International Journal of Pure and Applied Mathematics
Volume 84 No. 1 2013, 159-169
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v84i1.12

THE w-MODULAR FUNCTION AND THE EVALUATION OF ROGERS RAMANUJAN CONTINUED FRACTION

Nikolaos Bagis
Department of Informatics
Aristotle University of Thessaloniki
Thessaloniki, Greece

Abstract: In previous article we have considered attached to elliptic singular moduli k_r, the parameter w_r, in which if one knows its value then can evaluate explicit the elliptic singular moduli and the fifth degree singular moduli k_{25r} in radicals. In the present article we give modular equations of this parameter w_r and a table of some lower values and use this parameter to solve the Rogers-Ramanujan continued fraction.

AMS Subject Classification: 11Y65, 11A55, 35J75, 33E05
Key Words: modular equations, singular moduli, continued fractions, Ramanujan, algebraic equations, algebraic numbers, elliptic functions

1. Introduction

In ([2],[3],[5]) we have considered the function $w_r = \sqrt{k_r k_{25r}}$. Also it have been proved that if one knows w_r then can find in radicals the value of k_r and k_{25r}.

Values of k_{25r} are useful in the construction of π formulas as also for the evaluation of the Rogers-Ramanujan continued fraction-(RRCF). Here we consider the case of RRCF.

Ramanujan was the first who consider and study systematically these theories. For the Ramanujan's results one can see [8], [9], [10], [11].

Received: January 14, 2013

We note to the unfamiliar reader that the elliptic integral of the first kind K and the singular moduli k_r are defined respectively as (see [1],[19],[16])

$$K(x) = \int_0^{\pi/2} \frac{dt}{\sqrt{1-x^2\sin(t)^2}} = \frac{\pi}{2} {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;x^2\right) \tag{1}$$

and

$$\frac{K\left(\sqrt{1-k_r^2}\right)}{K(k_r)} = \sqrt{r} \tag{2}$$

also when r is positive rational then k_r is algebraic.

The RRCF is defined as

$$R(q) = \frac{q^{1/5}}{1+}\frac{q}{1+}\frac{q^2}{1+}\frac{q^3}{1+}\cdots \tag{3}$$

where $q = e^{-\pi\sqrt{r}}$, $r > 0$. For this it have been proved that (see [2],[5]):

$$R(q^2)^{-5} - 11 - R(q^2)^5 = \frac{(k_r k_r')^2}{(w_r w_r')^2}\left(\frac{w_r}{k_r} + \frac{w_r'}{k_r'} - \frac{w_r w_r'}{k_r k_r'}\right)^3 \tag{4}$$

Hence evaluating $R(q)$ it is better to know w_r.

For the case of π formulas except of k_r we need and another function called elliptic alpha function $a(r)$ which have been defined and treated in [14] see also [6]. The evaluation of k_r and k_{25r}, can formulated as follows (see [2], [3], [5]).

Assume that we know $w = w_r$, then solve the equation

$$w = \sqrt{\frac{L(18+L)}{6(64+3L)}} \tag{5}$$

then set

$$M = \frac{18+L}{64+3L} \tag{6}$$

It have been shown that

$$\frac{\sqrt{k_{25r}}}{\sqrt{w}} = \frac{\sqrt{w}}{\sqrt{k_r}} = \frac{1}{2}\sqrt{4+\frac{2}{3}\left(\frac{L^{1/6}}{M^{1/6}} - 4\frac{M^{1/6}}{L^{1/6}}\right)^2} +$$

$$+\frac{1}{2}\sqrt{\frac{2}{3}}\left(\frac{L^{1/6}}{M^{1/6}} - 4\frac{M^{1/6}}{L^{1/6}}\right) \tag{7}$$

2. Lower Modular Equations of w

We set the function $A(x)$ as

$$A(w) := \frac{S}{\sqrt{6}} + \frac{\sqrt{4+\frac{2}{3}S^2}}{2} \tag{8}$$

where

$$S = \frac{L^{1/6}}{M^{1/6}} - 4\frac{M^{1/6}}{L^{1/6}} \tag{9}$$

and the L, M are that of (5) and (6).

Hence

$$k_r = \frac{w_r}{A(w_r)^2} \tag{10}$$

Theorem 1. *The modular equations who relates w_{4r} with w_r (duplication formula) are*

$$w_{4r} = \sqrt{\frac{\left(1-\sqrt{1-\frac{w_r^2}{A(w_r)^4}}\right)\left(1-\sqrt{1-A(w_r)^4w_r^2}\right)}{\left(1+\sqrt{1-\frac{w_r^2}{A(w_r)^4}}\right)\left(1+\sqrt{1-A(w_r)^4w_r^2}\right)}} \tag{11}$$

$$w_{r/4} = \frac{4A(w_r)\sqrt{w_r}}{A(w_r)+\sqrt{w_r}+A(w_r)^2\sqrt{w_r}+A(w_r)w_r}. \tag{12}$$

Proof.

$$w_{4r} = \sqrt{k_{4r}k_{100r}} \tag{13}$$

but

$$k_{4r} = \frac{1-k'_r}{1+k'_r} \tag{14}$$

Hence

$$w_{4r}^2 = \frac{1-k'_r-\frac{(w'_r)^2}{k'_r}+w'_r}{1+k'_r+\frac{(w'_r)^2}{k'_r}+w'_r} \tag{15}$$

hence using (10) we get (11). For to prove (12) we have

$$k_r = \frac{4\sqrt{k_{4r}}}{(1+\sqrt{k_{4r}})^2} \tag{16}$$

From

$$w_r^2 = k_r k_{25r} \tag{17}$$

using (15) and (10) in (17) we get the desired result.

For the fifth degree modular equation of w_r we have the next

Theorem 2.

$$\left(w_{25r}w'_{25r}\right)^2=\frac{w_r^2\left(A(w_r)^4-w_r^2\right)}{A(w_r)^8}H\left(\frac{A(w_r)^8\left(w_rw'_r\right)^2}{w_r^2\left(A(w_r)^4-w_r^2\right)}\right)^2\times$$

$$\times H\left(\frac{w_r^2\left(A(w_r)^4-w_r^2\right)}{A(w_r)^8\left(w_{r/25}w'_{r/25}\right)^2}\right). \tag{18}$$

Proof. If $r\in\mathbf{R}_+^*$ set H to be the function (see relation (20) below), such that

$$\frac{k_{25r}k'_{25r}}{k_rk'_r}=H\left(\frac{k_rk'_r}{k_{r/25}k'_{r/25}}\right) \tag{19}$$

Then

$$\left(w_{25r}w'_{25r}\right)^2=\frac{k_{25^2r}k'_{25^2r}k_{25r}k'_{25r}}{k_{25r}k'_{25r}k_rk'_r}k_{25r}k'_{25r}k_rk'_r=$$

$$=k_{25r}k'_{25r}k_rk'_rH\left(\frac{k_{25r}k'_{25r}}{k_rk'_r}\right)H\left(\frac{k_rk'_r}{k_{r/25}k'_{r/25}}\right)=$$

$$=\frac{k_{25r}k'_{25r}}{k_rk'_r}(k_rk'_r)^2H\left(\frac{(w_rw'_r)^2}{(k_rk'_r)^2}\right)H\left(\frac{(k_rk'_r)^2}{(w_{r/25}w'_{r/25})^2}\right)=$$

$$=(k_rk'_r)^2H\left(\frac{(w_rw'_r)^2}{(k_rk'_r)^2}\right)^2H\left(\frac{(k_rk'_r)^2}{\left(w_{r/25}w'_{r/25}\right)^2}\right)$$

But

$$\left(k_rk'_r\right)^2=\frac{w_r^2\left(A(w_r)^4-w_r^2\right)}{A(w_r)^8}$$

From the above we get the desired result. The function H in view of [4] is given as

$$H(x)=U\left(Q\left(U^*\left(x^{1/6}\right)^6\right)^{1/6}\right)^6 \tag{20}$$

where

$$Q(x)=\frac{\left(-1-e^{\frac{1}{5}y}+e^{\frac{2}{5}y}\right)^5}{\left(e^{\frac{1}{5}y}-e^{\frac{2}{5}y}+2e^{\frac{3}{5}y}-3e^{\frac{4}{5}y}+5e^{y}+3e^{\frac{6}{5}y}+2e^{\frac{7}{5}y}+e^{\frac{8}{5}y}+e^{\frac{9}{5}y}\right)} \tag{21}$$

and

$$y = \operatorname{arcsinh}\left(\frac{11+x}{2}\right).$$

$$U(x) = \sqrt{-\frac{5}{3x^2} + \frac{25}{3x^2 h(x)} + \frac{x^4}{h(x)} + \frac{h(x)}{3x^2}} \tag{22}$$

where

$$h(x) = \left(-125 - 9x^6 + 3\sqrt{3}\sqrt{-125x^6 - 22x^{12} - x^{18}}\right)^{1/3} \tag{23}$$

$$U^*(x) = \sqrt{-\frac{1}{2x^2} + \frac{x^4}{2} + \frac{\sqrt{1+18x^6+x^{12}}}{2x^2}}. \tag{24}$$

In general assume that we have an arbitrary n-th degree modular equation of the singular moduli k_r, such that

$$f\left(k_r, k_{n^2 r}\right) = \text{const}, \forall r > 0 \tag{25}$$

then

$$f\left(k_{25r}, k_{25n^2 r}\right) = \text{const} \tag{26}$$

using now (10) we get

$$f\left(\frac{w_r^2 A(w_r)^2}{w_r}, \frac{w_{n^2 r}^2 A(w_{n^2 r})^2}{w_{n^2 r}}\right) = \text{const} \tag{27}$$

$$f\left(w_r A(w_r)^2, w_{n^2 r} A(w_{n^2 r})^2\right) = \text{const} \tag{28}$$

Hence

Theorem 3. *If the n-th degree modular equation of k_r is*

$$f\left(k_r, k_{n^2 r}\right) = const \tag{29}$$

then

$$f\left(w_r A(w_r)^2, w_{n^2 r} A(w_{n^2 r})^2\right) = const \tag{30}$$

and

$$f\left(\frac{w_r}{A(w_r)^2}, \frac{w_{n^2 r}}{A(w_{n^2 r})^2}\right) = const. \tag{31}$$

Theorem 3 gives us the general case of the reduction of a modular equation of k_r to that of w_r. As someone can see it is very complicated taking in mind that we have the argument of $w_{n^2 r}$ inside $A(x)$. However there exist a straight forward evaluation, if one knows the solution of either side of k_r or k_{n^2r}. More precisely we have

Theorem 4. *Suppose that*

$$k_{n^2r} = f(k_r) \tag{32}$$

then

$$w_{n^2r}^2 = f\left(w_r A(w_r)\right) f\left(\frac{w_r}{A(w_r)^2}\right). \tag{33}$$

Proof. From (32) we get

$$k_{25n^2r} = f(k_{25r})$$

Hence

$$\frac{w_{n^2r}^2}{k_{n^2r}} = f\left(\frac{w_r^2}{k_r}\right)$$

or

$$w_{n^2r}^2 = f\left(\frac{w_r^2}{k_r}\right) k_{n^2r}$$

or

$$w_{n^2r}^2 = f\left(\frac{w_r^2 A(w_r)^2}{w_r}\right) f(k_r)$$

or

$$w_{n^2r}^2 = f\left(w_r A(w_r)\right) f\left(\frac{w_r}{A(w_r)^2}\right)$$

which is the desired result.

As application one can get the evaluation of w_{9r} which follows from the solvable equation (see [10], [3]):

$$\sqrt{k_r k_{9r}} + \sqrt{k_r' k_{9r}'} = 1 \tag{34}$$

or equivalently

$$u^4 - 256uv + 384u^2v - 132u^3v + 384uv^2 - 762u^2v^2 + 384u^3v^2 -$$
$$-132uv^3 + 384u^2v^3 - 256u^3v^3 + v^4 = 0 \tag{35}$$

where $u = k_{9r}^2$, $v = k_r^2$.

We call the solution of (35) T, then $u = T(v)$ and form Theorem 4 we get

Theorem 5. *If* $f(x) = \sqrt{T(x^2)}$ *then*

$$w_{9r}^2 = f\left(w_r A(w_r)\right) f\left(\frac{w_r}{A(w_r)^2}\right). \tag{36}$$

3. The Rogers-Ramanujan Continued Fraction and the w-Moduli

From the above discussion and identities (4),(7),(8) and (10) it is clear that RRCF can be put in the form

$$R(q^2)^{-5} - 11 - R(q^2)^5 = G(w_r), \tag{37}$$

Hence it is better to describe $R(q)$ using the w-moduli and not the classical elliptic singular moduli k_r.

Theorem 6. *If* $q = e^{-\pi\sqrt{r}}$, *then*

$$R(q^2)^{-5} - 11 - R(q^2)^5 = \frac{\left(\sqrt{A(w_r)^4 - w_r^2} + w_r' - A(w_r)^2 w_r'\right)^3}{(A(w_r)w_r')^2\sqrt{A(w_r)^4 - w_r^2}} \tag{38}$$

and

$$w_r' = \sqrt{k_r' k_{25r}'} = \sqrt{\sqrt{1 - \frac{w_r^2}{A(w_r)^4}}\sqrt{1 - A(w_r)^4 w_r^2}}. \tag{39}$$

As someone can see the modular equations of the above theorems, for example that of two degree, are very complicated. They give us the desired result, but the output is a rough evaluation rather an elegant simplified form. In literature the existing evaluations are special values and not straight forward general evaluations. One can compare them with the modular equations given by Ramanujan for the RRCF (see [10],[11],[13]), which are also, give not elegant results.

Note also that the evaluation (18) of w_{25r} as also of w_{4r} and $w_{r/4}$ can be done using existing equations. For example to evaluate $w_{4r} = \sqrt{k_{4r}k_{100r}}$ we first evaluate k_{4r} and k_{100r}, then from (14) we evaluate w_{4r}. But this happens only if we know k_r and k_{25r}. Here with the knowledge of one value only (that of w_r) we evaluate RRCF successfully and both k_r and k_{25r}.

4. Table of w_r

For $r = 2/35$

$$w_{2/35} = -168 + 45\sqrt{14} + 5\sqrt{2245 - 600\sqrt{14}} \tag{40}$$

For $r = 3/20$

$$w_{3/20} = 2\sqrt{-8304 + 2626\sqrt{10} - 2\sqrt{30\left(1149197 - 363408\sqrt{10}\right)}} \tag{41}$$

For $r = 13/10$

$$w_{13/10}^2 = -12\sqrt{15673137249530 - 1374625071240\sqrt{130}} +$$

$$+34\left(-988019 + 86655\sqrt{130}\right) \tag{42}$$

For $r = 7/10$

$$w_{7/10} =$$

$$\sqrt{-108096 + 28890\sqrt{14} - 2\sqrt{35\left(166925663 - 44612760\sqrt{14}\right)}} \tag{43}$$

For $r = 3/10$

$$w_{3/10} = \sqrt{-416 + 170\sqrt{6} - 2\sqrt{86565 - 35340\sqrt{6}}} \tag{44}$$

For $r = 1/30$

$$w_{1/30} = \sqrt{2\left(-208 + 85\sqrt{6} + \sqrt{86565 - 35340\sqrt{6}}\right)} \tag{45}$$

For $r = 1/25$

$$w_{1/25} = \sqrt{-3 + \frac{3\sqrt{5}}{2} + \sqrt{-20 + 9\sqrt{5}}} \tag{46}$$

For $r = 1/20$

$$w_{1/20} = 2\sqrt{-14 - 6\sqrt{5} + \sqrt{380 + 170\sqrt{5}}} \tag{47}$$

For $r = 1/15$

$$w_{1/15} = \frac{\sqrt{4 + \sqrt{15}}}{4} \tag{48}$$

For $r = 1/10$

$$w_{1/10} = \sqrt{2\left(-3 + \sqrt{10}\right)} \tag{49}$$

For $r = 1/5$

$$w_{1/5} = \sqrt{\frac{1}{2}\left(-2+\sqrt{5}\right)} \tag{50}$$

For $r = 2/5$

$$w_{2/5} = -3+\sqrt{10} \tag{51}$$

For $r = 3/5$

$$w_{3/5} = \frac{\sqrt{4-\sqrt{15}}}{4} \tag{52}$$

For $r = 4/5$

$$w_{4/5} = 5+2\sqrt{5}-2\sqrt{11+5\sqrt{5}} \tag{53}$$

For $r = 5/5 = 1$

$$w_1 = \frac{1}{\sqrt{2}\left(51841+23184\sqrt{5}+12\sqrt{37325880+16692641\sqrt{5}}\right)^{1/4}} \tag{54}$$

For $r = 6/5$

$$w_{6/5} = -12+5\sqrt{6}-\sqrt{5\left(49-20\sqrt{6}\right)} \tag{55}$$

For $r = 8/5$

$$w_{8/5} = 25+8\sqrt{10}-4\sqrt{79+25\sqrt{10}} \tag{56}$$

For $r = 9/5$

$$w_{9/5} = 25+8\sqrt{10}-4\sqrt{79+25\sqrt{10}} \tag{57}$$

For $r = 11/5$

$$w_{11/5} = \frac{1}{4\sqrt{1194+360\sqrt{11}+\sqrt{2851435+859740\sqrt{11}}}} \tag{58}$$

For $r = 12/5$

$$w_{12/5} = 95-30\sqrt{10}-2\sqrt{6\left(721-228\sqrt{10}\right)} \tag{59}$$

For $r = 15/5 = 3$

$$w_{14/5} = -168+45\sqrt{14}-5\sqrt{2245-600\sqrt{14}} \tag{60}$$

For $r=17/5$

$$w_{17/5}=\sqrt{-41+\frac{9\sqrt{85}}{2}-6\sqrt{5\left(-378+41\sqrt{85}\right)}} \tag{61}$$

For $r=18/5$

$$w_{18/5}=u \tag{62}$$

where

$$1/u-u=2\left(240+98\sqrt{6}-5\sqrt{5\left(921+376\sqrt{6}\right)}\right) \tag{63}$$

For $r=20/5=4$

$$v+1/v=6\left(69121+30912\sqrt{5}+16\sqrt{37325880+16692641\sqrt{5}}\right) \tag{64}$$

$$w_4=\sqrt{(3-2\sqrt{2})v} \tag{65}$$

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