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GENERALIZED HYPERBOLIC METRICS AND DISTORTION OF QUASICONFORMAL MAPPINGS

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Abstract: In this paper, we investigate the distortion of generalized hyperbolic metrics under the class $Id_K(\partial D)$. We also obtain improved distortion theorems whenever generalized hyperbolic metrics are defined in convex domains.

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1. Introduction

There is a classical result of Teichmüller's concerning the distortion of normalized quasiconformal mappings.

Theorem 1. *([1]) Let $h(z,w)$ denote the hyperbolic metric of constant curvature -4 in the three punctured sphere $\mathbb{C}\backslash\{0,1\}$. If f is a K-quasiconformal mapping of the Riemann sphere fixing 0,1 and ∞, then for any $z \in \mathbb{C}\backslash\{0,1\}$, $h(z, f(z)) \leq \log K$.*

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Let D be a proper subdomain of $\mathbb{R}^n$, and let

$$Id_K(\partial D) = \{f : \mathbb{R}^n \to \mathbb{R}^n \text{ is K-quasiconformal} : f(x) = x, \forall x \in \mathbb{R}^n \backslash D\}.$$

It is well known that the class $Id_K(\partial D)$ has played an important role both in quasiconformal mapping theory and Teichmüller theory with the case $n = 2$. There are some generalized hyperbolic metrics which have been studied recently. In this paper, we investigate the distortion of generalized hyperbolic metrics under the class $Id_K(\partial D)$. We also obtain improved distortion theorems whenever generalized hyperbolic metrics are defined in convex domains.

2. Preliminary Results

Given E,$F \subset \mathbb{R}^n$ we denote by $\Delta(E, F)$ the family of all curves that join the sets E,F in $\mathbb{R}^n$ and denote by $M(\Delta(E, F))$ its modulus. For a ring domain $R(C_0, C_1)$ with complementary components C_0 and C_1, we define the modulus of $R(C_0, C_1)$ by

$$\mathrm{mod}R(C_0, C_1) = \left(\frac{\omega_{n-1}}{M(\Delta(C_0, C_1))}\right)^{1/(n-1)},$$

where ω_{n-1} is the surface area of the unit sphere in $\mathbb{R}^n$.

The Teichmüller ring domain $R_{T,n}(t)$, $t > 0$ is the doubly connected domains with complementary components $[-e_1, 0]$ and $[te_1, \infty]$. For the capacity of the Teichmüller ring domain we write $\tau_n(t) = \mathrm{cap}R_{T,n}(t) = M(\Delta([-e_1, 0], [te_1, \infty]))$.

Lemma 2. *([2]) Let $R = R(C_0, C_1)$ be a ring in $\overline{\mathbb{R}}^n$, and let a,$b \in C_0$, c,$\infty \in C_1$ be distinct points. Then*

$$capR \geq capR_{T,n}(t), t = \frac{|a - c|}{|a - b|}.$$

Equality holds for the Teichmüller ring $R_{T,n}(t)$, with $a = 0$, $b = -e_1$, and $c = te_1$,$t > 0$.

Let x, $y \in D \subset \mathbb{R}^n$. We denote by $d(x, \partial D)$ the distance between x and the boundary ∂D. Let

$$j'_D(x, y) = \log\left(1 + \frac{|x - y|}{d(x, \partial D)}\right)\left(1 + \frac{|x - y|}{d(y, \partial D)}\right).$$

The distance ratio metric j'_D is well known in geometry function theory. It is introduced by Gehring and Osgood in [3].

The cross ratio of z_1, z_2, z_3, z_4 is defined as

$$|z_1, z_2, z_3, z_4| = \frac{|z_1 - z_3||z_2 - z_4|}{|z_1 - z_2||z_3 - z_4|}.$$

Apollonian metric is defined as

$$\alpha_D(x, y) = \sup_{a,b \in \partial D} \log\left(|a, x, y, b|\right).$$

Apollonian metric was first studied by Beardon[4].

Seittentanta metric[5] is defined as

$$\delta_D(x, y) = \sup_{a,b \in \partial D} \log\left(1 + |x, a, y, b|\right).$$

Denote

$$\delta_D^p(x, y) = \sup_{a,b \in \partial D} \log\left(1 + (|x, a, y, b|^p + |x, b, y, a|^p)^{1/p}\right),$$

and

$$j_D^p(x, y) = \sup_{a \in \partial D} \log\left(1 + \left(\frac{|x-y|^p}{|x-a|^p} + \frac{|x-y|^p}{|y-a|^p}\right)^{1/p}\right).$$

j_D^p and δ_D^p were studied by Hästö in [6]. Seittenranta has proved the following relation between α_D and δ_D.

Lemma 3. *([5]) Let $D \subset \mathbb{R}^n$ be an open set and Card $\partial D \geq 2$. Then* $\alpha_D \leq \delta_D \leq \alpha_D + \log 3$.

In paper [7],the author has proved the following relation among α_D, j_D' and δ_D^p.

Lemma 4. *([7]) Let $D \subset \mathbb{R}^n$ be an open set. Then $j_D' - \log 9 \leq \alpha_D \leq j_D'$.*

Lemma 5. *([7]) Let $D \subset \mathbb{R}^n$ be an open set and Card $\partial D \geq 2$. Then* $\alpha_D \leq \delta_D^p \leq \alpha_D + \log(1 + 2^{\frac{1}{p}+1})$.

3. The Main Results and Proofs

In this section we denote by $c(n, K)$ a constant depending only on n and K.

Theorem 6. *Let $D \subset \mathbb{R}^n$ be a domain and $f \in Id_K(\partial D)$. Then for all $x \in D$*

$$j_D^p(x, f(x)) \leq 2^{\frac{1}{p}} \log(2 + c(n, K)).$$

Proof. Let $y = f(x)$. Since $f^{-1} \in Id_K(\partial D)$, we may assume $d(x, \partial D) \le d(y, \partial D)$. Fix $z \in D$ such that $d(x, \partial D) = |x - z|$. For $t > 0$, write $F_t = \{z + u(z - x) : u \ge t\}$. Let $\Gamma_t = \Delta([x, z], F_t)$ be the family of all curves in $\mathbb{R}^n$ joining $[x, z]$ to F_t. $\Gamma'_t = f(\Gamma_t) = \Delta(f([x, z]), F_t)$. From lemma 2, it follows that

$$\tau_n\left(\frac{t|x-z|}{|y-z|}\right) \le M(\Gamma'_t) \le KM(\Gamma_t) = K\tau_n(t).$$

Setting $t = 1$, we have

$$\frac{|y-z|}{|x-z|} \le \frac{1}{\tau_n^{-1}(K\tau_n(1))}.$$

Setting $t = |y - z|/|x - z|$, we have

$$\frac{|y-z|}{|x-z|} \le \tau_n^{-1}\left(\frac{\tau_n(1)}{K}\right).$$

Hence it follows that

$$\frac{|y-z|}{|x-z|} \le \min\left\{\frac{1}{\tau_n^{-1}(K\tau_n(1))}, \tau_n^{-1}\left(\frac{\tau_n(1)}{K}\right)\right\} = c(n, K),$$

and

$$\frac{|x-y|}{d(x,\partial D)} = \frac{|x-y|}{|x-z|} \le \frac{|x-z| + |y-z|}{|x-z|} \le 1 + c(n, K).$$

For any $x > 0$ and $p > 0$, it is easy to see that $\log(1 + 2^{\frac{1}{p}}x) \le 2^{\frac{1}{p}}\log(1 + x)$. Hence

$$\begin{aligned} j_D^p(x,y) &= \sup_{a\in\partial D}\log\left(1 + \left(\tfrac{|x-y|^p}{|x-a|^p} + \tfrac{|x-y|^p}{|y-a|^p}\right)^{1/p}\right) \\ &\le 2^{\frac{1}{p}}\log\left(1 + \tfrac{|x-y|}{\min\{d(x,\partial D), d(y,\partial D)\}}\right) \\ &= 2^{\frac{1}{p}}\log\left(1 + \tfrac{|x-y|}{d(x,\partial D)}\right) \\ &\le 2^{\frac{1}{p}}\log(2 + c(n,K)). \end{aligned}$$

□

Similarly, we can prove the following

Theorem 7. *Let $D \subset \mathbb{R}^n$ be a domain and $f \in Id_K(\partial D)$. Then for all $x \in D$*

$$j'_D(x, f(x)) \le 2\log(2 + c(n, K)).$$

By Theorem 7 and Lemma 4, we have the following

Corollary 8. *Let $D \subset \mathbb{R}^n$ be a domain and $f \in Id_K(\partial D)$. Then for all $x \in D$*

$$\alpha_D(x, f(x)) \le 2\log(2 + c(n, K)).$$

By lemma 3 and corollary 8, we have

Corollary 9. *Let $D \subset \mathbb{R}^n$ be a domain and $f \in Id_K(\partial D)$. Then for all $x \in D$*

$$\delta_D(x, f(x)) \le 2\log(2 + c(n, K)) + \log 3.$$

By lemma 5 and corollary 8, we have

Corollary 10. *Let $D \subset \mathbb{R}^n$ be a domain and $f \in Id_K(\partial D)$. Then for all $x \in D$*

$$\delta_D^p(x, f(x)) \le 2\log(2 + c(n, K)) + \log(1 + 2^{\frac{1}{p}+1}).$$

In what follows, we will obtain improved distortion theorems of generalized hyperbolic metrics for convex domains.

Theorem 11. *Let $D \subset \mathbb{R}^n$ be a convex domain and $f \in Id_K(\partial D)$. Then for all $x \in D$*

$$j_D^p(x, f(x)) \le 2^{\frac{1}{p}} \log(1 + \sqrt{c(n, K)^2 - 1}).$$

Proof. Let $y = f(x)$. Since $f^{-1} \in Id_K(\partial D)$, we may assume $d(x, \partial D) \le d(y, \partial D)$. Fix $z \in D$ such that $d(x, \partial D) = |x - z|$. For $t > 0$, write $F_t = \{z + u(z - x) : u \ge t\}$. Let $\Gamma_t = \Delta([x, z], F_t)$ be the family of all curves in $\mathbb{R}^n$ joining $[x, z]$ to F_t. $\Gamma_t' = f(\Gamma_t) = \Delta(f([x, z]), F_t)$. From lemma 2, it follows that

$$\tau_n\left(\frac{t|x - z|}{|y - z|}\right) \le M(\Gamma_t') \le KM(\Gamma_t) = K\tau_n(t).$$

Setting $t = 1$, we have

$$\frac{|y - z|}{|x - z|} \le \frac{1}{\tau_n^{-1}(K\tau_n(1))}.$$

Setting $t = |y - z|/|x - z|$, we have

$$\frac{|y - z|}{|x - z|} \le \tau_n^{-1}\left(\frac{\tau_n(1)}{K}\right).$$

Hence it follows that

$$\frac{|y - z|}{|x - z|} \le \min\left\{\frac{1}{\tau_n^{-1}(K\tau_n(1))}, \tau_n^{-1}\left(\frac{\tau_n(1)}{K}\right)\right\} = c(n, K).$$

Since D is convex, $|y-z|^2 \geq |x-y|^2 + |x-z|^2$. Hence

$$\frac{|x-y|}{d(x,\partial D)} = \frac{|x-y|}{|x-z|} \leq \sqrt{\left(\frac{|y-z|}{|x-z|}\right)^2 - 1}.$$

For any $x > 0$ and $p > 0$, it is easy to see that $\log(1 + 2^{\frac{1}{p}} x) \leq 2^{\frac{1}{p}} \log(1+x)$. Hence

$$\begin{aligned} j_D^p(x,y) &= \sup_{a\in\partial D} \log\left(1 + \left(\tfrac{|x-y|^p}{|x-a|^p} + \tfrac{|x-y|^p}{|y-a|^p}\right)^{1/p}\right) \\ &\leq 2^{\frac{1}{p}} \log\left(1 + \tfrac{|x-y|}{\min\{d(x,\partial D), d(y,\partial D)\}}\right) \\ &= 2^{\frac{1}{p}} \log\left(1 + \tfrac{|x-y|}{d(x,\partial D)}\right) \\ &\leq 2^{\frac{1}{p}} \log(1 + \sqrt{c(n,K)^2 - 1}). \end{aligned}$$

□

Similarly, we can prove the following

Theorem 12. *Let $D \subset \mathbb{R}^n$ be a convex domain and $f \in Id_K(\partial D)$. Then for all $x \in D$*

$$j'_D(x, f(x)) \leq 2\log(1 + \sqrt{c(n,K)^2 - 1}).$$

By Theorem 12 and Lemma 4, we have the following

Corollary 13. *Let $D \subset \mathbb{R}^n$ be a convex domain and $f \in Id_K(\partial D)$. Then for all $x \in D$*

$$\alpha_D(x, f(x)) \leq 2\log(1 + \sqrt{c(n,K)^2 - 1}).$$

By lemma 3 and corollary 13, we have

Corollary 14. *Let $D \subset \mathbb{R}^n$ be a convex domain and $f \in Id_K(\partial D)$. Then for all $x \in D$*

$$\delta_D(x, f(x)) \leq 2\log(1 + \sqrt{c(n,K)^2 - 1}) + \log 3.$$

By lemma 5 and corollary 13, we have

Corollary 15. *Let $D \subset \mathbb{R}^n$ be a convex domain and $f \in Id_K(\partial D)$. Then for all $x \in D$*

$$\delta_D^p(x, f(x)) \leq 2\log(1 + \sqrt{c(n,K)^2 - 1}) + \log(1 + 2^{\frac{1}{p}+1}).$$

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