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MINIMAL IDEALS OF ABEL-GRASSMANN GROUPOIDS

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Abstract: The properties of Abel-Grassmann groupoids have been attracted the attention of many authors. The aim of this paper is to study the properties of the minimal left ideals of an Abel-Grassmann groupoid (in brevity, an AG-groupoid) with left identity. It is proved that if L is a minimal left ideal of an AG-groupoid S with left identity then Lc is a minimal left ideal of S for all $c \in S$. We also show that the kernel K of an AG-groupoid S (the intersection of all two sided ideals of S if exists)is simple and the class sum Σ of all minimal left ideals of S containing at least one minimal left ideal of S is precisly the kernel K of S. Finally, we show that if S is an AG-groupoid with left identity then $Sa^2S = Sa^2$ for all $a \in S$. Finally, if S is an AG-groupoid with left identity and does not contain any non-trivial nilpotent ideals, then every minimal ideal of S is simple.A number of classical results of L. M. Gluskin and O. Steinfeld given in 1978 [3] concerning the minimal one sided ideals of semigroups and rings are consequently extended to and strengthened in AG-groupoids.

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1. Introduction

Throughout this paper, unless otherwise state,we denote an AG-groupoid by S. In the literature, a left almost semigroup (in brevity, a LA-semigroup) is said to be an Abel-Gassmann groupoid (i.e. an AG-groupoid). By an AG-groupoid S, we mean a groupoid satisfying the following left invertive law

$$(ab)c = (cb)a, \quad \text{for all } a, b, c \in S. \tag{1}$$

The following medial law is always satisfied in S :

$$(ab)(cd) = (ac)(bd), \quad \text{for all } a, b, c, d \in S. \tag{2}$$

It is known that the left identity of a groupoid S if exists is unique [8].

In an AG-groupoid S with left identity, the following paramedial law holds.

$$(ab)(cd) = (db)(ca) \quad \text{holds, for all } a, b, c, d \in S. \tag{3}$$

We recall that an AG-groupoid is a non-associative algebraic structure which is mid way between a groupoid and a commutative semigroup. A groupoid satisfying the above equations (2) and (3) is called a medial and paramedial groupoid respectively, see [5] and [6].It is interesting to note that an AG-groupoid S with a left identity is a medial and paramedial groupoid. However, an AG-groupoid with a right identity will be a commutative semigroup with an identity [7]. We now call S a locally associative AG-groupoid if for all $a \in S$, the equality $(aa)a = a(aa)$ holds. The general properties of AG-grouoids have been studied by P. V. Protic and N. Stevanovic in [20] and [21]. We notice that the AG-groupoids and the left regular partial ordered AG-groupoids have also been investigated and discussed in [16],[19] and [17]. In an AG-groupoid S, the following equality holds.

$$(a(bc) = b(ac) \quad \text{for all } a, b, c \in S \tag{4}$$

We now call an $AG - groupoid$ an $AG^{**} - groupoid$.

The above groupoid was first studied by Q. Iqbal and Q. Mushtaq [15] that the equality

$$(ab)^n = a^n b^n$$

holds for all $a, b \in S$ in a locally associative AG-groupoid S. Also,it has been proved in [16] that if S is a locally associative $AG^{**} - groupoid$, then $a^n b^m = b^m a^n$ holds for all $a, b \in S$ and $m, n \geq 2$, while these identities hold in a commutative semigroup.

It is well known that the AG-groupoids have many applications in the structure theory of semigroups. In particular, P.Holgate [3] called an AG-groupoid a simple invertive groupoid. We observe that a commutative group $(S, \circ)$ will become an AG-groupoid $(S, \cdot)$ under the following relation $a \cdot b{=}b \circ a^{-1}$, see([7] and [8]). Clearly,an AG-groupoid $(S, \cdot)$ will become a semigroup under a new binary operation "$\circ$" defined on S. Thus, there exists an element $a \in S$ such that $x \circ y = (xa)y$ for all $x, y \in S$.

It is easy to show that the binary operation "$\circ$" is an associative operation and hence we have the following equality:

$$\begin{aligned}(x \circ y) \circ z =& (((xa)y)a)z = (za)((xa)y) = (xa)((za)y)\\ =& (xa)((ya)z) = x \circ (y \circ z).\end{aligned}$$

Obviously, the system "$(S, \circ)$" forms a semigroup under the binary operation "$\circ$". In connection with the above discussion, the following example of a non-associative system is an interesting and useful example of an AG-groupoid.

In the following, we consruct an example of an AG-groupoid with a left identity.

Example 1.1. Let $S = \{1, 2, 3, 4, 5\}$ be a set with the binary operation "$\cdot$" defined on S as follows:

$\cdot$	1	2	3	4	5
1	1	2	3	4	5
2	5	1	2	3	4
3	4	5	1	2	3
4	3	4	5	1	2
5	2	3	4	5	1

Then we can easily verify that the system "$(S, \cdot)$" forms an AG-groupoid with a left identity 1.

We now call a subset I of S a left (right) ideal of S if $SI \subseteq I$ and $(IS \subseteq I)$, and a left (right) ideal of S is said to be a minimal left (right) ideal of S if I does not contain any other left (right) ideal other than itself.

According to D. Rees [22], the kernel K of S may be described as the intersection of all two sided ideals of S. Later on, A. H. Clifford [2] noticed that if M is a minimal ideal of a groupoid S then M is the kernel of S. Moreover, K is a simple sub-groupoid of S. We describe here the non-associative structure of S. We shall call an AG-groupoid S left (right) simple if S does not contain any proper left (right) ideals of S. If S is an AG-groupoid with left identity then Sa is a principal left ideal of S generated by a for all $a \in S$, and we can also show that Sa is an ideal of S for all $a \in E(S)$, where $E(S)$ is the set of all the idempotents of S.

In the following example, we construct an AG-groupoid S with a kernel K of S.

Example 1.2. Let $S = \{1, 2, 3, 4, 5, 6\}$ be a set on which a binary operation "·" defined as follows:

·	1	2	3	4	5	6
1	3	3	3	3	3	3
2	3	3	3	3	3	3
3	3	3	3	3	3	3
4	3	3	3	3	3	3
5	3	3	3	3	3	3
6	3	2	3	3	3	3

It is celar to see that the above system $(S, \cdot)$ forms an AG-groupoid S. Obviously, $\{2, 3\}$ is an ideal of S and $\{3\}$ is a minimal ideal of S and is also the kernel of S.

2. Minimal Ideals and the Kernel of an *AG*-Groupoid

The left and right minimal ideals of a semigroup and a ring have been extensively studied and investigated by L. M. Gluskin and O. Steinfeld in 1978, see [3]. We now begins with several lemmas of minimal ideals of an AG-groupoid S.

Lemma 2.1. *Let S be an AG-groupoid with left identity containing a minimal ideal M. Then M is the kernel of S.*

Proof. Suppose that A is an ideal of S. Then $AM \subseteq A \cap M$ implies $A \cap M$ is non empty and so $AM \subseteq M$ and also $AM \subseteq A$ but since M is minimal,$M = AM \subseteq A$. This implies $M \subseteq A$ and M is hence contained in every ideal of S, that is, M is the kernel of S. □

Lemma 2.2. *Let S be an AG-groupoid with a left identity. Then the kernel K of S is a simple AG-groupoid.*

Proof. Let K be the kernel of S with left identity. If A is an ideal of K, then $((KA)K)^2$ is obvious an ideal of S contained in K, but since K is the intersection of all the ideals of S, $K \subseteq ((KA)K)^2$. This implies $((KA)K)^2 = K$

and since $((KA)K)^2 \subseteq A \subseteq K$, $A = K$. This proves that K is a simple AG-groupoid. □

We state below a crucial Lemma of a minimal left ideal of an AG-groupoid.

Lemma 2.3. *Let S be an AG-groupoid with left identity and L a minimal left ideal of S. Then Lc is a minimal left ideal of S, for all $c \in S$.*

Proof. Let L be a minimal left ideal of S with left identity. Then, it is trivial to see that Lc is a left ideal of S for all $c \in S$, since $S(Lc) = (Se)(Lc) = (SL)(ec) \subseteq Lc$. Now, let M be a minimal left ideal of S contained in Lc. If $L_1 \subseteq L$ is the set of all elements l_1 of S such that $l_1c \in M$ and $M = L_1c$, then we see immediately that $S(l_1c) \subseteq M$. This implies that $S(l_1c) \subseteq L_1c$ and we further deduce that $(Sl_1)(ec) \subseteq L_1c$, for every $l_1 \in L_1$, that is $(SL_1)c \subseteq L_1c$. This is clearly a contradiction. Thus, Lc is a minimal left ideal of S. □

In corresponding to Lemma 2.3, we are able to give the following Lemma.

Lemma 2.4. *Let S be an AG-groupoid with left identity containing a zero element 0. If L is a minimal left ideal of S and c is an arbitray element of S then either Lc is a minimal left ideal of S or $Lc = \{0\}$.*

Proof. Let $L \neq \{0\}$ be a minimal left ideal of S. Then, Lc is clearly a minimal left ideal of S, for all $c \in S$. The proof of this Lemma is similar to Lemma 2.3. Hence Lc is a minimal left ideal of S and $Lc \neq \{0\}$. □

3. 0-Minimal Ideals and the Class Sum of Left Minimal Ideals

In this section, we call an ideal of an Abel-Grassmann groupoid S 0-minimal if it is minimal in the set of all non-zero ideals of S. The set union of all minimal left ideals of S is called the class sum of all minimal left ideals of S and is denoted by Σ.

For o-minimal left ideal of an AG-groupoid S. We have the following Lemma.

Lemma 3.1. *Any member in the set union LS of all 0-minimal left ideals of S is of the form Lx for some $x \in S$.*

For 0-minimal left ideals of an AG-groupoid S, we futher have the following lemmas.

Lemma 3.2. *Let L' be any 0-minimal left ideal of S contained in LS. Then there exists some x in S such that $L' = Lx$.*

Proof. Since every left ideal of LS is of the form Lx for some $x \in S$ and all the 0-minimal left ideals of S are 0-disjoint. Let L' be some 0-minimal left ideal of S contained in LS. Then, there exists some x in S such that $L' = Lx$. □

Lemma 3.3. *If $x \in LS \backslash \{0\}$, then Sx is an 0-minimal left ideal of S contained in LS and $Sx = Ly$ for $y \in S$.*

Proof. By Lemma 2.4, we deduce immediately that LS is the set union of all 0-minimal left ideals of S which are of the form Ly for any $y \in S$. Now let $x \in LS \backslash \{0\}$, then Ly is in LS such that $x \in Ly$ and the equation $ly = x$ has a solution in LS for $y \in S$ and $l \in L \backslash \{0\}$. Thus $S(ly) = Sx$ which implies $(Sl)y = Sx$, and since Sl is a left ideal of S contained in L and L is an 0-minimal left ideal of S. Thus, $Sl = L$. This implies $Ly = Sx$ for $y \in S$. Since Ly is an 0-minimal left ideal of S, Sx is 0-minimal left ideal of S. □

The following lemmas are useful lemmas for minimal ideals of an *AG*-groupoid S with left identity.

Lemma 3.4. *Let M be an 0-minimal ideal of S. Then either M is an 0-simple AG-subgroupoid or $M^2 = \{0\}$.*

Proof. The proof of the above lemma is exactly the same as in [18]. □

Lemma 3.5. *Let L be an 0-minimal left ideal of S containing a zero element 0 and $L^2 \neq \{0\}$. Then $L = Sa$, for all $a \in L \backslash \{0\}$.*

Proof. Suppose that L is an 0-minimal left ideal of S and $L^2 \neq \{0\}$. Now let a be any non-zero element of L. Then $Sa \subseteq L$ is a left ideal of S. Hence, either $Sa = \{0\}$ or $L = Sa$, for all $a \in L \backslash \{0\}$.Suppose that $Sa = \{0\}$. Then, $a = 0$. This is clearly impossible. Thus, $L = Sa$ for all $a \in L \backslash \{0\}$. □

For the o-minimal left ideals of the groupoid S, we deduce the following corollary.

Corollary 3.6. *If L is an 0-minimal left ideal of S containing an idempotent of S,then L is an ideal of S and $L = Sa$, for all $a \in E(L)$.*

Proof. Suppose that $a \neq 0$ belongs to L and is also an idempotent. Then Sa is an ideal of S and $Sa \subseteq L$ but since L is 0-minimal left ideal of S, we see either $L = Sa$ or $Sa = \{0\}$ for all $a \in L \backslash \{0\}$. If $Sa = \{0\}$ then $a = 0$, contradicts to our hypothesis. Thus, $L = Sa$. □

We now consider the minimal ideals of an AG-groupoid S.

We start with the following Lemma.

Lemma 3.7. *Let S be an AG-groupoid with left identity and A a two sided ideal of S containing every minimal left ideal of S.Then $L \subseteq A$ for any minimal left ideal L of S.*

Proof. Let L be a minimal left ideal of S. If A is a two sided ideal of S, then, $LA \subseteq A, AL \subseteq A$ and $AL \subseteq L$. Obviously, AL is an left ideal of S contained in A and L. But since L is a minimal left ideal of S, $AL = L$. Hence, we have shown that $L \subseteq A$. □

For the class sum of all minimal left ideals of S, we have the following theorem.

Theorem 3.8. *Let S be an AG-groupoid with left identity. If S contains at least one minimal left ideal and Σ is the class sum of all minimal left ideals of S, then Σ is the kernel of S.*

Proof. Suppose that Σ is the class sum of all minimal left ideals of S. Then, Σ is an left ideal of S. Since by hypothesis, $\Sigma \neq \phi$ and so for every $c \in S$, Σc is also a left ideal of S, and for each $k \in \Sigma$ there exists some minimal left ideal L of S such that $k \in L$ and $kc \in Lc\ \forall c \in S$, where Lc is a minimal left ideal of S. Hence, $kc \in \Sigma$ which implies $(kc)S \subseteq \Sigma$. This leads to $\Sigma S \subseteq \Sigma$. Thus, Σ is a right ideal of S and hence a two sided ideal of S. By the above Lemma,Σ is contained in every ideal of S and therefore Σ itself is an ideal of S. We hence conclude that Σ is contained in every ideal of S. Thus, Σ is contained in the intersection of all ideals of S. In other words, Σ is the kernel of S. □

For the left ideals of an AG-groupoid S, we have the following Lemmas.

Lemma 3.9. *Let S be an AG-groupoid. Then every left ideal of the kernel K of S is still a left ideal of S.*

Proof. Let K be the kernel of S and A a left ideal of K, that is $KA \subseteq A$. Since each element a of A belongs to some minimal left ideal L of S, Ka is a minimal left ideal of S contained in L for every $a \in A$ but since L is the minimal left ideal of S,$Ka = L$. This implies $a \in Ka$ and hence, we further deduce that $A \subseteq KA$. Thus $A = KA$ as required. □

Remark 3.10. Every minimal left ideal of S is also a minimal left ideal of K and vice versa.

4. Minimal Left Ideals

In this section, we consider the minimal left ideals of an *AG*-groupoid S.

We first give the following Lemma.

Lemma 4.1. *Every left ideal A of an AG-groupoid S contains at least one minimal left ideal of S.*

Proof. Let K be the class sum of all the minimal left ideals of S. If A is an left ideal of S, then by Lemma 2.2, KA is also a left ideal of S contained in K. But $KA = A$, we therefore conclude that $A \subseteq K$.This shows that A contains at least one left ideal of S. □

For the minimal left ideals of an *AG*-groupois S, We have the following theorem.

Theorem 4.2. *A minimal left ideal L of S is a left simple AG-sub-groupoid.*

Proof. Suppose that L is a minimal left ideal of S and $a \in L$. Then La is a minimal left ideal of S contained in L. But since L is minimal, $La = L$. Thus L is a left simple *AG* sub-groupoid. □

In the remaining part of this paper, we simply denote the minimal left ideal of S by L. For the minimal right ideal R of S,we have $RL = L$.

Lemma 4.3. *If $a \in R$ and $b \in RL$, then the equation $a^2x = b$ has a solution x in RL.*

Proof. Suppose that $a \in R$. Then, $a^2 \in R$. Now, it is obvious that a^2R is a right ideal of S contained in R. Since R is a minimal right ideal of S, $a^2R = R$. This implies $(a^2R)(eL) = RL$ and we further deduce that $(a^2e)(RL) = RL$, that is, $a^2(RL) = RL$. Thus the equation $a^2x = b$ has a solution x in RL for all $b \in RL$. □

Remark 4.4. If S is an *AG*-groupoid with a left identity e and R is a right ideal of S, then $Re = eR = R$.

Lemma 4.5. *If $a \in L$ and $b \in RL$ then the equation $xa = b$ has a solution x in RL for all $b \in RL$.*

Proof. Suppose that L is a minimal left ideal of S. Clearly, La is a minimal left ideal of S and if $a \in L$ then $La \subseteq L$. But since L is minimal, $La = L$ for all $a \in L$ which implies $R(La) = RL$, and thereby, $(RL)a = RL$. Thus, $xa = b$ has a solution x in RL for every $b \in RL$. □

Corollary 4.6. *If $a \in L$ and $b \in RL$, then the equation $xa^2 = b$ has a solution x in RL, for all $b \in RL$.*

The following Lemma is an easy consequence of the above Corollary.

Lemma 4.7. *Let S be a groupoid with left identity. If $a \in R \cap L$ and $b \in RL$. then the equations $a^2x = b$ and $ya^2 = b$ has solutions x, y in RL.*

5. Nil and Nipotent Ideals

We state below the defintion of nil-ideals in an *AG*-groupoid.

Definition 5.1. Let S be an *AG*-groupoid with left idetity. Then an ideal A(left or right) of S is said to be a nill-ideal of S if for each $a \in A$, there exists some $n \in N$ such that $a^n = \underbrace{(...(((aa)a)a)...)a}_{n-times} = 0$.

In the following theorem, we consider an *AG*-groupoid without nilpotent ideals.

Theorem 5.2. *Suppose that an AG-groupoid S does not contain any nilpotent ideals. Then every minimal ideal of S is simple.*

Proof. Let M be a minimal ideal of S. Suppose that B, properly contained in M, is an ideal of M. Then, $MBM \subseteq M$ is a left ideal of M. In this case, $(MBM)^2$ is clerly an ideal of S contained in M, but M is minimal and therefore either $(MBM)^2 = \{0\}$ or $(MBM)^2 = M$. In either case, $(MBM)^2 = \{0\}$ is not possible since S does not contain any nilpotent ideals. Hence we deduce that $(MBM)^2 = M$. This implies that $M \subseteq B$. Thus, $M = B$ and consequently, M is simple. □

In the following lemmas, we further study the left minimal ideals of an *AG*-groupoid S.

Lemma 5.3. *If every ideal A(left or right) of S contains an idempotent then S contains no non-trivial nil-ideals.*

Proof. Suppose that L is a left ideal of S containing an idempotent a of S. Now, we assume that L is a nill-ideal of S, that is, for each element of L, there exists a positive integer n such that $a^n = \underbrace{(...(((aa)a)a)...)a}_{n-times} = 0$ but from our hypothesis, we know that $a^2 = a$, for some $a \in A$, a contradiction. Thus, S does not contain any nil-ideals. □

Lemma 5.4. *Let M be a minimal ideal of S. Then every left ideal of M is also a minimal left ideal of S.*

Proof. Let M be a minimal ideal of S. Then, by Lemma 3.7, M contains at least one minimal left ideal L of S. Now let $B \neq \{0\} \subseteq L$ be a left ideal of M, that is, $MB \subseteq B$ and MB is an left ideal of S. Hence, $MB \subseteq L$, but L is a minimal left ideal of S and therefore, $L = MB \subseteq B$. This leads to $L \subseteq B$. Thus $L = B$. □

The following results in AG-groupoids are similar to the well known classical results in semigroups.

Lemma 5.5. *Let S be an AG-groupoid S with left identity. Then Sa is a principal ideal of S generated by a for all $a \in E(S)$, where $E(S)$ is the set of all idempotents of S.*

Proof. Suppose that S has a left identity e, and a an idempotentof S. Then, $(Sa)S = (Sa^2S)(Se) = (SS)(a^2e) \subseteq Sa^2S = Sa$. Hence, we can verify that Sa is a right ideal of S, and this proves that Sa is an ideal of S. □

Lemma 5.6. *Let S be an AG-groupoid with left identity. Then $(Sa^2)S = Sa^2$, for all $a \in S$.*

Proof. Let S be an AG-groupoid with a left identity e. Then

$$Sa^2S = (Sa^2)(Se) = (SS)(a^2e) = Sa^2.$$ □

6. Characterizations of 0-Simple and Simple AG-Groupoids

In this section, we give a characterization theorem for 0-simple and simple AG-groupoids.

Theorem 6.1. *An AG-groupoid S with left identity is 0-simple if and only if $Sa^2 = S$ for all $a \in S \setminus \{0\}$.*

Proof. By Lemma 5.6, we have $(Sa^2)S = Sa^2$ for all $a \in S$. The rest of the proof is exactly the same as in [18]. We hence omit the details. □

Corollary 6.2. *An AG-groupoid S with left identity is simple if and only if $Sa^2 = S$ for all $a \in S$.*

Proof. The proof of the above corollary is exactly the same as in [18] and we hence omit the details. □

The following lemmas are some useful lemmas for minimal ideals of an *AG*-groupoid S with left identity.

Lemma 6.3. *Let M be an 0-minimal ideal of S. Then either M is an 0-simple AG-subgroupoid or $M^2 = \{0\}$.*

Proof. The proof of the above Lemma is exactly the same as in [18]. □

Lemma 6.4. *Let L be a 0-minimal left ideal of S containing a zero element 0 and $L^2 \neq \{0\}$. Then $L = Sa$, for all $a \in L \backslash \{0\}$.*

Proof. Suppose that L is an 0-minimal left ideal of S and $L^2 \neq \{0\}$. Now let a be any non-zero element of L. Then $Sa \subseteq L$ is a left ideal of S. Hence, either $Sa = \{0\}$ or $L = Sa$, for all $a \in L\backslash\{0\}$.Suppose that $Sa = \{0\}$. Then, $a = 0$. This is clearly impossible. Thus $L = Sa$ for all $a \in L\backslash\{0\}$. □

Corollary 6.5. *If L is an 0-minimal left ideal of S containing an idempotent of S. then L is an ideal of S and $L = Sa$, for all $a \in E(L)$.*

Proof. Suppose that $a \neq 0$ belongs to L and is also an idempotent of S. Then Sa is clearly an ideal of S and $Sa \subseteq L$ but since L is 0-minimal left ideal of S, we see that either $L = Sa$ or $Sa = \{0\}$ for all $a \in L\backslash\{0\}$. If $Sa = \{0\}$ then $a = 0$, which contradicts to our hypothesis. Thus $L = Sa$. □

The proof of the following corollary is trivial and we hence omit the details.

Corollary 6.6. *Every left ideal of S containing an idempotent contains an ideal of S.*

For 0-minimal ideals, we have the following additional Lemmas.

Lemma 6.7. *If M is an 0-minimal ideal of S and $M^2 \neq \{0\}$. Let $L \neq \{0\}$ contained in M be a left ideal of S. Then either $L^2 = M$ or $L^2 = \{0\}$.*

Proof. Suppose that M is an 0-minimal ideal of S and $M^2 \neq \{0\}$. We assume that $L \neq \{0\}$ is a left ideal of S contained in M. Then, ${}^2 \subseteq M$ is an ideal of S. But since M is minimal and so either $L^2 \subseteq M$ or $L^2 = \{0\}$. □

Lemma 6.8. *Let M be an 0-minimal ideal of S containing at least one 0-minimal left ideal of S. Then M is the set union of all the 0-minimal left ideals of S contained in M.*

Proof. Let M be an 0-minimal ideal of S and M containing an 0-minimal left ideal L of S. Now suppose that $A \subseteq M$ is the union of all 0-minimal left ideals of S contained in M. Clearly, A is a left ideal of S. Now let $a \in A\backslash\{0\}$ and $c \in S$. Then, by the definition of A, a belongs to some 0-minimal left ideal L of S, that is, $a \in L$, and $ac \in Lc$. Since L is an 0-minimal left ideal of S. Therefore, Lc is also an 0-minimal left ideal of S. Thus, $Lc \subseteq M$. This implies $ac \in M$ and $ac \in A$ as well which further implies that $Ac \subseteq A$ for all $c \in S$. Hence, $AS \subseteq A$, that is, A is the right ideal of S and hence an ideal of S. As $A \subseteq M$ but M is 0-minimal, and therefore $M = A$. □

Let A be a non empty subset of S. Then the intersection K of all the left ideals L of S containing A is a left ideal. Hence $A \subseteq K$ as well.

Lemma 6.9. *Suppose that S contains a left identity e and A is a non-empty subset of S. Then $A \cup SA$ is a left ideal of S containing A.*

Proof. Let A be a non-empty subset of S. Then the following equalities hold:

$$\begin{aligned} S(A \cup SA) =& SA \cup S(SA) = (SA) \cup (SS)(SA) = (SA) \cup (AS)(SS) \\ =& (SA) \cup (AS)S = (SA) \cup (SS)A = SA \cup SA = SA \\ \subseteq& A \cup SA. \end{aligned}$$

From the above equality, $A \cup SA$ is a left ideal of S containing A and we call it the left ideal of S generated by A. It is now clear that $A \cup (SA) \cup (AS) \cup (SAS)$ is also an ideal of S generated by A. If A contains only one element a, then we write $A =< a >$, and $A \cup SA = Sa$. □

In summarizing the above results, we state a theorem of an *AG*-groupoid S without nipotent ideals.

Theorem 6.10. *Let S be an AG-groupoid with left identity and contains no nilpotent ideals. Then every minimal left ideal of S is contained in some minimal two sided ideals of S.*

Proof. Let L be minimal left ideal of S and $L \neq \{0\}$. Then, $L \cup LS$ is an ideal of S containing L. Since the following equality holds

$$(L \cup LS)S = LS \cup (LS)S = (LS) \cup ((eL)S)S = (LS) \cup ((SL)e)S$$

$$\subseteq (LS) \cup (Le)S \subseteq (LS) \cup L$$
$$\subseteq L \cup LS.$$

Thus $LS \cup L$ is a right ideal of S, and hence an ideal of S. Now we still need to prove that $L \cup LS$ is a minimal ideal of S. Let $B \neq \{0\} \subseteq L \cup LS$ be an ideal of S. Then $BL \subseteq L$ is also a left ideal of S. But since L is minimal, we either have $BL = \{0\}$ or $BL = L$. Suppose that $BL = \{0\}$. Then we deduce the following equation.

$$BB \subseteq B(L \cup LS) = BL \cup B(LS) = \{0\} \cup (Be)(LS) = \{0\} \cup (BL)S = \{0\},$$

Thereby $B^2 = \{0\}$, which contradicts our hypotheisis. This leads to $BL = L$ and $B \subseteq LS$. Since $Lc \subseteq B$, for all $c \in S$, and so $LS \subseteq B$, and consequently $LS = B$. This shows that $L \cup LS = B$ and $L \cup LS$ is a minimal ideal of S. In other words, every minimal left ideal of S is contained in some minimal two sided ideal of S.This ends the proof. □

Since quasi ideals of rings and semigroups have also been extensively studied by O.Stenfeld in his well known monograph [23].

In closing this paper, we ask for an AG-groupoid S with left identity containing no nilpotent ideals, whether every minimal left quasi left ideal of S will be properly contained in some miminal two sided quasi ideals of S?

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References

[1] A.H. Clifford, Semigroup without nilpotent ideals, *American Journal of Mathematics*, **71**, No. 4 (1949), 834-844.

[2] A.H. Clifford, Semigroups containing minimal ideals, *American Journal of Mathematics*, **70** (1948), 521-526.

[3] L.M. Gluskin, O. Steinfeld, Rings (semigroups) containing minimal (minimal) right and left ideals, *Publ. Math. Debrecen*, **25**, No-s: 3-4 (1978), 275-280.

[4] P. Holgate, Groupoids satisfying a simple invertive law, *The Math. Stud.*, **61**, No-s: 1-4 (1992), 101-106.

[5] J. Ježek, T. Kepka, A note on medial division groupoids, *Proc. Amer. Math. Soc.*, **119**, No. 2 (1993), 423-426.

[6] J. Ježek, T. Kepka, The equational theory of paramedial cancellation groupoids, *Czech. Math. J.*, **50**, No. 125 (2000), 25-34.

[7] M.A. Kazim, M. Naseeruddin, On almost semigroups, *The Alig. Bull. Math.*, **2** (1972), 1-7.

[8] Q. Mushtaq, S. M. Yusuf, On LA-semigroups, *The Alig. Bull. Math.*, **8** (1978), 65-70.

[9] Q. Mushtaq, S.M. Yusuf, On LA-semigroup defined by a commutative inverse semigroup, *Math. Bech.*, **40** (1988), 59-62.

[10] Q. Mushtaq, Q. Iqbal, Decomposition of a locally associative LA-semigroup, *Semigroup Forum*, **41** (1990), 155-164.

[11] Q. Mushtaq, M. Khan, Ideals in AG-band and AG*-groupoid, *Quasigroups and Related Systems*, **14** (2006), 207-215.

[12] Q. Mushtaq, M. Khan, A note on an Abel Grassmann's-3-band, *Quasigroups and Related Systems*, **15** (2007), 295-301.

[13] Q. Mushtaq, M. Khan, M-systems in left almost semigroups, *Southeast Asian Bull. Math.*, **33**, No. 3 (2009), 321-327.

[14] Q. Mushtaq, M. Khan, Semilattice decomposition of locally associative AG**-groupoids, *Algebra. Colloq.*, **16**, No. 1 (2009), 17-22.

[15] Q. Mushtaq, Madad Khan, Ideals in left almost semigroups, In: *Proceedings of 4th International Pure Mathematics Conference* (2003), 65-77.

[16] M. Khan, S. Anis, On semilattice decomposition of an Abel-Grassmann's groupoid, *Acta Math. Sin. (Engl. Ser.)*, **28**, No. 7 (2012), 1461-1468.

[17] M. Khan, S. Anis, K. P. Shum, Characterizations of left regular ordered Abel-Grassmann groupoids, *Int. J. Algebra*, **5**, No-s: 9-12 (2011), 499-452.

[18] M. Khan, *Some Studies in AG*-Groupoids*, Ph.D. Thesis, Quaid-i-Azam University Islamabad, Pakistan (2008).

[19] M. Khan, K. P. Shum and Shamas-ur-Rehman, On generalized fuzzy ideals and semigroups which are semilattices of left Groups, *Southeast Asian Bull. Math.*, **37**, No. 1 (2013), To Appear.

[20] P.V. Protic, N. Stevanovic, AG-test and some general properties of Abel-Grassmann's groupoids, *P.U.M.A.*, **6**, No. 4 (1995), 371-383.

[21] N. Stevanović, P.V. Protić, Composition of Abel-Grassmann's 3-bands, *Novi Sad J. Math.*, **34**, No. 2 (2004), 175-182.

[22] D. Rees, On semigroups, *Mathematical Proceedings of the Cambridge Philosphical Society*, **36** (1940), 387-400.

[23] O. Steinfeld, *Quasi-Ideals in Rings and Semigroups*, With a foreword by L. Redei. Disquisitiones Mathematicae Hungaricae [Hungarian Mathematics Investigations], 10. Akademiai Kiado, Budapest (1978). xi+154.