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ON KROPINA CHANGE OF TWO-DIMENSIONAL FINSLER SPACES

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Abstract: The purpose of the present paper is to obtained the relationship between the main scalars, geodesic and scalar curvature among two-dimensional Finsler spaces F^2 and a Finsler space F^{*2} due to Kropina change.

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1. Introduction

Let (M^n, L) be an n-dimensional Finsler space on a differentiable manifold M^n, equipped with the fundamental function L(x, y). In 1971, Matsumoto [1] introduced the transformation of Finsler metric:

$$L^{'}(x,y) = L(x,y) + \beta(x,y) \tag{1}$$

where, $\beta(x,y) = b_i(x)y^i$, $b_i(x)$ are components of a covariant vector which is a function of position alone. If L(x, y) is a metric function of Riemannian

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space then $L^{'}(x, y)$ reduces to the metric function of Randers space. Such a Finsler metric was first introduced by G. Randers [2] from the stand point of general theory of relativity and applied to the theory of electron microscope by R. S. Ingarden [3], who first named it as Randers space. The geometrical property of this space has been studied by various authors [4, 5, 6, 7, 8]. In 1978, Numata [9] has studied the properties of $(M^n, L^{'})$ which is obtained from Minkowskian space (M^n, L) by the transformation (1). In 1984, Shibata [10] has studied the properties of Finsler space (M^n, L^*) whose metric function $L^*(x, y)$ is obtained from L(x, y) by the relation $L^*(x, y) = f(L, \beta)$ where f is positively homogeneous of degree one in L and β. This change of metric function is called a β-change. The change (1) is a particular case of β-change called Randers change.

Another particular β-change of Finsler metric function is a Kropina change of metric function given by

$$L^*(x, y) = \frac{L^2(x, y)}{\beta(x, y)} \tag{2}$$

If L(x, y) reduces to the metric function of Riemannian space then $L^*(x, y)$ reduces to the metric function of Kropina space [11]. Due to this reason the transformation (2) has been called the Kropina change of Finsler metric.

2. The Finsler Space (M^n, L^*)

Let (M^n, L) be a given Finsler space and let $b_i(x)y^i$ be a one form on M^n. We shall define on M^n a function $L^*(x, y)$ 0 by the equation,

$$L^*(x, y) = \frac{L^2(x, y)}{\beta(x, y)} \tag{3}$$

where, we put $\beta(x, y) = b_i(x)y^i$.

To find the metric tensor g^*_{ij}, the angular metric tensor h^*_{ij}, the Cartan tensor C^*_{ijk}, and the v-curvature tensor of (M^n, L^*) [12], we use the following results,

$$\dot{\partial}_i\beta = b_i, \ \ \dot{\partial}_i L = l_i, \ \ \dot{\partial}_i l_i = L^{-1}h_{ij} \tag{4}$$

where, $\dot{\partial}_i$ stands for partial derivative with respect to y^i and h_{ij} are components of angular metric tensor of (M^n, L) given by $h_{ij} = g_{ij} - l_i l_j = L\dot{\partial}_i\dot{\partial}_j L$. The successive differentiation of (3) with respect to y^i and y^j gives,

$$l^*_i = \frac{2L}{\beta}l_i + \frac{L^2}{\beta^2}b_i \tag{5}$$

$$h^*_{ij} = \frac{2L^2}{\beta^2}h_{ij} + \frac{2L^2}{\beta^2}l_il_j + \frac{2L^3}{\beta^3}(l_ib_j + l_jb_i) + \frac{2L^4}{\beta^4}b_ib_j \tag{6}$$

From (5) and (6) we get the following relation between metric tensors of (M^n, L) and (M^n, L^*),

$$g^*_{ij} = \frac{2L^2}{\beta^2}g_{ij} + \frac{4L^2}{\beta^2}l_il_j + \frac{4L^3}{\beta^3}(l_ib_j + l_jb_i) + \frac{3L^4}{\beta^4}b_ib_j \tag{7}$$

The contravariant component of the metric tensor of (M^n, L^*) will be derived from (7) as follows,

$$g^{*ij} = \frac{\beta^2}{2L^2}g^{ij} + \frac{\beta^2}{L^2}(1 - \frac{2\beta^2}{L^2b^2})l^il^j - \frac{\beta^2}{2L^2b^2}b^ib^j + \frac{\beta^3}{L^3b^2}(l^ib^j + l^jb^i) \tag{8}$$

where, we put $b^2 = g^{ij}b_ib_j, b^i = g^{ij}b_j, l^i = g^{ij}l_j$. Differentiating (7) with respect to y^k and using (4), we get the following relation between the Cartan tensors of (M^n, L) and (M^n, L^*)

$$C^*_{ijk} = \frac{2L^2}{\beta^2}C_{ijk} - \frac{2L^2}{\beta^2}(h_{ij}d_k + h_{jk}d_i + h_{ki}d_j) - \frac{6L^4}{\beta^5}d_id_jd_k \tag{9}$$

where, $d_i = b_i - \frac{\beta}{L}l_i$. It is to be noted that,

$$d_il^i = 0, \;\; d_ib^i = b^2 - \frac{\beta^2}{L^2}, \;\; h_{ij}l^j = 0, \;\; h_{ij}d^j = h_{ij}b^j = d_i \tag{10}$$

where, $d^i = g^{ij}d_j = b^i - \frac{\beta}{L}l^i$
To find $C^{*i}_{jk} = g^{*ih}C^*_{jhk}$, we use (8), (9) and (10), we have,

$$C^{*i}_{jk} = C^i_{jk} - \beta^{-1}(h^i_jd_k + h^i_kd_j + h^i_{jk}) - 3L^2\beta^{-3}d_jd_kd^i - \tag{11}$$
$$b^{-2}C_{.jk}e^i + (3L^2\beta^{-3} - \beta^{-1}b^{-2})d_jd_ke^i + (\beta^{-1} - \beta L^{-2}b^{-2})h_{jk}e^i$$

where, $e^i = b^i - 2\beta L^{-1}l^i = d^i - \beta L^{-1}l^i$ and $C_{.jk} = C_{ijk}b^i$.
Throughout this paper we use the symbol $'.'$ to denote the contraction with b^i.

Proposition 1. *Let $F^{*n} = (M^n, L^*)$ be an n-dimensional Finsler space obtained from the Kropina change of the Finsler space $F^n = (M^n, L)$, then their normalized supporting element l^*_i, angular metric tensor h^*_{ij}, fundamental metric tensor g^*_{ij} and (h)hv-torsion tensor C^*_{ijk} is given by (5), (6), (7) and (9) respectively.*

3. Kropina Change of Main Scalar

The (h)hv-torsion tensor for a two-dimensional Finsler space F^2 is given by,

$$LC_{ijk} = Im_i m_j m_k \tag{12}$$

where, $I = C_{222}$ is the main scalar [13] in F^2.
Similarly, the (h)hv-torsion tensor for a two-dimensional Finsler space F^{*2} is given by,

$$L^* C^*_{ijk} = I^* m^*_i m^*_j m^*_k \tag{13}$$

where, I^* is the main scalar [13] in F^{*2}, and m^*_i is unit vector orthogonal to l^*_i in two-dimensional Finsler space.

Putting, i = k in equation in (11), we get

$$C^*_i = C_i - \beta^{-1}(n+1)d_i - b^{-2}C_{..i} \tag{14}$$

The normalized torsion vector $m^i = \frac{C^i}{C}$, in F^2 and $m^{*i} = \frac{C^{*i}}{C^*}$ in F^{*2} is the length of C^i and C^{*i} respectively.
The equation (14) can also be written

$$m^*_i = \lambda m_i + \mu d_i + \phi C_{..i} \tag{15}$$

where, $\lambda = \frac{C}{C^*}$, $\mu = \frac{-(n+1)\beta^{-1}}{C^*}$, $and \phi = \frac{-b^{-2}}{C^*}$. since,

$$\begin{aligned} C^{*2} = g^{*ij}C^*_i C^*_j = \frac{\beta^2}{2L^2}C^2 - \frac{\beta^2}{2L^2 b^2}B^2 - \beta^{-1}(n+1)[\frac{\beta^2}{L^2}D - \\ \frac{\beta^2}{L^2 b^2}B(b^2 - (\frac{\beta^2}{L^2}))] - b^{-2}\frac{\beta^2}{L^2}D - \frac{\beta^2}{L^2 b^2}BC_{...} + \frac{(n+1)^2}{2L^2}F - \\ \frac{(n+1)^2}{2L^2 b^2}(b^2 - (\frac{\beta^2}{L^2}))^2 + \beta^{-1}(n+1)b^{-2}[\frac{\beta^2}{L^2}H - \frac{\beta^2}{L^2 b^2}(b^2 - \\ (\frac{\beta^2}{L^2}))C_{...}] + \frac{b^{-4}\beta^2}{2L^2}J - \frac{\beta^2}{2L^2}b^{-2}C^2_{...} \end{aligned} \tag{16}$$

where, $B = b^i C_i$, $D = g^{ij}C_i d_j = g^{ij}C_j d_i$, $E = g^{ij}C_i C_{..j} = g^{ij}C_{..i}C_j$, $F = g^{ij}d_i d_j$, $H = g^{ij}d_i C_{..j} = g^{ij}C_{..i}d_j$, $J = g^{ij}C_{..i}C_{..j}$ are an scalars.
The contravarient component of l^{*i} and m^{*i} is given by

$$l^{*i} = g^{*ij}l^*_j = \frac{\beta}{L}l^i \tag{17}$$

where, $l^i l_i = 1$.
Again,

$$m^{*i} = g^{*ij} m^*_j = Mm^i + Nb^i + Pl^i + Qd^i + SC^i_{..} \tag{18}$$

where, $M = \lambda \frac{\beta^2}{2L^2}$, $N = -(\frac{\lambda\beta^2}{2L^2 b^2} K + \frac{\mu\beta^2}{2L^2 b^2}(b^2 - \frac{\beta^2}{L^2}) + \frac{\phi\beta^2}{2L^2 b^2} C_{...})$, $P = \frac{\lambda\beta^2}{L^3 b^2} K + \frac{\mu\beta^2}{L^3 b^2}(b^2 - \frac{\beta^2}{L^2}) + \frac{\phi\beta^2}{L^3 b^2} C_{...}$, $Q = \frac{\mu\beta^2}{2L^2}$, $S = \frac{\phi\beta^2}{2L^2}$, $b^i m_i = K$ is an certain scalar.

Hence

Proposition 2. *Let $F^{*n} = (M^n, L^*)$ be an n-dimensional Finsler space obtained from the Kropina change of the Finsler space $F^n = (M^n, L)$, then contravariant and covariant components of the Berwald frame (l, m) in two-dimensional Finsler space is given by (17), (18), (5) and (15) respectively.*

Proposition3. *Let $F^{*n} = (M^n, L^*)$ be an n-dimensional Finsler space obtained from the Kropina change of the Finsler space $F^n = (M^n, L)$, then the relationship between the length of the components C_i and C^*_i is given by (16).*

Since, the (h)hv-torsion tensor given by (9) can be rewritten as in two-dimensional as follows:

$$I^* m^*_i m^*_j m^*_k = \frac{2L^2}{\beta^2} I m_i m_j m_k - \frac{6L^2}{\beta^3} b_2 m_i m_j m_k + \frac{2L}{\beta^2}(m_i m_j l_k + m_k m_j l_i + m_k m_i l_j) - \frac{6L^4}{\beta^5} b_2^3 m_i m_j m_k - \frac{6L^2}{\beta^3} b_2 (l_i l_j m_k + l_k l_j m_i + l_k l_i m_j) + \frac{6L^3}{\beta^4} b_2^2 (l_i m_j m_k + l_k m_j m_i + l_k m_i m_j) + \frac{6L}{\beta^2} l_i l_j l_k \tag{19}$$

where, $h_{ij} = m_i m_j$ and $b_i = b_1 l_i + b_2 m_i$, then $b_i y^i = 0 \Longrightarrow b_1 = 0$. So, $b_i = b_2 m_i$, b_1 and b_2 are certain scalars.
From equation (15) and (19), we have

$$I^*(\lambda + \mu b_2 + \phi I K^2)^3 m_i m_j m_k = \frac{2L^2}{\beta^2} I m_i m_j m_k - \frac{6L^2}{\beta^3} b_2 m_i m_j m_k + \frac{2L}{\beta^2}(m_i m_j l_k + m_k m_j l_i + m_k m_i l_j) - \frac{6L^4}{\beta^5} b_2^3 m_i m_j m_k - \frac{6L^2}{\beta^3} b_2 (l_i l_j m_k + l_k l_j m_i + l_k l_i m_j) + \frac{6L^3}{\beta^4} b_2^2 (l_i m_j m_k + l_k m_j m_i + l_k m_i m_j) + \frac{6L}{\beta^2} l_i l_j l_k - \frac{6L^4}{\beta^5} b_2^3 \tag{20}$$

Contracting (20) by $m_i m_j m_k$, we have,

$$(\lambda + \mu b_2 + \phi I K^2)^3 I^* = \frac{2L^2}{\beta^2} I - \frac{6L^2}{\beta^3} b_2 \tag{21}$$

Theorem 1. *Let $F^{*n} = (M^n, L^*)$ be an n-dimensional Finsler space obtained from the Kropina change of the Finsler space $F^n = (M^n, L)$, then the relationship between the main scalar I^* and I of the Finsler space is given by (21).*

4. Kropina Change of Geodesic

Let us consider s be the arc-length, then the equations of a geodesic [14] of $F^n = (M^n, L)$ is written in the well-known from,

$$\frac{d^2x^i}{ds^2} + 2G^i(x, \frac{dx}{ds} = 0 \tag{22}$$

where, functions $G^i(x, y)$ are given by

$$2G^i = g^{ir}(y^j \dot{\partial}_r \partial_j F - \partial_r F), \quad F = \tfrac{L^2}{2}$$

Now, suppose s^* be the arc-length in the Finsler space $F^{*n} = (M^n, L^*)$, then the equations of a geodesic can be written as,

$$\frac{d^2x^i}{ds^{*2}} + 2G^{*i}(x, \frac{dx}{ds^*}) = 0 \tag{23}$$

where, functions $G^{*i}(x, y)$ are given by

$$2G^{*i} = g^{*ir}(y^j \dot{\partial}_r \partial_j F^* - \partial_r F^*), \quad F^* = \tfrac{L^{*2}}{2}$$

Since, $ds^* = L^*(x, dx)$, this is also be written as,

$$ds^* = \tfrac{L^2(x,y)}{\beta(x,y)} = \tfrac{ds^2}{b_i dx^i}$$

Since, ds= L(x,dx)
Thus we have,

$$\frac{dx^i}{ds} = \frac{dx^i}{ds^*}[\frac{2ds}{\beta} - (\frac{ds}{\beta})^2 b_i \frac{dx^i}{ds}] \tag{24}$$

Differentiating (24) with respect to ds, we have,

$$\frac{d^2x^i}{ds^2} = \frac{d^2x^i}{ds^{*2}}[\frac{2ds}{\beta} - (\frac{ds}{\beta})^2 b_i \frac{dx^i}{ds}]^2 + \frac{dx^i}{ds^*}[\frac{2}{\beta} - \frac{2ds}{\beta^2} b_i \frac{dx^i}{ds} - (\frac{ds}{\beta})^2 b_i \frac{d^2x^i}{ds^2} - b_i \frac{dx^i}{ds}(\frac{2ds}{\beta^2} - \frac{2(ds)^2}{\beta^3} b_i \frac{dx^i}{ds})] \tag{25}$$

Since, $2G^{*i} = g^{*ir}(y^j \dot{\partial}_r \partial_j \frac{L^{*2}}{2} - \partial_r \frac{L^{*2}}{2})$
Then,

$$G_i^* = \frac{2L^2}{\beta^2} G_i + y^j [\frac{2L^2}{\beta}(\frac{l_i}{\beta} - \frac{L}{\beta^2} b_i) \partial_j L - \frac{L^4}{\beta^3} \dot{\partial}_i \partial_j \beta - \frac{2L^3}{\beta^2} l_i \partial_j \beta + \frac{2L^4}{\beta^4} b_i \partial_j \beta + \frac{2L^3}{\beta^3} l_i \partial_j \beta - \frac{2L^3}{\beta^3} b_i \partial_j L] + \frac{L^4}{\beta^3} \partial_j \beta \tag{26}$$

Now, we have

$$G^{*i} = g^{*ir} G_r^* = G^i + \frac{2L^2}{\beta^2} G_r [\frac{\beta^2}{L^2}(1 - \frac{2\beta^2}{L^2 b^2}) l^i l^r - \frac{2\beta^2}{2L^2 b^2} b^i b^r + \frac{\beta^3}{L^3 b^2}(l^i b^r + l^r b^i)] + [\frac{\beta^2}{2L^2} g^{ij} + \frac{\beta^3}{L^3 b^2}(l^i b^r + l^r b^i) + \frac{\beta^2}{L^2}(1 - \frac{2\beta^2}{L^2 b^2}) l^i l^r - \frac{2\beta^2}{2L^2 b^2} b^i b^r][y^j [\frac{2L^2}{\beta}(\frac{l_i}{\beta} - \frac{L}{\beta^2} b_i) \partial_j L - \frac{L^4}{\beta^3} \dot{\partial}_i \partial_j \beta - \frac{2L^3}{\beta^2} l_i \partial_j \beta + \frac{2L^4}{\beta^4} b_i \partial_j \beta + \frac{2L^3}{\beta^3} l_i \partial_j \beta - \frac{2L^3}{\beta^3} b_i \partial_j L] + \frac{L^4}{\beta^3} \partial_j \beta] \tag{27}$$

Proposition 4. *Let $F^{*n} = (M^n, L^*)$ be an n-dimensional Finsler space obtained from the Kropina change of the Finsler space $F^n = (M^n, L)$, then the relationship between the Berwald connection function G^{*i} and G^i is given by (27).*

Proposition 5. *Let $F^{*n} = (M^n, L^*)$ be an n-dimensional Finsler space obtained from the Kropina change of the Finsler space $F^n = (M^n, L)$, then the relationship between the arc-length s^* and s is given by (24).*

Theorem 2. *Let $F^{*n} = (M^n, L^*)$ be an n-dimensional Finsler space obtained from the Kropina change of the Finsler space $F^n = (M^n, L)$, then the equation of geodesic is given by (23) where $\frac{d^2 x^i}{ds^{*2}}$ and G^{*i} is given by (25) and (27) respectively.*

5. Kropina Change of Scalar Curvature

The (v)h-torsion tensor R^i_{jk} in two-dimensional Finsler space may be written as,

$$R^i_{jk} = LRm^i(l_j m_k - l_k m_j) \tag{28}$$

where, R is the h-scalar curvature.
Again, the (v)h-torsion tensor R^{*i}_{jk} in Finsler space F^{*2} is,

$$R^{*i}_{jk} = L^* R^* m^{*i}(l^*_j m^*_k - l^*_k m^*_j) \tag{29}$$

The equation (29) can also be written as,

$$\frac{R^{*i}_{jk}}{R^*} = L^* m^{*i}(l^*_j m^*_k - l^*_k m^*_j)$$

In view of (3), (5), (15) and (17), we have,

$$\frac{R^{*i}_{jk}}{R^*} = \frac{2L^2\lambda}{\beta^2} MLm^i(l_j m_k - l_k m_j) + \frac{2L^3\lambda}{\beta^2}(Nb^i + Pl^i + Qd^i + SC^i_{..})(l_j m_k - l_k m_j) + \frac{L^2}{\beta}(Mm^i + Nb^i + Pl^i + Qd^i + SC^i_{..})[\frac{2L_\mu}{\beta}(l_j d_k - l_k d_j) + \frac{2L_\phi}{\beta}(l_j C_{..k} - l_k C_{..j}) - \frac{L^2\lambda}{\beta^2}(b_j m_k - b_k m_j) - \frac{L^2\mu}{\beta^2}(b_j d_k - b_k d_j) - \frac{L^2\phi}{\beta^2}(b_j C_{..k} - b_k C_{..j})] \tag{30}$$

Using (28) in (30), we have,

$$\frac{R^{*i}_{jk}}{R^*} = \frac{2L^2\lambda}{\beta^2} M \frac{R^i_{jk}}{R} + \frac{2L^3\lambda}{\beta^2}(Nb^i + Pl^i + Qd^i + SC^i_{..})(l_j m_k - l_k m_j) + \frac{L^2}{\beta}(Mm^i + Nb^i + Pl^i + Qd^i + SC^i_{..})[\frac{2L_\mu}{\beta}(l_j d_k - l_k d_j) + \frac{2L_\phi}{\beta}(l_j C_{..k} - l_k C_{..j}) - \frac{L^2\lambda}{\beta^2}(b_j m_k - b_k m_j) - \frac{L^2\mu}{\beta^2}(b_j d_k - b_k d_j) - \frac{L^2\phi}{\beta^2}(b_j C_{..k} - b_k C_{..j})] \tag{31}$$

Theorem 3. *Let $F^{*n} = (M^n, L^*)$ be an n-dimensional Finsler space obtained from the Kropina change of the Finsler space $F^n = (M^n, L)$, then the relationship between (v)h-torsion tensor and scalar curvature is given by (31).*

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