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ON THE SPECIAL IDEALS IN KK-ALGEBRAS

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Abstract: In this paper, we define the notions of q-ideal, a-ideal and p-ideal in KK-algebras. We give several characterizations and the extensive theorems for the q-ideal, a-ideal and p-ideal.

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1. Introduction

In [2], S. Asawasamrit and A. Sudprasert introduced a new algebraic structure which is called KK-algebras. And we described the relation between ideals and congruences. Furthermore, we defined quotient KK-algebra and studied its properties. In addition, L.L.Yong and M.Jie [5] introduced the notion of q-ideals and a-ideals in BCI-algebras. They gave several characterizations and the extensive theorems about q-ideals and a-ideals.

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In this paper, we define the notion of q-ideals and a-ideals in KK-algebras and investigated some related properties. The purpose of this paper is to derive some straightforward consequences of the relations between q-ideals, a-ideals and p-ideal. We also investigate some of its properties.

2. Preliminaries

In this section we introduced an algebraic structure called a *KK-algebra* which is an algebra $(X, *, 0)$ with a binary operation $*$ and a nullary operation 0 such that for all $x, y, z \in X$, satisfies the following properties:

(KK-1) $(x * y) * ((y * z) * (x * z)) = 0$;

(KK-2) $0 * x = x$;

(KK-3) $x * y = 0$ and $y * x = 0$ if and only if $x = y$.

On KK-algebra $(X, *, 0)$, we defined a binary relation $\leq$ on X by putting $x \leq y$ if and only if $y * x = 0$. Then $(X, \leq)$ is a partially ordered set. It is easy to show that the following properties are true for a KK-algebra. For any x, y, z in X :

(P-1) $x * ((x * y) * y) = 0$;

(P-2) $x * x = 0$;

(P-3) $x * (y * z) = y * (x * z)$;

(P-4) $((x * y) * y) * y = x * y$;

(P-5) $(x * y) * 0 = (x * 0) * (y * 0)$;

(P-6) $(x * y) * ((z * x) * (z * y)) = 0$;

(P-7) $x \leq y$ implies $y * z \leq x * z$;

(P-8) $x \leq y$ implies $z * x \leq z * y$.

A subset A of a KK-algebra X is called *closed* of X if $x * y \in A$ whenever $x, y \in A$. A non-empty subset A of a KK-algebra X is called an *ideal* of X if it satisfies the following conditions:

(I-1) $0 \in A$

(I-2) for any $x, y \in X$, $x * y \in A$ and $x \in A$ imply $y \in A$.

Let I be an ideal of KK-algebra X. Define the relation $\sim$ on X by $x \sim y$ if and only if $x * y \in I$ and $y * x \in I$. Then the relation $\sim$ is an equivalence relation on X and $[0]_I = \{x \in X \mid x \sim 0\}$ is an ideal of X.

Let $\sim$ be an equivalence relation on a KK-algebra X and I be an ideal of X. Define $[x]_I$ by $[x]_I = \{y \in X \mid x \sim y\} = \{y \in X \mid x * y \in I, y * x \in I\}$. Then the family $\{[x]_I | x \in X\}$ gives a partition of X which is denoted by X/I. For any $x, y \in X$, we defined $[x]_I \circ [y]_I = [x * y]_I$, then the binary operation $\circ$ is a mapping from $X/I \times X/I$ to X/I. It is easily checked that $(X/I, \circ, [0]_I)$ is a KK-algebra. Moreover, the set X/I is called the *quotient KK-algebra.* And if I is a closed ideal of KK-algebra X, then it is clear that $[a]_I = I$ for all a in I.

3. q-Ideals and Its Properties

In this section, we describe properties of q-ideals.

Definition 3.1. A non-empty subset I of KK-algebras X is said to be a *q-ideal* of X if it satisfies the following conditions (I-1) and
(I-3) for any $x, y, z \in X$, $(x * y) * z \in I$ and $y \in I$ imply $x * z \in I$.

First, give example shows that the q-ideal of X exists.

Example 3.2. Let $X = \{0, 1, 2\}$. Define an operation $*$ on X with the Cayley table given by

$*$	0	1	2
0	0	1	2
1	0	0	2
2	2	2	0

Then the Cayley table clearly $(X, *, 0)$ is a KK-algebras and it is easily checked that $I = \{0, 1\}$ is a q-ideal. □

Next, we give the relations between q-ideal and ideal and closed are considered as the following theorem.

Theorem 3.3. *A q-ideal is a closed ideal.*

Proof. Suppose that I is an q-ideal of KK-algebra and let $x * y \in I$ and $x \in I$. It follows that $(0 * x) * y \in I$ imply $0 * y \in I$. Thus, $y \in I$, proving (I-2) holds. Combining (I-1), we conclude that I is an ideal of X. □

Next, I will show example the converse of theorem 3.3 is not true.

Example 3.4. Let $X = \{0, 1, 2, 3\}$. Define an operation $*$ on X with the Cayley table given by

$*$	0	1	2	3
0	0	1	2	3
1	3	0	1	2
2	2	3	0	1
3	1	2	3	0

Then it is easily checked that $(X, *, 0)$ is a KK-algebra and $I = \{0\}$ is a closed ideal of X, but not a q-ideal of X. Since $(3 * 0) * 1 = 1 * 1 = 0 \in \{0\}$ and $0 \in \{0\}$ but $3 * 1 = 2 \notin \{0\}$. □

Now, we investigate the characterization of q-ideal.

Theorem 3.5. *If I is an ideal of KK-algebras X, then the following are equivalent:*

(1) I is an q-ideal of X;

(2) for any $x, y \in X$, $(x * 0) * y \in I$ implies $x * y \in I$;

(3) for any $x, y, z \in X$, $(x * y) * z \in I$ implies $x * (y * z) \in I$.

Proof. Assume that I is an ideal of KK-algebra X and $x, y, z \in X$.

(1) $\Rightarrow$ (2) Let I be an q-ideal of X and $(x*0)*y) \in I$. Since $0 \in I$, by (I-3), $x * y \in I$.

(2) $\Rightarrow$ (3) Suppose that (2) holds and $(x*y)*z \in I$. We see that $(x*y)*z)*((x*0)*(y*z)) \leq (x*0)*(y*(x*y)) = (x*0)*(x*(y*y)) = (x*0)*(x*0) = 0 \in I$. Since I is an ideal of X, so $((x * 0) * (y * z)) \in I$. By (2), so $x * (y * z) \in I$.

(3) $\Rightarrow$ (1) Let $(x*y)*z \in I$ and $y \in I$. From (3), we obtain that $x*(y*z) \in I$. Thus $y * (x * z) \in I$ by (P-3). Since $y \in I$ and I is an ideal, hence $x * z \in I$, proving that I is a q-ideal of X. □

Theorem 3.6. *Let A and I be ideals of a KK-algebra X with $I \subseteq A$. If I is a q-deal of X, then so is A.*

Proof. Let I is a q-deal of a KK-algebra X and set $s = (x*0)*y \in A$. Since $(x * 0) * (s * y) = s * ((x * 0) * y) = 0 \in I$. By 3.5(2), we get that $x * (s * y) \in I$. And since I is a $q-$ideal, then $s * (x * y) \in I$. Thus $s * (x * y) \in A$ and A is an ideal, so $x * y \in A$. Therefore A is a q-ideal. □

Corollary 3.7. *If zero ideal $\{0\}$ of KK-algebra X is a q-ideal, then every ideal of X is a q-ideal.*

Theorem 3.8. *Let I be an ideal of KK-algebra X. If for any $x \in I$ and $y \in X$ imply $x * y \in I$, then I is q-ideal of X.*

Proof. Assume that $(x * y) * z \in I$ and $y \in I$. By hypothesis, we obtain $x * ((x * z) * z) \in I$ and $x * y \in I$. Then $(x * y) * (x * z) \in I$, since I is an ideal of X, so $x * z \in I$, proving that I is $q-$ideal of X. □

Lemma 3.9. *If I is a q-ideal of KK-algebra X, then $x * (x * 0) \in I$ for all $x \in X$.*

Proof. Assume that I is a q-ideal of KK-algebra X. Since $(x * 0) * (x * 0) = 0 \in I$, then it follows that $x * (x * 0) \in I$ because theorem 3.5(2). □

4. a-Ideals and Its Properties

Definition 4.1. A non-empty subset I of KK-algebra X is called an *a-ideal* of X if it satisfies the following conditions (I-1) and
(I-4) for all $x, y, z \in X$, $(x * 0) * (z * y) \in I$ and $z \in I$ imply $y * x \in I$.

Example 4.2. Let $X = \{0, 1, 2, 3\}$. Define an operation $*$ on X with the Cayley table given by

*	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Then it is easily checked that $(X, *, 0)$ is a KK-algebras and $I = \{0, 1\}$ an a-ideal of X. □

The following theorem show the relations between a-ideals and ideals and between a-ideals and closed.

Theorem 4.3. *If I is an a-ideal of KK-algebra X, then I is a closed ideal.*

Proof. Let I be an a-deal of X. First, we will show that I is an ideal of X. Assume that $x * y \in I$ and $x \in I$. It follows that $(0 * 0) * (x * y) \in I$ and $x \in I$, by (I-4), we obtain $y * 0 \in I$. Substituting $x = 0 = z$ in (I-4), we get that $y \in I$ and $0 \in I$, then $y * 0 \in I$. So, it follows that $(y * 0) * 0 \in I$. Putting $y = z = 0$ in (I-4), it follows that if $(x * 0) * (0 * 0) \in I$ and $0 \in I$, then $0 * x \in I$. Now $(x * 0) * 0 \in I$, implies that $x \in I$. Since $(y * 0) * 0 \in I$, so $y \in I$. Proving that

I is an ideal of X. Finally, to show that I is a closed. Now, assume that $x \in I$ and $y \in I$. We see that $x * 0 \in I$ and $y * 0 \in I$. Since $x * (y * x) = y * 0 \in I$ and $x \in I$, then $y * x \in I$. Similarly, $x * y \in I$. Therefore I is a closed, proving our theorem. □

The following theorem gives us some equivalences of a-ideals.

Theorem 4.4. *Let I be an ideal of KK-algebra X. The following conditions are equivalent:*

(1) I is an a-ideal of X;

(2) $(x * 0) * (z * y) \in I$ implies $(z * y) * x \in I$, for any $x, y, z \in X$;

(3) $(x * 0) * y \in I$ implies $y * x \in I$ for any $x, y \in X$.

Proof. Let I be an ideal of KK-algebra X.

(1) ⇒ (2), Assume that I is an a-ideal of X and set $s = (x * 0) * (z * y) \in I$. We can write $(x * 0) * (s * (z * y)) = s * ((x * 0) * (z * y)) = 0 \in I$. By (I-4) and $s \in I$, thus $(z * y) * x \in I$. Proving that (2) holds.

(2) ⇒ (3), Putting $z = 0$ in (2), we obtain (3).

(3) ⇒ (1), Let $(x * 0) * (z * y) \in I$ and $z \in I$. We see that $((x * 0) * (z * y)) * ((x * 0) * y) \leq (z * y) * y \leq z \in I$. Since I is an ideal of X, thus $((x*0)*(z*y))*((x*0)*y) \in I$, and imply that $(x*0)*y \in I$. By (3), $y*x \in I$, and so I is an a-ideal of X. □

Definition 4.5. An ideal I of KK-algebra X is called an *p-ideal* of X if it satisfies $(x * 0) * 0 \in I$ implies $x \in I$.

Next, we give the relations between a-ideal and p-ideal are considered as the following theorem.

Theorem 4.6. *Any a-ideal of KK-algebra is a p-ideal.*

Proof. Suppose that I is an a-ideal of KK-algebras. By theorem 4.3, it follows that I is an ideal. Putting $y = z = 0$ in theorem 4.2(2), so $(x*0)*(0*0) \in I$ implies $(x * 0) * 0 \in I$. By theorem 4.4(2), we get that $(x * 0) * 0 \in I$ implies $x \in I$. Therefore I is an q-ideal. □

Next, I will show example the converse of theorem 4.6 is not true.

Example 4.7. Let $X = \{0, 1, 2, 3\}$. Define an operation $*$ on X with the Cayley table given by

*	0	1	2
0	0	1	2
1	2	0	1
2	1	2	0

Then it is easily checked that $(X, *, 0)$ is a KK-algebras and $I = \{0\}$ is a p-ideal of X, but it is not a $a-$ ideal of X. Since $(2*0)*(0*1) = 1*1 = 0 \in \{0\}$ and $0 \in \{0\}$ but $1 * 2 = 1 \notin \{0\}$. The proof is complete. □

Theorem 4.8. *Any a-ideal of KK-algebra is a q-ideal.*

Proof. Suppose that I is an a-ideal of KK-algebras. It follows that I is an ideal. Now, let $(x * 0) * y \in I$. We obtain that

$$\begin{aligned}((x*0)*y)*((((y*0)*x)*0)*0) &= ((x*0)*y)*((((y*0)*0)*(x*0))*0)\\ &=((x*0)*y)*((((y*0)*0)*0)*((x*0)*0))\\ &=((x*0)*y)*((y*0)*((x*0)*0))\\ &\leq((x*0)*y)*((x*0)*y)\\ &=0 \in I.\end{aligned}$$

Since I is an ideal, so $(((y * 0) * x) * 0) * 0 \in I$. It follows that I is a p-ideal as theorem 4.6. Then $(y * 0) * x \in I$ and by theorem 4.4(3), we get $x * y \in I$. The proof is complete. □

The converse of theorem 4.8 is not true. From example 3.2 we can show that $I = \{0\}$ is not a-ideal of X. And it easy to show $I = \{0\}$ is a q-ideal. The following from theorem 4.8, that if I is an a-ideals, then it is a q-ideals. For the converse part we need the condition that I is an p-ideals as follows.

Theorem 4.9. *A non-empty subset I of KK-algebra X is an a-ideal if and only if it is both a q-ideal and a p-ideal.*

Proof. Let I be an a-deal of X. It is clear that I is both a q-ideal and a p-ideal because theorem 4.6 and theorem 4.8.

On the other hand, suppose that I is both a q-ideal and a p-ideal. It follows that I is a closed. Now, assume that $(x*0)*y \in I$. By theorem 3.5(2), we have that $x * (0 * y) \in I$, and hence $x * y \in I$. Consider the equation

$$\begin{aligned}(x*y)*[(y*0)*(x*0)] &= (y*0)*[(x*y)*(x*0)]\\ &= (y*0)*[x*((x*y)*0)]\\ &= (y*0)*[x*((x*0)*(y*0)]\end{aligned}$$

$$\begin{aligned} &= x * [(y * 0) * ((x * 0) * (y * 0)] \\ &= x * [(x * 0) * ((y * 0) * (y * 0)] \\ &= x * [(x * 0) * 0] \\ &= 0 \in I. \end{aligned}$$

From I is an ideal and $x * y \in I$, implies $(y * x) * 0 \in I$. It follows that $((y*x)*0)*0 \in I$ because I is a closed. And since I is a p-ideal, then $y*x \in I$, and so I is an a-ideal. □

The extensive theorem of a-ideal was given by the following theorem.

Theorem 4.10. *Let A and I be two ideals of KK-algebra X with $I \subseteq A$. If I is an a-ideal of X, then so is A.*

Proof. Let I be an a-ideal of X, then I is both a q-ideal and a p-ideal. Now, we need to show A is a q-ideal and a p-ideal of X. Assume that $s = (x*0)*y \in A$, then we have $(x * 0) * (s * y) = s * ((x * 0) * y) = 0 \in I$. By theorem 3.5(2), $x*(s*y) \in I$, and it follows $s*(x*y) \in I$. Thus $s*(x*y) \in A$ and since A is an ideal, so $x*y \in A$. Therefore A is a q-ideal of X. Finally, to show that A is a p-ideal of X and let $t = (x*0)*0 \in A$. Then $(x*0)*(t*0) = t*((x*0)*0) = 0 \in I$, and we can write

$$\begin{aligned} ((x * 0) * (t * 0)) * (((t * x) * 0) * 0) &= ((x * 0) * (t * 0)) * (((t * 0) * 0) \\ &\quad * ((x * 0) * 0)) \\ &\leq ((x * 0) * (t * 0)) * ((x * 0) * (t * 0)) \\ &= 0 \in I. \end{aligned}$$

Since I is an ideal, we get that $((t*x)*0)*0 \in I$. And since I is a p-ideal, thus $t * x \in I \subseteq A$, and so $x \in A$ as A is an ideal. Hence A is a p-ideal of X. The proof is complete. □

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