{c_2(c_1+1)}{c_1}e_n^2 + \frac{c_2^2(c_1+2)-c_1c_3(2+3c_1+c_1^2)}{c_1^2}e_n^3 + ... + O(e_n^7). \tag{16}$$

$$Let \qquad A = \frac{f(y_n)}{f[w_n,x_n]} - \frac{2f(y_n)f(x_n)}{f[w_n,x_n]\big(f(x_n)-f(y_n)\big)}.$$

Using (14) and (16) in the second step of (11) gives us

$$\begin{aligned} z_n - \alpha &= y_n + A - \alpha \\ &= -\frac{c_1c_2^2(c_1+1)}{c_1^2}e_n^3 + ... + O(e_n^7). \end{aligned} \tag{17}$$

This shows that at the end of the second step the method is of third order convergence. Therefore, the third step is introduced to achieve the higher order. The Taylor expansion about the simple root for $f(z_n)$ is given

$$f(z_n) = -c_2^2(c_1+1)e_n^3 - \frac{c_1c_2c_3(1+6c_1+7c_1^2+2c_1^3) - c_1c_2^3(9+7c_1+2c_1^2) - 3c_2^3}{c_1^2}e_n^4 + ... + O(e_n^7). \tag{18}$$

Using (15) and (18) in

$$f[y_n,z_n] = \frac{f(y_n)-f(z_n)}{y_n-z_n}$$

we get

$$f[y_n,z_n] = c_1 + \frac{(c_1+1)c_2^2}{c_1}e_n^2 + ... + O(e_n^7).$$

Similarly

$$f[x_n,y_n] = c_1 + c_2e_n + \frac{c_1c_3+c_2^2+c_1c_2^2}{c_1}e_n^2 + ... + O(e_n^7).$$

$$f[x_n,z_n] = c_1 + c_2e_n + c_3e_n^2 + ... + O(e_n^7).$$

Combining the above terms

$$f'(z_n) = f[y_n, z_n] - f[x_n, y_n] + f[x_n, z_n]$$
$$= c_1 - \frac{(1+c_1)(c_1c_2c_3(1+c_1-c_2)+2c_1c_2^3)}{c_1^2}e_n^3 + ... + O(e_n^7). \quad (19)$$

Now dividing (19) by (18) and using the last step of (11), we have

$$e_{n+1} = \frac{c_1^2c_2^3(c_1+1)^2(c_3((c_1+1)-c_2)+c_2^2)}{c_1^5}e_n^6 + O(e_n^7).$$

This proves that our first proposed method defined by (11) is a sixth order derivative free algorithm and satisfies the above error equation. This completes the proof. □

2.2. Second Sixth Order Method (M2)

Let us consider a fourth order method presented by Ezzati and Saleki's in [5].

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$
$$x_{n+1} = y_n - f(y_n)\left(\frac{1}{f'(x_n)} - \frac{4}{f'(x_n)+f'(y_n)}\right), \quad (20)$$

which requires four function evaluations per iteration and its efficiency index is 1.414., by combining the method (20) with Newton's method, we obtain a new three-step iterative algorithm without memory as follows

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$
$$z_n = y_n - f(y_n)\left(\frac{1}{f'(x_n)} - \frac{4}{f'(x_n)+f'(y_n)}\right),$$
$$x_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}, \quad (21)$$

which includes three evaluations of the function derivative. we remove $f'(x_n)$ and $f'(z_n)$ by the approximations (9) and (10) and for $f'(y_n)$ we use the following approximation [6]

$$f'(y_n) \approx \frac{f'(x_n)(f(x_n)-f(y_n))}{f(x_n)+f(y_n)},$$

Our three-step derivative free iterative algorithm without memory is given by

$$y_n = x_n - \frac{f(x_n)}{f[w_n, x_n]},$$

$$z_n = y_n - f(y_n)\big(\frac{1}{f[w_n, x_n]} - \frac{4(f(x_n) + f(y_n))}{f[w_n, x_n] + f[w_n, x_n](f(x_n) - f(y_n))}\big),$$

$$x_{n+1} = z_n - \frac{f(z_n)}{f[y_n, z_n] - f[x_n, y_n] + f[x_n, z_n]}. \tag{22}$$

Theorem 2 shows that the derivative free method defined by (22) has sixth order convergence.

Theorem 2. *If $\alpha \in I$ be a simple root of a sufficiently differentiable function $f : I \subseteq \mathbf{R} \to \mathbf{R}$ in an open interval I. If the initial approximation x_0 is sufficiently close to α, then the derivative free method defined by (22) has order of convergence six.*

Proof. By applying same assumptions and symbolic calculations as done in the proof of Theorem 1 and then by substituting them into (22), we get the following error equation

$$e_{n+1} = \frac{c_1^2 c_2^3 (c_1 + 1)^2 (c_3((c_1 + 1) - c_2) + c_2^2)}{c_1^5} e_n^6 + O(e_n^7).$$

This shows that our second proposed method defined by (22) is of sixth order derivative free algorithm and satisfies the above error equation. This completes the proof. □

2.3. Seventh Order Method (M3)

In [7], a method of fifth order convergence which requires four function evaluations has been presented by the following scheme

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$x_{n+1} = y_n - \big(1 + (\frac{f(y_n)}{f(x_n)})^2\big)\frac{f(y_n)}{f'(y_n)}, \tag{23}$$

where its efficiency index is 1.495, by combining the method (23) with the Newton's method, we obtain a new three-step iterative algorithm without memory

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = y_n - \big(1 + (\frac{f(y_n)}{f(x_n)})^2\big)\frac{f(y_n)}{f'(y_n)},$$

$$x_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}, \tag{24}$$

which uses three evaluations of the function derivative. We approximate $f'(x_n)$ and $f'(z_n)$ by (9) and (10). we also approximate $f'(y_n)$ by the divided difference

$$f'(y_n) \approx \frac{f[x_n, y_n]f[w_n, y_n]}{f[w_n, x_n]}.$$

Thus, our new three-step derivative free iterative algorithm without memory is given by

$$y_n = x_n - \frac{f(x_n)}{f[w_n, x_n]},$$

$$z_n = y_n - \big(1 + (\frac{f(y_n)}{f(x_n)})^2\big)\frac{f(y_n)f[w_n, x_n]}{f[x_n, y_n]f[w_n, y_n]},$$

$$x_{n+1} = z_n - \frac{f(z_n)}{f[y_n, z_n] - f[x_n, y_n] + f[x_n, z_n]}. \tag{25}$$

Theorem 3 demonstrates the convergence analysis of the method given by (2.19).

Theorem 3. *If $\alpha \in I$ be a simple root of a sufficiently differentiable function $f : I \subseteq \mathbf{R} \to \mathbf{R}$ in an open interval I. If the initial approximation x_0 is sufficiently close to α, then the derivative free method defined by (25) has order of convergence seven.*

Proof. Again by using same assumptions and symbolic calculations as done in the proof of Theorem 1 and by substituting them into (25). It is only need to obtain

$$\frac{f[w_n, x_n]}{f[x_n, y_n]f[w_n, y_n]} = \frac{1}{c_1} + \frac{c_1^2c_3 - 3c_2^2 - 3c_1c_2^2 + c_1c_3}{c_1^3}e_n^2 + ... + O(e_n^8). \tag{26}$$

From(15) and (26)

$$\frac{f[w_n,x_n]f(y_n)}{f[x_n,y_n]f[w_n,y_n]} = \frac{c_2(c_1+1)}{c_1}e_n^2 + \frac{c_1c_3((c_1+1)^2+c_1+2)-c_2^2((c_1+1)^2+1)}{c_1^2}e_n^3+...+O(e_n^8). \tag{27}$$

also

$$\big(1+(\frac{f(y_n)}{f(x_n)})^2\big) = 1+\frac{c_2(c_1+1)^2}{c_1^2}e_n^2 + \frac{2c_1c_2c_3(2+5c_1+4c_1^2+c_1^3)-2c_2^3(-3+6c_1-4c_1^2-c_1^3)}{c_1^3}e_n^3 +...+O(e_n^8). \tag{28}$$

From (27) and (28)

$$\frac{f[w_n,x_n]f(y_n)}{f[x_n,y_n]f[w_n,y_n]}\big(1+(\frac{f(y_n)}{f(x_n)})^2\big) = \frac{c_2(c_1+1)}{c_1}e_n^2 + \frac{c_1c_3((c_1+1)^2+c_1+2)-c_2^2((c_1+1)^2+1)}{c_1^2}e_n^3 +...+O(e_n^8). \tag{29}$$

$$Let \qquad B = \frac{f[w_n,x_n]f(y_n)}{f[x_n,y_n]f[w_n,y_n]}\big(1+(\frac{f(y_n)}{f(x_n)})^2\big)$$

Using (14) and (29) in the second step of (25) gives us

$$z_n-\alpha = y_n - B - \alpha = \frac{(1+c_1)(-2c_1c_2c_3-(c_1^2-1)c_2^3)}{c_1^3}e_n^4+...+O(e_n^8). \tag{30}$$

This shows that at the end of the second step the method is of optimal fourth order. Therefore, the third step is introduced to achieve the higher order of convergence. Taylor expansion of $f(z_n)$ about α

$$f(z_n) = \frac{(1+c_1)(-2c_1c_2c_3-(c_1^2-1)c_2^3)}{c_1^2}e_n^4+...+O(e_n^8). \tag{31}$$

Now dividing (19) by (31) and using last step of (25), we have

$$\begin{aligned} e_{n+1} = \frac{1}{c_1^6}&((c_1c_2^6 - c_1c_2^5c_3)((1-2c_1) + c_1^3(2+c_1)) \\ &+ c_1c_2^2c_3(c_2^2(2c_1^2(2c_1+1) + c_1^4(c_1+1) + (c_1+1)) \\ &+ c_1c_3(2c_1^2(c_1+3) + 2c_1c_2(c_1-2)) + 2c_2)e_n^7 + O(e_n^8). \end{aligned}$$

This means that our third proposed method defined by (25) has seventh order convergence which is a derivative free algorithm and satisfies the above error equation. □

3. Algorithms

In this section, we describe the algorithms of the proposed methods M1(11), M2(22) and M3(25). The algorithm of method M1(11) is described as follows:

Algorithm M1:

Step 1. For initial guess x_0, a tolerance $\varepsilon > 0$, and for iterations N set $n = 0$.

Step 2: Calculate

$$y_n = x_n - \frac{f(x_n)}{f[w_n, x_n]},$$

$$z_n = y_n + \frac{f(y_n)}{f[w_n, x_n]} - \frac{2f(y_n)f(x_n)}{f[w_n, x_n]\big(f(x_n) - f(y_n)\big)},$$

$$x_{n+1} = z_n - \frac{f(z_n)}{f[y_n, z_n] - f[x_n, y_n] + f[x_n, z_n]}.$$

Step 3: If $|x_{n+1} - x_n| < \varepsilon$ then stop, otherwise go to step four.

Step 4: Set $n = n + 1$ and go to **Step 1.**

In the same way, the algorithms of the methods M1(22) and M3(25) can be obtained.

Method	Order	Efficiency index
Steffensen (SM)	2	1.4142
Dehghan-Hajarian (DHM)	3	1.4422
Cordero (CM4)	4	1.4142
Cordero (CM6)	6	1.4301
M1(11)	6	1.5650
M2(22)	6	1.5650
M3(25)	7	1.6265

Table 1: Order and efficiency indices of some derivative free methods

4. Numerical Results

In this section, we test the effectiveness of our new methods **M1 (11)**, **M2 (22)** and **M3 (25)**. We have used second order method of Steffensen (SM) [2], third order method of Dehghan-Hajarian (DHM) [3] and fourth and sixth order methods of Cordero et al. (CM4) and (CM6) [4] for comparison with our methods by using same initial approximation to find the simple roots of the nonlinear equations. For every method, we find the order of convergence, efficiency index, the number of iterations, the number of function evaluations (NFE), the difference between the approximated root x_n and the exact root α, the absolute values of the function and the computational order of convergence (COC) defined by [8]

$$\rho \approx \frac{\ln |(x_{n+1} - \alpha)/(x_n - \alpha)|}{\ln |(x_n - \alpha)/(x_{n-1} - \alpha)|}.$$

All of the numerical computations have been carried out using variable precision arithmetic, with 700 significant digits and $N = 1000$ in Maple 7.1 and the stopping criterion used is $|x_{n+1} - x_n| < 10^{-17}$. Table 1 shows, the comparison of order of convergence and the efficiency indices of derivative free methods mentioned above with our methods. In Table 2 the nonlinear equations and their roots are given. Table 3 shows the number of iterations and the number of function evaluations (NFE) of different derivative free methods. Table 4 shows the difference between the approximated root x_n and the exact root α, the absolute values of the function $|f(x_n)|$ and the computational order of convergence (COC).

Nonlinear equations	Roots
$f_1(x) = (1 + \cos(x))(\exp(x) - 2)$	$\alpha \approx 0.693147180559945$
$f_2(x) = x/exp(x) - 0.1$	$\alpha \approx 0.111832559158962$
$f_3(x) = x^2 + \sin(x) + x$	$\alpha = 0$
$f_4(x) = 10x \exp(-x^2) - 1$	$\alpha \approx 0.167963061042844$
$f_5(x) = x^5 - 8x^4 + 24x^3 - 1$	$\alpha \approx 0.361079894525022$
$f_6(x) = \sqrt{x^2 + 2x + 5} - 2\sin(x) - x^2 + 3$	$\alpha \approx 2.331967655883964$
$f_7(x) = \ln(x) - \sqrt{x} + 5$	$\alpha \approx 8.309432694231571$
$f_8(x) = \arcsin(x^2 - 1) - x/2 + 1$	$\alpha \approx 0.594810968398398$
$f_9(x) = \sin(x)^2 - x^2 + 1$	$\alpha \approx 1.404491648215341$
$f_{10}(x) = x^3 + 4x^2 - 10$	$\alpha \approx 1.365230013414096$

Table 2: Nonlinear equations and their roots

5. Conclusion

In this paper, we have given simple yet powerful derivative free iterative algorithms without memory for solving nonlinear equations. Numerical analysis proves that the new methods preserve their order of convergence. Our Steffensen like two sixth order and one seventh order derivative free algorithms require four functional evaluations per iteration and have better efficiency index than the Steffensen's method [2], Dehghan-Hajarian's (DHM) [3] and Cordero et al. method [4]. The main characteristic of our new Steffensen like methods is that they only by increasing the evaluation of one more function its order of convergence increases from four to six and from five to seven. These methods are also more efficient than all those methods which have the same order of convergence and efficiency index involving the first derivative [9], [10], [11].

$f(x)$	x_0	SM	DHM	$CM4$	$CM6$	$M1$	$M2$	$M3$
$f_1(x)$	0.5							
$Iterations$		6	5	4	3	3	3	3
NFE		12	15	16	15	12	12	12
$f_2(x)$	0.2							
$Iterations$		6	4	3	3	3	3	3
NFE		12	12	12	15	12	12	12
$f_3(x)$	0.3							
$Iterations$		7	5	4	3	3	3	3
NFE		14	15	16	15	12	12	12
$f_4(x)$	1.5							
$Iterations$		6	4	4	3	3	3	3
NFE		12	12	16	15	12	12	12
$f_5(x)$	0.35							
$Iterations$		6	5	3	3	3	3	3
NFE		12	15	12	15	12	12	12
$f_6(x)$	2.9							
$Iterations$		5	4	4	4	3	3	3
NFE		10	12	16	20	12	12	12
$f_7(x)$	11.8							
$Iterations$		6	5	4	3	3	3	3
NFE		12	15	16	15	12	12	12
$f_8(x)$	1.2							
$Iterations$		8	4	5	4	4	4	4
NFE		16	12	20	20	16	16	16
$f_9(x)$	1.0							
$Iterations$		7	4	4	3	4	4	3
NFE		14	12	16	15	16	12	12
$f_{10}(x)$	1.5							
$Iterations$		8	6	4	4	4	3	3
NFE		16	18	16	20	16	12	12

Table 3: Numerical results of different derivative free methods

$f(x)$	x_0	SM	DHM	$CM4$	$CM6$	$M1$	$M2$	$M3$
$f_1(x)$	0.5							
COC		1.9999	3.0000	3.9999	5.9999	5.9999	6.7810	6.9999
$\|f(x_n)\|$		1.2e-52	3.7e-125	3.7e-177	1.3e-132	1.5e-169	2.4e-94	3.7e-235
$\|x_n - \alpha\|$		3.6e-53	1.0e-125	1.0e-177	3.7e-133	4.3e-170	6.9e-93	1.0e-235
$f_2(x)$	0.2							
COC		1.9999	3.0000	3.9999	5.9999	5.9999	6.2192	7.0000
$\|f(x_n)\|$		2.2e-49	2.6e-52	1.6e-77	6.3e-239	2.0e-183	2.9e-111	2.2e-339
$\|x_n - \alpha\|$		2.8e-49	3.3e-52	2.0e-77	8.0e-239	2.6e-183	3.7e-111	2.8e-339
$f_3(x)$	0.3							
COC		1.9999	2.9999	3.9999	5.9999	5.9999	5.5034	6.9999
$\|f(x_n)\|$		3.7e-63	9.8e-113	6.3e-197	1.9e-152	8.3e-185	1.9e-243	1.1e-279
$\|x_n - \alpha\|$		1.8e-63	4.9e-113	3.1e-197	9.5e-153	4.1e-185	5.9e-244	5.6e-280
$f_4(x)$	1.5							
COC		1.9999	2.9999	4.0000	5.9999	6.0000	6.1352	7.0000
$\|f(x_n)\|$		1.0e-48	1.8e-77	5.7e-253	3.4e-281	1.2e-152	1.7e-110	4.1e-292
$\|x_n - \alpha\|$		3.8e-49	6.7e-78	2.0e-253	1.2e-281	4.3e-153	6.2e-111	1.5e-292
$f_5(x)$	0.35							
COC		1.9999	3.0000	3.9999	6.0000	5.9999	6.2718	7.0000
$\|f(x_n)\|$		3.0e-37	9.8e-126	3.1e-78	4.0e-241	8.2e-161	1.7e-96	2.4e-286
$\|x_n - \alpha\|$		3.7e-38	1.2e-126	3.9e-79	5.0e-242	1.0e-161	2.2e-97	3.0e-287
$f_6(x)$	2.9							
COC		2.0000	3.0000	3.9999	6.0000	6.0000	5.9612	6.9999
$\|f(x_n)\|$		6.1e-35	2.9e-112	1.0e-108	1.5e-425	5.6e-204	4.6e-130	2.5e-298
$\|x_n - \alpha\|$		2.5e-35	1.2e-112	4.3e-109	6.5e-426	2.3e-204	1.9e-130	1.0e-298
$f_7(x)$	11.8							
COC		2.0000	2.9999	3.9999	6.0000	6.0000	6.0885	6.9999
$\|f(x_n)\|$		4.3e-48	8.9e-150	1.8e-233	2.5e-213	2.9e-180	3.7e-109	9.7e-268
$\|x_n - \alpha\|$		1.4e-47	3.0e-149	6.3e-233	8.7e-213	1.0e-179	1.2e-108	3.3e-267
$f_8(x)$	1.2							
COC		2.0000	2.9999	4.0000	5.9999	5.9999	6.0010	6.9999
$\|f(x_n)\|$		1.4e-47	1.3e-101	2.4e-217	1.3e-251	2.0e-232	3.7e-139	6.8e-402
$\|x_n - \alpha\|$		1.3e-47	1.2e-101	2.3e-217	1.2e-251	9.2e-232	4.3e-139	6.5e-402
$f_9(x)$	1.0							
COC		1.9999	2.9999	4.0000	6.0000	6.0000	5.6126	7.0000
$\|f(x_n)\|$		1.0e-62	1.1e-82	2.1e-175	1.2e-146	1.6e-472	1.1e-243	4.1e-169
$\|x_n - \alpha\|$		4.3e-63	4.7e-83	8.5e-176	4.9e-147	6.7e-473	5.9e-244	1.6e-169
$f_{10}(x)$	1.5							
COC		1.9999	2.9999	3.9999	6.0000	5.9999	5.3014	7.0031
$\|f(x_n)\|$		1.0e-35	4.5e-108	4.2e-99	5.1e-365	7.8e-516	1.1e-82	6.0e-153
$\|x_n - \alpha\|$		6.4e-37	2.7e-109	2.5e-100	3.1e-366	4.7e-517	6.9e-84	3.6e-154

Table 4: Numerical results of different derivative free methods

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