International Journal of Pure and Applied Mathematics
Volume 82 No. 5 2013, 825-828
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v82i5.11
ijpam.eu

COMPOSED PENCILS ON A SMOOTH CURVE WITH A SINGULAR MODEL IN A QUADRIC SURFACE

E. Ballico
Department of Mathematics
University of Trento
38 123 Povo (Trento) - Via Sommarive, 14, ITALY

Abstract: Let C be the normalization of an integral curve of type (a, a') on $\mathbb{P}^1 \times \mathbb{P}^1$. We give conditions on $\mathrm{Sing}(Y)$ and y for the non-existence of a pencil on C partially composed with the g^1_a or the $g^1_{a'}$ obtained in C from the projections $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$.

AMS Subject Classification: 14H51, 14H50

1. Introduction

Let $Q \subset \mathbb{P}^3$ be a smooth quadric surface. Let $\pi : Q \to \mathbb{P}^1$ and $\pi' : Q \to \mathbb{P}^1$ be the two projections. We have $\mathrm{Pic}(Q) \cong \mathbb{N}^2$. We have $h^0(\mathcal{O}_Q(a, b)) = (a + 1)(b + 1)$ and $h^1(Q, \mathcal{O}_Q(a, b)) = 0$ if $a \geq -1$ and $b \geq -1$. A curve $|\mathcal{O}_Q(a, b)|$ is said to have type (a, b). The lines of Q are the curve with either type $(1, 0)$ or type $(0, 1)$. We use the convention that the fibers of π have type $(0, 1)$, while the fibers of π' have type $(1, 0)$. Fix an integral $Y \in |\mathcal{O}_Q(a, a')|$. Let $w : C \to Y$ be the normalization map. Let $m : C \to \mathbb{P}^1$ (resp. $m' : C \to \mathbb{P}^1$) be the composition of w with $\pi|Y$ (resp. $\pi'|Y$). We have $\deg(m) = a$ and $\deg(m') = a'$. Set $R := m^*(\mathcal{O}_{\mathbb{P}^1}(1))$

Received: December 18, 2012

and $R' := m'^{*}(\mathcal{O}_{\mathbb{P}^1}(1))$. R and R' are spanned line bundles of degree a and a', respectively. Under mild assumptions on Y we have $h^0(R) = h^0(R') = 2$ (see Remark 1). In this note we consider the following classical question. Under which assumptions on Y there is no spanned $M \in \mathrm{Pic}(C)$, $M \neq R$ and $M \neq R'$, such that $h^0(C, M) = 2$ and the morphism $\phi : C \to \mathbb{P}^1$ induced by $|M|$ is composed either with m or with m', i.e. one of the two maps (m, ϕ) or (m', ϕ) from C into $\mathbb{P}^1 \times \mathbb{P}^1$ is not birational onto its image? Assume that M exists and let D be the normalization of either (m, ϕ) or (m', ϕ). Set $y := \deg(M)$. The map (m, ϕ) (or (m', ϕ) induces a non-constant map $\beta : C \to D$ and (m, ϕ) (resp. (m', ϕ)) factor though w. Set $b := \deg(\beta)$. We have $b|y$, $b < y$; we have $b|a$ and $b < a$ (resp. $b|a'$ and $b < a'$). Since $M \notin \{R, R'\}$, C has genus $g \geq 2$ and D has positive genus. In this note we prove the following result.

Theorem 1. *Assume the existence of a spanned $M \in \mathrm{Pic}^y(C)$, $M \neq R$ (resp. $M \neq R'$) such that ϕ is composed with m (resp. m'), i.e. the map (m, ϕ) (resp. (m', ϕ)) has degree $b \geq 2$ onto its image. We have $b|y$ and $b < y$. We have $b|a$ and $b < a$ (resp. $b|a'$ and $b < a'$). Let $\mathcal{J}$ the conductor of C, seen as an ideal sheaf of Q, with $\mathcal{O}_Z := \mathcal{O}_Q/\mathcal{J}$. Then none of these conditions is satisfied:*

1. *$h^1(Q, \mathcal{J}(a-2, a'-2-y/b)) = 0$ (resp. $h^1(Q, \mathcal{J}(a-2-y/b, a'-2)) = 0$);*

2. *Y has only ordinary nodes and ordinary cusps as singularities, $\mathrm{Sing}(Y)$ is formed by general points of Q and $\sharp(\mathrm{Sing}(Y)) \leq (a-1)(a'-1-y/b)$ (resp. $\sharp(\mathrm{Sing}(Y)) \leq (a-1-y/b)(a'-1)$);*

3. *assume $a' \geq 2 + y/b$ (resp. $a \geq 2 + y/b$; set $v := \max\{a-2, a'-2-y/b\}$, $u := \min\{a-2, a'-2-y/b\}$ (resp. $v := \max\{a-2-y/b, a'-2\}$ and $u := \min\{a-2-y/b, a'-2\}$). Set $\alpha := \lfloor u/3 \rfloor$. Y has only ordinary nodes and ordinary cusps as singularities, no two of the points of $\mathrm{Sing}(Y)$ are contained in a line of Q, at most $u+v$ of the points of Z are contained in a curve of type $(1,1)$ and at most $3u+1$ of the points of Z are contained in a curve of type $(2,1)$ or $(1,2)$ and $\sharp(\mathrm{Sing}(Y)) \leq v - u + 10\alpha - 1$;*

4. *Y has only ordinary nodes and ordinary cusps as singularities, $a' \geq 2+y/b$ (resp. $a \geq 2 + y/b$) and $\sharp(\mathrm{Sing}(Y)) \leq \min\{a-1, a'-1-y/b\}$ ((resp. $\sharp(\mathrm{Sing}(Y)) \leq \min\{a-1-y/b, a'-1\}$).*

We work over an algebraically closed field $\mathbb{K}$.

2. The Proof

Let $Z \subset Q$ be a zero-dimensional scheme. Let Δ_Z be the union of all lines $L \subset Q$ such that $L \cap Z = \emptyset$. Notice that Δ_Z is a finite union of lines. This is a fundamental difference between Q and $\mathbb{P}^2$.

Lemma 1. *Fix $(x, v) \in \mathbb{N}^2$ and the ideal sheaf $\mathcal{J}$ of a zero-dimensional scheme Z such that $h^1(\mathcal{J}(u, v)) = 0$. Fix a set $B \subset Q$ such that $B \cap \Delta_Z = \emptyset$ and there is an integer $b > 0$ such that for each $I \in |\mathcal{O}_Q(0, 1)|$ either $I \cap B = \emptyset$ or $\sharp(I \cap B) = b$. Set $y := \sharp(B)$. Assume $b \leq u + 1$. Then $Z \cap B = \emptyset$, $b|y$ and $h^1(\mathcal{I}_{\mathcal{J} \cup B}(u, v + y/b)) = 0$.*

Proof. Since $Z \subseteq \Delta_Z$ and $B \cap \Delta_Z = \emptyset$, we have $B \cap Z = \emptyset$. By assumption there is a curve $F \in |\mathcal{O}_Q(0, y/b)|$, F union of y/b distinct lines such that $B \subset F$ and $\sharp(F \cap I) = b$ for each connected component I of F. Since $B \cap \Delta_Z = \emptyset$, we have $Z \cap F = \emptyset$. Hence $\deg((Z \cup B) \cap I) = b$ for each component I of F. Since $b \leq u+1$, we get $h^1(F, \mathcal{I}_{Z\cup B}(u, v+y/b)) = 0$. Since $Z = (Z \cup B) \setminus (Z \cup B) \cap F$. Use the exact sequence

$$0 \to \mathcal{I}_Z(u, v) \to \mathcal{I}_{Z\cup B}(u, v + y/b) \to \mathcal{I}_{(Z\cup B)\cap F, F}(u, v + y/b) \to 0$$

and the assumption $h^1(\mathcal{J}(u, v)) = 0$. □

Remark 1. The adjunction formula gives that $h^0(R) = 2$ (resp. $h^0(R') = 2$) if and only if $h^1(\mathcal{J}(a - 2, a' - 3)) = 0$ (resp. $h^1(\mathcal{J}(a - 3, a' - 2)) = 0$). Just the existence of a reduced curve $Y \in |\mathcal{O}_Q(a, a')|$ implies $h^1(\mathcal{J}(a-2, a'-2)) = 0$.

Proof of Theorem 1. We first assume that $|M|$ is composed with the g^1_a on C induced by π. Fix a general $A \in |M|$ and set $B := w(A) \subset Y$. Since $|M|$ is a complete linear system, even in positive characteristic we see that ϕ is not composed with a Frobenius. Hence ϕ is separable. Since ϕ is separable and A is general in $|M|$, A is formed by y distinct points. Since $|M|$ is spanned and A is general, we have $A \cap w^{-1}(\mathrm{Sing}(Y)) = \emptyset$. Hence $B \subset Y$, $B \cap Z = \emptyset$ and $\sharp(B) = y$. Since $|M|$ has no base point, Δ_Z is a finite union of lines and A is general, we have $B \cap \Delta_Z = \emptyset$. Fix $O \in A$. Since M has no base points, we have $h^0(C, \mathcal{O}_C(A \setminus \{O\})) = h^0(C, \mathcal{O}_C(A)) - 1$, i.e. $h^0(C, \omega_C(-A)) = h^0(C, \omega_C(-A)(O))$ (Riemann-Roch and Serre duality). The adjunction formula gives $\omega_Y \cong \mathcal{O}_Y(a - 2, a' - 2)$. Since $h^i(\mathcal{O}_Q(-2, -2)) = 0$, $i = 0, 1$, we get that $|\omega_C|$ is induced by the linear system $|\mathcal{J}(a - 2, a' - 2)|$ on Q. Since $A \cap w^{-1}(\mathrm{Sing}(Y)) = \emptyset$, we get $h^1(C, \omega_C(-A)) = h^1(\mathcal{I}_{Z\cup B}(a - 2, a' - 2))$. Hence $h^1(\mathcal{I}_{Z\cup B}(a - 2, a' - 2)) > 0$ Since $h^1(\mathcal{I}_{Z\cup B}(a - 2, a' - 2)) > 0$, lemma 1 gives a contradiction.

Now assume that $|R|$ is composed with the $g^1_{a'}$ induced by π'. We conclude as above taking a' instead of a.

Hence we proved Theorem 1 under the assumption (1). We only need to prove that in the remaining cases the assumption of (1) is satisfied. If Y has only ordinary nodes and ordinary cusps, then Z is the set $\mathrm{Sing}(Y)$ with its reduced structure. In case (2) uses the definition of " general set ". In case (3) use [1], lemma 2. The proof that (4) implies (1) is an easy exercise. □

Acknowledgements

The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

References

[1] E. Ballico, On the gonality sequence of smooth curves: normalizations of singular curves in a quadric surface, *Preprint*.