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COEFFICIENT BOUNDS FOR A CLASS MULTIVALENT FUNCTION DEFINED BY SALAGEAN OPERATOR

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Abstract: The aim of the present paper is to define a subclass of analytic p-valent function in the open unit disk $U = \{z : |z| < 1\}$ namely $S_p^\lambda(A, B, b)$. For the class defined, we obtain the upper bounds for the Fekete Szego functional,$|a_{p+2} - \mu a_{p+1}^2|$.

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1. Introduction

Let S be the class of analytic univalent functions $f(z)$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

that are defined in the open unit disk $U = \{z : |z| < 1\}$. Let S_p denote the class of all analytic p-valent functions $f(z)$ of the form of

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$$f(z) = z^p + \sum_{k=2}^{\infty} a_{p+k} z^{p+k}. \tag{2}$$

Let $S_p^\lambda(A,B,b)$ be the subclass that consists of functions $f(z) \in S_p$ that satisfy the condition

$$1 + \frac{1}{b}\left(\frac{1}{p}\frac{z(D^\lambda f(z))'}{D^\lambda f(z)} - 1\right) \prec \frac{1+Az}{1+Bz}$$

where $\prec$ denotes subordination, b is any non-zero complex number. A and B are the arbitrary fixed numbers, $-1 \leq B < A \leq 1$ and $z \in U$. $D^\lambda f(z)$ is the operator introduced by Shenen et al. [11] which is the extension of Salagean operator, $D^n f(z)$ defined by Salagean [10] where

$$D^\lambda f(z) = D(D^{\lambda-1} f(z)) = z^p + \sum_{k=1}^{\infty} \left(\frac{p+k}{p}\right)^\lambda a_{p+k} z^{p+k}$$

with $\lambda \in N_0 = \{0\} \cup N$. For different choices of parameters p, A, B, b and λ we obtain special relationships with the previous known classes as shown below:

1) $S_p^\lambda(A,B,1) = K_{p,n}^0(A,B)$ which was studied by Aouf et al. [3].

2) $S_p^0(A,B,1) = S_1^0(0,A,B)$ which is the class studied by Cho and Kim [4].

3) $S_1^0(A,B,b) = S^*(A,B,b)$ and $S_1^1(A,B,b) = K(A,B,b)$ which are the classes studied by Ravichandran et al. [9].

4) $S_1^0(A,B,b) = M_{0,b}(\phi)$ which was the class studied by Suchitra et al. [12].

5) $S_1^1(A,B,b) = M_1(A,B,b,0)$ which was studied by Akbarally and Darus [1].

6) $S_1^0(1,-1,b) = S(b)$ which was studied by Nasr and Aouf [8].

7) $S_p^1(1,-1,b) = C(b,p)$ which was introduced by Aouf [2].

8) $S_1^1(1,-1,b) = C(b)$ which was investigated by Wiatrowski in 1971 (Nasr and Aouf [7]).

9) $S_1^0(1,-1,1) = S^*$ and $S_1^1(1,-1,1) = K$ are the well known classes of starlike and convex functions.

The purpose of this paper is to find the upper bounds of the Fekete Szego functional for the class $S_p^\lambda(A,B,b)$.

2. Fekete Szego Theorem

We first state a lemma which will be used in the proof of our theorem.

Lemma 1. *(Ma & Minda [6]). If $p(z) = 1 + c_1z + c_2z^2 + ...$ is a function with positive real part, then for any complex number μ,*

$$|c_2 - \mu c_1^2| \leq 2\max\{1, |2\mu - 1|\}$$

and the result is sharp for functions given by

$$p(z) = \frac{1+z^2}{1-z^2} \quad \text{and} \quad p(z) = \frac{1+z}{1-z}.$$

Theorem 2. *Let $\frac{1+Az}{1+Bz} = 1 + F_1z + F_2z^2 + F_3z^3 + ...$ If $f(z)$ given by (2) belongs to $S_p^\lambda(A, B, b)$, then for some complex number μ,*

$$|a_{p+2} - \mu a_{p+1}^2| \leq \frac{|b|F_1p^{\lambda+1}}{(A-B)(p+1)^\lambda} \times \max$$

$$\{1, |\frac{1}{A-B}[(A+B) + \frac{2F_2}{F_1} + 2bpF_1\{\frac{(p+1)^{2\lambda} - 2\mu[p(p+2)]^\lambda}{(p+1)^{2\lambda}}\}]|\}.$$

The result is sharp.

Proof. If $S_p^\lambda(A, B, b)$, then there exists a Schwarz function with $w(0) = 0$ and $|w(z)| < 1$, analytic in the open unit disk such that

$$1 + \frac{1}{b}(\frac{1}{p}\frac{z(D^\lambda f(z))'}{D^\lambda f(z)} - 1) = \frac{1 + Aw(z)}{1 + Bw(z)}. \tag{3}$$

Let $\frac{1+Aw(z)}{1+Bw(z)} = 1 + c_1z + c_2z^2 + ...$, we obtain

$$w(z) = \frac{c_1}{A-B}z + [\frac{c_2}{A-B} + \frac{c_1^2B}{(A-B)^2}]z^2 + ... \tag{4}$$

Since $\frac{1+Az}{1+Bz} = 1 + F_1z + F_2z^2 + F_3z^3 + ...$, therefore from (4)

$$\frac{1+Aw(z)}{1+Bw(z)} = 1 + \frac{F_1c_1}{A-B}z + [\frac{BF_1c_1^2}{(A-B)^2} + \frac{F_1c_2}{A-B} + \frac{F_2c_1^2}{(A-B)^2}]z^2 + ...$$

Now, let

$$1 + \frac{1}{b}(\frac{1}{p}\frac{z(D^\lambda f(z))'}{D^\lambda f(z)} - 1) = 1 + h_1z + h_2z^2 + ... \tag{5}$$

Therefore,

$$1+\frac{F_1c_1}{A-B}z+[\frac{BF_1c_1^2}{(A-B)^2}+\frac{F_1c_2}{A-B}+\frac{F_2c_1^2}{(A-B)^2}]z^2+... \\ =1+h_1z+h_2z^2+... \tag{6}$$

Comparing the coefficients for z and z^2, we obtain

$$h_1=\frac{F_1}{A-B}c_1 \tag{7}$$

and

$$h_2=\frac{BF_1}{(A-B)^2}c_1^2+\frac{F_1}{A-B}c_2+\frac{F_2}{(A-B)^2}c_1^2. \tag{8}$$

From (5),

$$1+\frac{1}{b}(\frac{\sum_{k=1}^{\infty}\frac{k}{p}(\frac{p+k}{p})^\lambda a_{p+k}z^k}{1+\sum_{k=1}^{\infty}(\frac{p+k}{p})^\lambda a_{p+k}z^k})=1+h_1z+h_2z^2+...$$

which yields

$$\frac{1}{p}(\frac{p+1}{p})^\lambda a_{p+1}z+\frac{2}{p}(\frac{p+2}{p})^\lambda a_{p+2}z^2+... \\ =bh_1z+[bh_1(\frac{p+1}{p})^\lambda a_{p+1}+bh_2]z^2+....$$

By comparing the coefficients for z we obtain

$$\frac{1}{p}(\frac{p+1}{p})^\lambda a_{p+1}=bh_1$$

and solving for h_1 we obtain

$$h_1=\frac{1}{bp}(\frac{p+1}{p})^\lambda a_{p+1}. \tag{9}$$

By comparing the coefficients for z^2 we obtain

$$\frac{2}{p}(\frac{p+2}{p})^\lambda a_{p+2}=bh_1(\frac{p+1}{p})^\lambda a_{p+1}+bh_2$$

and solving for h_2 we obtain

$$h_2 = \frac{2}{bp}(\frac{p+2}{p})^{\lambda} a_{p+2} - \frac{1}{bp}(\frac{p+1}{p})^{2\lambda} a_{p+1}^2. \tag{10}$$

Equating (7) and (9),

$$\frac{F_1}{A-B} c_1 = \frac{1}{bp}(\frac{p+1}{p})^{\lambda} a_{p+1}.$$

Then, solving for a_{p+1} we have that

$$a_{p+1} = \frac{bpF_1}{A-B} c_1 (\frac{p}{p+1})^{\lambda}. \tag{11}$$

Equating (8) and (10),we obtain

$$\frac{BF_1}{(A-B)^2} c_1^2 + \frac{F_1}{(A-B)} c_2 + \frac{F_2}{(A-B)^2} c_1^2$$
$$= \frac{2}{bp}(\frac{p+2}{p})^{\lambda} a_{p+2} - \frac{1}{bp}(\frac{p+1}{p})^{2\lambda} a_{p+1}^2.$$

Solving for a_{p+2} yields

$$a_{p+2} = \frac{bp}{2}(\frac{p}{p+2})^{\lambda}(\frac{F_1}{A-B})[\frac{Bc_1^2}{A-B} + c_2 + \frac{F_2 c_1^2}{F_1(A-B)} + \frac{bpF_1 c_1^2}{A-B}]. \tag{12}$$

From (11) and (12), we obtain

$$|a_{p+2} - \mu a_{p+1}^2| = |\frac{bF_1 p^{\lambda+1}}{2(p+2)^{\lambda}(A-B)}(c_2 - vc_1^2)|$$

where

$$v = -\frac{B}{A-B} \times [1 + \frac{F_2}{BF_1} + \frac{bpF_1}{B} - \mu(\frac{2bpF_1}{B})(\frac{p+2}{p})^{\lambda}(\frac{p}{p+1})^{2\lambda}].$$

From Lemma 1,

$$|a_{p+2} - \mu a_{p+1}^2| \leq \frac{|b|F_1 p^{\lambda}}{2(p+2)^{\lambda}(A-B)}[2\max\{1, |2v-1|\}].$$

Therefore

$$|a_{p+2} - \mu a_{p+1}^2| \leq \frac{|b|F_1 p^{\lambda+1}}{(A-B)(p+1)^\lambda} \times \max$$
$$\{1, |\frac{1}{A-B}[(A+B) + \frac{2F_2}{F_1} + 2bpF_1\{\frac{(p+1)^{2\lambda} - 2\mu[p(p+2)]^\lambda}{(p+1)^{2\lambda}}\}]|\}.$$

The result is sharp for the functions defined by

$$1 + \frac{1}{b}(\frac{1}{p}\frac{z(D^\lambda f(z))'}{D^\lambda f(z)} - 1) = \frac{1+Az^2}{1+Bz^2}$$

and

$$1 + \frac{1}{b}(\frac{1}{p}\frac{z(D^\lambda f(z))'}{D^\lambda f(z)} - 1) = \frac{1+Az}{1+Bz}.$$

That completes the proof of Theorem 2. □

We obtain the results of Ravichandran et al. [9] in the following corollary.

Corollary 3. *If $f \in S_1^0(1,-1,b) \equiv S^*(b)$, then for some complex number* μ,$|a_3 - \mu a_2^2| \leq 2\max\{1, |\frac{F_2}{F_1} + (1-2\mu)bF_1|\}$.

Also, we obtain the following results by setting $\lambda = 1$.

Corollary 4. *If $f \in S_1^1(1,-1,b) \equiv K^*(b)$, then for some complex number* μ,

$$|a_3 - \mu a_2^2| \leq \frac{|b|F_1}{3(A-B)} \times \max\{1, |\frac{1}{A-B}[(A+B) + \frac{2F_2}{F_1} + bF_1(2-3\mu)]|\}.$$

Results of Keogh & Merkes [5] are obtained in the following corollaries.

Corollary 5. *If $f \in S_1^0(1,-1,1) \equiv S^*$, then* $|a_3 - \mu a_2^2| \leq \max\{1, |4\mu - 3|\}$.

Corollary 6. *If $f \in S_1^1(1,-1,1) \equiv K$, then* $|a_3 - \mu a_2^2| \leq \max\{\frac{1}{3}, |1-\mu|\}$.

Note : Setting $A = 1$ and $B = -1$ implies $F_1 = F_2 = 2$.

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