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A LIMIT TO REPRESENT BERNOULLI NUMBERS USING EULERIAN NUMBERS

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Abstract: The Bernoulli Numbers are a sequence of rational numbers applied in various mathematics fields and can be written as a limit with t tending toward zero applied to the nth derivative of the function $t/\left(e^t-1\right)$. From this result, the Bernoulli Numbers are written in terms of the Eulerian Numbers. From sucessives derivatives of the function $t/\left(e^t-1\right)$ the Eulers Triangle is obtained and a second triangle which allow to construct a limit to represent Bernoulli Numbers using Eulerian Numbers.

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1. Introduction

In 1303 the chinese mathematician Zhu Shijie constructed a matrix of numbers, arranged in the shape of a triangle, from which one can draw several interesting numerical relations. Three hundred and fifty years after is published *Traite du triangle arithmetique* by Blaise Pascal, which again is treated exactly the same triangle of numbers discovered by Zhu Shijie. This triangle of numbers nowadays is better knowm as Pascal's Triangle. The work Katsuyo Sampo, due to the japanese mathematician Takakazu Seki Kowa, published posthumously

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in the year 1712, presents a sequence of numbers that plays an important role in several mathematics branches. In 1713 the work *Ars Conjectandi*, due to the swiss mathematician Jacob Bernoulli, appears again the sequence of numbers discovered by Seki Kowa, in Chapter 3 can be find some of these numbers, written as $A = 1/6$, $B = -1/30$, $C = 1/42$ and $D = -1/30$.

The sequence of numbers discovered by Seki Kowa and Jacob Bernoulli is called Bernoulli Numbers, the first nine numbers are: $B_0 = 1$, $B_1 \pm 1/2$, $B_2 = 1/6$, $B_3 = 0$, $B_4 = -1/30$, $B_5 = 0$, $B_6 = 1/42$, $B_7 = 0$ and $B_8 = -1/30$.

The Bernoulli Numbers can be obtained, for example, from Pascal's Triangle and one of its main applications refers to one of the greatest mathematical challenge of the seventeenth century, called "The Problem of Basel which was to determine the exact value of the infinite series:

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{25} + \cdots + \frac{1}{n^2} + \cdots . \tag{1}$$

Leonhard Euler, presented a more general solution to this problem considering the case $\sum_{n=1}^{\infty} \frac{1}{n^{\mathbf{2k}}}$ and summarized in the expression

$$\zeta(2k) = \frac{(-1)^{k-1} 2^{2k-1} \pi^{2k}}{(2k)!} B_{2k}. \tag{2}$$

Here B_{2k} exactly corresponds to the pairs Bernoulli Numbers.

2. Analysis of Derivative $\frac{d^n}{dt^n} \frac{t}{e^t - 1}$

Definition 2.1. The Bernoulli polynomials comprise a sequence of polynomials $\{B_n(x)\}_{n=0}^{\infty}$, which have the following properties:

$$B_0(x) = 1,$$

$$\frac{d}{dx} B_n(x) = n B_{n-1}(x),$$

$$\int_0^1 B_n(x)\, dx = 0, \quad \text{for} \quad n \geq 1.$$

The numbers obtained for $B_n(0)$, with $n = 0,1, 2, \ldots$, are called Bernoulli Numbers. The generating functions commonly used to describe the the Bernoulli Numbers and the Bernoulli polynomials are respectively given by

$$\sum_{n=0}^{\infty} B_n \frac{t^n}{n!} = \frac{t}{e^t - 1} \tag{3}$$

and

$$\sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} = \frac{te^{tx}}{e^t-1}. \tag{4}$$

To confirm the result (3) we can expand the right side using Taylor series, which yields:

$$\frac{t}{e^t-1} = \lim_{a\to 0}\frac{a}{e^a-1} + \lim_{a\to 0}\left[\left(\frac{d}{da}\frac{a}{e^a-1}\right)\frac{(t-a)}{1!}\right] + \lim_{a\to 0}\left[\left(\frac{d^2}{da^2}\frac{a}{e^a-1}\right)\frac{(t-a)^2}{2!}\right] + \lim_{a\to 0}\left[\left(\frac{d^3}{da^3}\frac{a}{e^a-1}\right)\frac{(t-a)^3}{3!}\right] + \cdots .$$

In the case where a is equal to zero we have a MacLaurin series, but if we make a tend to zero at $a/(e^a-1)$ we have an undetermined limit, which implies the use of the L'Hopital rule. Thus, we have that:

$$\lim_{a\to 0}\frac{a}{e^a-1} = 1,$$

$$\lim_{a\to 0}\left(\frac{d}{da}\frac{a}{e^a-1}\right) = -\frac{1}{2},$$

$$\lim_{a\to 0}\left(\frac{d^2}{da^2}\frac{a}{e^a-1}\right) = \frac{1}{6},$$

$$\lim_{a\to 0}\left(\frac{d^3}{da^3}\frac{a}{e^a-1}\right) = 0,$$

$$\lim_{a\to 0}\left(\frac{d^4}{da^4}\frac{a}{e^a-1}\right) = -\frac{1}{30}.$$

With these results and the expression (3) we find:

$$\sum_{n=0}^{\infty} B_n\frac{t^n}{n!} = B_0\frac{t^0}{0!} + B_1\frac{t^1}{1!} + B_2\frac{t^2}{2!} + B_3\frac{t^3}{3!} + B_4\frac{t^4}{4!} + \ldots \tag{5}$$

and knowing that

$$\frac{t}{e^t-1} = 1 - \frac{1}{2}\frac{t}{1!} + \frac{1}{6}\frac{t^2}{2!} + 0\frac{t^3}{3!} - \frac{1}{30}\frac{t^4}{4!} + \ldots \tag{6}$$

We can then conclude that the Bernoulli Numbers can be expressed using the following limit:

$$B_n = \lim_{t\to 0}\left(\frac{d^n}{dt^n}\frac{t}{e^t-1}\right). \tag{7}$$

Figure 1 shows the behavior of some derivatives of the function $f(t) = t/\left(e^t-1\right)$ near the Bernoulli Numbers. The first column of Figure 1 shows the function, the second column have the graphs shown in the interval $[-10, 10]$ and the third column have been changed to reduced intervals to visualize the behavior of the function next to the B_n corresponding. We can observe the fluctuation of functions representing those derivatives of $t/\left(e^t-1\right)$ near, respectively, to the three first Bernoulli Numbers $B_0 = 1$, $B_1 = -1/2$ and $B_2 = 1/6$. Considering some of the derivatives of $t/\left(e^t-1\right)$and rewriting the numerator such that we have first the terms containing te^t and then the terms containing e^t, we have the following results:

$$\frac{d^0}{dt^0}\frac{t}{e^t-1} = \frac{t}{e^t-1},$$

$$\frac{d}{dt}\frac{t}{e^t-1} = \frac{-te^t+e^t-1}{\left(e^t-1\right)^2},$$

$$\frac{d^2}{dt^2}\frac{t}{e^t-1} = \frac{te^{2t}+te^t-2e^{2t}+2e^t}{\left(e^t-1\right)^3},$$

$$\frac{d^3}{dt^3}\frac{t}{e^t-1} = \frac{-te^{3t}-4te^{2t}-te^t+3e^{3t}-3e^t}{\left(e^t-1\right)^4},$$

$$\frac{d^4}{dt^4}\frac{t}{e^t-1} = \frac{te^{4t}+11te^{3t}+11te^{2t}+te^t-4e^{4t}-12e^{3t}+12e^{2t}+4e^t}{\left(e^t-1\right)^5},$$

$$\frac{d^5}{dt^5}\frac{t}{e^t-1} = \frac{-te^{5t}-26te^{4t}-66te^{3t}-26te^{2t}-te^t+5e^{5t}+50e^{4t}-50e^{2t}-5e^t}{\left(e^t-1\right)^6}.$$

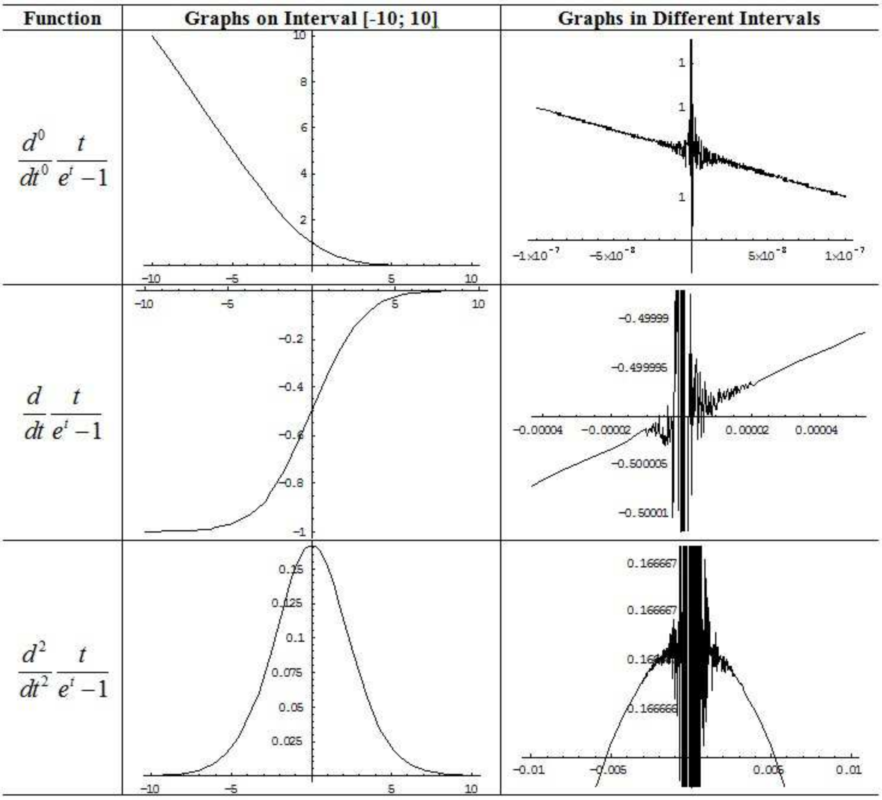

Figure 1: Graphs of derivatives of $t/\left(e^t-1\right)$ at different intervals.

We can observe that the nth derivative of the function $t/\left(e^t-1\right)$ can be represented as follows:

$$\frac{d^n}{dt^n}\frac{t}{e^t-1}=\frac{\sum\limits_{i=1}^{n}c_i te^{n-(i-1)t}+\sum\limits_{i=1}^{n}d_i e^{n-(i-1)t}}{\left(e^t-1\right)^{n+1}}. \tag{8}$$

Thus, we have the sum of the terms containing the product $te^{n-(i-1)t}$ given by:

$$\sum_{i=1}^{n}c_i te^{n-(i-1)t}=c_1te^{nt}+c_2te^{(n-1)t}+c_3te^{(n-2)t}+\ldots+c_nte^t, \tag{9}$$

in the case that we have only exponential portions $e^{n-(i-1)t}$ the sum may be represented as:

$$\sum_{i=1}^{n} d_i e^{n-(i-1)t} = d_1 e^{nt} + d_2 e^{(n-1)t} + d_3 e^{(n-2)t} + \ldots + d_n e^t. \tag{10}$$

From the results obtained, can be seen that in the case of c_i coefficients, we have the following relations:

$$\begin{aligned} |c_1| =& |c_n| = 1, \\ |c_2| =& |c_{n-1}|, \\ |c_3| =& |c_{n-2}|, \\ & \vdots \end{aligned}$$

Here —c_i— describes the absolute value of the coefficient c_i.

In the first derivative of $t/\left(e^t-1\right)$ the numerator is equal to $-te^t + e^t - 1$ and is the only case in which we have a constant, in this case -1, which is not multiplied by $e^{n-(i-1)t}$ or $te^{n-(i-1)t}$ where $n-(i-1) \neq 0$. This symmetry of values can also be observed in the case of coefficients d_i, allowing writing when $n \geq 3$, that:

$$\begin{aligned} |d_1| =& |d_n| = n, \\ |d_2| =& |d_{n-1}|, \\ |d_3| =& |d_{n-2}|, \\ & \vdots \end{aligned}$$

Here —d_i— describes the absolute value of the coefficient d_i.

When the order n of the derivative is odd and $n \geq 3$ we can verify that $d_{\frac{n+1}{2}} = 0$. Another feature that can also be extract is the fact that d_i is multiple of n.

From the derivatives of of $t/\left(e^t-1\right)$ we can construct a matrix containing only the signals, obtained from each of the portions of the numerators, shown in Table 1.

In Table 1, the first columns shows the order of the derivative of $t/\left(e^t-1\right)$, lines contain only the signals of the numerators, for example, the fourth-order derivative is:

$$\left(te^{4t} + 11te^{3t} + 11te^{2t} + te^t - 4e^{4t} - 12e^{3t} + 12e^{2t} + 4e^t\right)/\left(e^t-1\right)^5$$

0	+															
1	–	+	–													
2	+	+	–	+												
3	–	–	–	+	–											
4	+	+	+	+	–	–	+	+								
5	–	–	–	–	–	+	+	–	–							
6	+	+	+	+	+	+	–	–	–	+	+	+				
7	–	–	–	–	–	–	–	+	+	+	–	–	–			
8	+	+	+	+	+	+	+	+	–	–	–	–	+	+	+	+

Table 1: Matrix of signals of the numerator coefficients in the derivatives of $t/\left(e^t-1\right)$

such that the signals of the coefficients c_i and d_i, are given by $+,+,+,+,-,-,+$ and $+$. On the line that corresponds to the first derivative, the last signal belongs to the constant -1. The signals from the numerators of the coefficients obtained from derivatives of the function $t/\left(e^t-1\right)$, can be obtained for $n \geq 2$ using the expression:

$$\sum_{i=1}^{n} (-1)^n \left[t + sgn\left(\frac{n+1}{2} - i\right)\right] e^{it}, \tag{11}$$

being $sgn\,(x)$ the sign function given by:

$$sgn\,(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0. \end{cases}$$

Using the steps above and knowing that the sequence of signals for the coefficients $c_i^{(n)}$ and $d_i^{(n)}$ can be obtained from the expression (11), with results presented in Tables 2 and 3. The numbers presented in Table 2 are called Eulerian Numbers and the matrix formed by these numbers is called Euler's Triangle. A numeric matrix presented in Table 3 is built from the Euler's Triangle.

In Tables 2 and 3 we have that $\left|c_i^{(n)}\right|$and $\left|d_i^{(n)}\right|$ represents, respectively, the absolute value of the coefficient c_i and d_i from the nth derivative of the function $t/\left(e^t-1\right)$.

n	$\|c_1\|$	$\|c_2\|$	$\|c_3\|$	$\|c_4\|$	$\|c_5\|$	$\|c_6\|$	$\|c_7\|$	$\|c_8\|$	$\|c_9\|$	$\|c_{10}\|$	$\|c_{11}\|$	$\|c_{12}\|$	$\|c_{13}\|$	$\|c_{14}\|$
1	1													
2	1	1												
3	1	4	1											
4	1	11	11	1										
5	1	26	66	26	1									
6	1	57	302	302	57	1								
7	1	120	1191	2416	1191	120	1							
8	1	247	4293	15619	15619	4293	247	1						
9	1	502	14608	88234	156190	88234	14608	502	1					
10	1	1013	47840	455192	1310354	1310354	455192	47840	1013	1				
11	1	2036	152637	2203488	9738114	15724248	9738114	2203488	152637	2036	1			
12	1	4083	478271	10187685	66318474	162512286	162512286	66318474	10187685	478271	4083	1		
13	1	8178	1479726	45533450	423281535	1505621508	2275172004	1505621508	423281535	45533450	1479726	8178	1	
14	1	16369	4537314	198410786	2571742175	12843262863	27971176092	27971176092	12843262863	2571742175	198410786	4537314	16369	1

Table 2: Coefficients $\left|c_i^{(n)}\right|$

n	$\lvert d_1\rvert$	$\lvert d_2\rvert$	$\lvert d_3\rvert$	$\lvert d_4\rvert$	$\lvert d_5\rvert$	$\lvert d_6\rvert$	$\lvert d_7\rvert$	$\lvert d_8\rvert$	$\lvert d_9\rvert$	$\lvert d_{10}\rvert$	$\lvert d_{11}\rvert$	$\lvert d_{12}\rvert$	$\lvert d_{13}\rvert$	$\lvert d_{14}\rvert$
1	1													
2	2	2												
3	3	0	3											
4	4	12	12	4										
5	5	50	0	50	5									
6	6	150	240	240	150	6								
7	7	392	1715	0	1715	392	7							
8	8	952	8568	9800	9800	8568	952	8						
9	9	2214	36414	101934	0	101934	36414	2214	9					
10	10	5010	141060	736260	679560	679560	736260	141060	5010	10				
11	11	11132	515097	4480872	9406782	0	9406782	4480872	515097	11132	11			
12	12	24420	1807212	24610212	90415512	71833608	71833608	90415512	24610212	1807212	24420	12		
13	13	53066	6164444	126222382	729700257	1250519556	0	1250519556	729700257	126222382	6164444	53066	13	
14	14	114478	20601672	616752136	5288473190	15152759622	10773706944	10773706944	15152759622	5288473190	616752136	20601672	114478	14

Table 3: Coefficients $\left|d_i^{(n)}\right|$

3. Limit Representation of B_n using $E_{n,m}$

Definition 3.1. The numbers $E_{n,m}$ of orderings $(1,2,3,\ldots,n)$ with increasing m sequences are called Eulerian Numbers and obtained by the expresion:

$$E_{n,m}=\sum_{k=1}^{m}(-1)^{m-k}\begin{pmatrix} n+1 \\ m-k \end{pmatrix}k^n, m=1,2,3,\ldots \tag{12}$$

Proposition 3.1. Let B_n a Bernoulli Number and $E_{n,m}$ a Eulerian Number, we have that B_n can be represented from $E_{n,m}$ through the following limit:

$$B_n=\lim_{t\to 0}\frac{\sum_{i=1}^{n}(-1)^n\left[tE_{n,i}+n\left(E_{n-1,i}-E_{n-1,i-1}\right)\right]e^{it}}{\left(e^t-1\right)^{n+1}}, n>1. \tag{13}$$

Proof. From the results obtained in the previous section we can write the nth derivative of the function $t/\left(e^t-1\right)$ as follows:

$$\frac{d^n}{dt^n}\frac{t}{e^t-1}=\frac{\sum_{i=1}^{n}(-1)^n\left[c_it+d_isgn\left(\frac{n+1}{2}-i\right)\right]e^{it}}{\left(e^t-1\right)^{n+1}} \tag{14}$$

knowing that in expression (12) when m is zero we have $E_{n,0}=0$ and in (14) making the substitutions

$$c_i=E_{n,i} \tag{15}$$

and

$$d_isgn\left(\frac{n+1}{2}-i\right)=n\left(E_{n-1,i}-E_{n-1,i-1}\right), \tag{16}$$

enables the construction of the following expression for the nth derivative of the function $t/\left(e^t-1\right)$:

$$\frac{d^n}{dt^n}\frac{t}{e^t-1}=\frac{\sum_{i=1}^{n}(-1)^n\left[tE_{n,i}+n\left(E_{n-1,i}-E_{n-1,i-1}\right)\right]e^{it}}{\left(e^t-1\right)^{n+1}}, n>1 \tag{17}$$

from where the result (13) follows.

Example. Consider the Bernoulli number B_{14} which is equal to 7/6, using the expression (13) we have the following result:

$$B_{14}=\lim_{t\to 0}\frac{\sum_{i=1}^{14}(-1)^{14}\left[tE_{14,i}+n\left(E_{14-1,i}-E_{14-1,i-1}\right)\right]e^{it}}{\left(e^t-1\right)^{14+1}}$$

$$=\lim_{t\to 0}\frac{\sum_{i=1}^{14}(-1)^{14}\left[t\sum_{k=1}^{m}(-1)^{m-k}\begin{pmatrix} 14+1 \\ m-k \end{pmatrix}k^{14}+\right.}{\left(e^t-1\right)^{14+1}}$$

$$\frac{+14\left(\sum_{k=1}^{m}(-1)^{m-k}\binom{14-1+1}{m-k}k^{14-1}+\right.}{\left(e^{t}-1\right)^{14+1}}$$
$$\frac{\left.\left.-\sum_{k=1}^{m}(-1)^{m-1-k}\binom{14-1+1}{m-1-k}k^{14-1}\right)\right]e^{it}}{\left(e^{t}-1\right)^{14+1}}.$$

yielding $B_{14}=\frac{7}{6}$.

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