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A NEW BOUND ON POISSON APPROXIMATION FOR INDEPENDENT GEOMETRIC VARIABLES

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Abstract: The Stein-Chen method is used to improve the bound in [3] to be more appropriate for measuring the accuracy of Poisson approximation with mean $\lambda = \sum_{i=1}^{n} q_i$ for a sum of independently distributed geometric random variables.

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1. Introduction

Let $X_1, ..., X_n$ be n independently distributed geometric random variables, each with probability $P(X_i = k) = (1-p_i)^k p_i$ for $k = 0, 1, ...$, and let $W = \sum_{i=1}^{n} X_i$. If p_i's are identical to p, then W has the negative binomial distribution with parameters $n \in \mathbb{N}$ and p. It is well-known that if all $q_i = (1-p_i)$ are small, the distribution of W can be approximated by the Poisson distribution with mean $\lambda = E(W) = \sum_{i=1}^{n} q_i p_i^{-1}$. Correspondingly, the distribution function of W can also be approximated by the Poisson distribution function with mean λ. Let

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$\mathbb{PW}(w_0) = P(W \le w_0)$ and $\mathbb{P}_\lambda(w_0) = \sum_{k=0}^{w_0} \frac{\lambda^k e^{-\lambda}}{k!}$ be the distribution function of W and the Poisson distribution function at $w_0 \in \mathbb{N} \cup \{0\}$, respectively. In this case, Teerapabolarn and Wongkasem [5] used the Stein-Chen method to give a non-uniform bound for the difference of $\mathbb{PW}(w_0)$ and $\mathbb{P}_\lambda(w_0)$ as follows:

$$|\mathbb{PW}(w_0) - \mathbb{P}_\lambda(w_0)| \le \lambda^{-1}(e^\lambda - 1)\sum_{i=1}^{n} \min\left\{1, \frac{1}{p_i(w_0+1)}\right\} q_i^2 p_i^{-1}. \tag{1.1}$$

Later, Teerapabolarn [3] used this method to obtain a better result of (1.1),

$$|\mathbb{PW}(w_0) - \mathbb{P}_\lambda(w_0)| \le \lambda^{-1}(e^\lambda - 1)\sum_{i=1}^{n} \min\left\{1, \frac{1}{p_i(w_0+1)}\right\} q_i^2, \tag{1.2}$$

for $\lambda = \sum_{i=1}^n q_i$ and $w_0 \in \mathbb{N} \cup \{0\}$. In this study, we focus on improving the non-uniform bound of (1.2) to be more appropriate for measuring the accuracy of this approximation.

2. Method

Stein's method was originally formulated for normal approximation by Stein [2]. It was adapted and applied to the Poisson case by Chen [1], which is refer to as the Stein-Chen method. Following Teerapabolarn [3], Stein's equation of the Poisson cumulative distribution function with parameter $\lambda > 0$ is of the form

$$h_{w_0}(w) - \mathbb{P}_\lambda(w_0) = \lambda f_{w_0}(w+1) - w f_{w_0}(w), \tag{2.1}$$

where $w_0, w \in \mathbb{N} \cup \{0\}$, and for $h_{w_0}(w) = 1$ if $w \le w_0$ and $h_{w_0}(w) = 0$ if $w > w_0$, the solution f_{w_0} is

$$f_{w_0}(w) = \begin{cases} (w-1)!\lambda^{-w}e^\lambda[\mathbb{P}_\lambda(w-1)[1-\mathbb{P}_\lambda(w_0)]] & \text{if } w \le w_0, \\ (w-1)!\lambda^{-w}e^\lambda[\mathbb{P}_\lambda(w_0)[1-\mathbb{P}_\lambda(w-1)]] & \text{if } w > w_0, \\ 0 & \text{if } w = 0. \end{cases} \tag{2.2}$$

Note that $f_{w_0}(w) \ge 0$ for every $x_0, x \in \mathbb{N} \cup \{0\}$. The following lemma gives a non-uniform bound of (2.2), which is used to determine the main result.

Lemma 2.1. *For $w_0 \in \mathbb{N} \cup \{0\}$ and $k \in \mathbb{N} \setminus \{1\}$, let $p_\lambda(w_0) = \frac{e^{-\lambda}\lambda^{w_0}}{w_0!}$. Then the following inequality holds:*

$$\sup_{w \ge k} f_{w_0}(w) \le \frac{\mathbb{P}_\lambda(w_0)(1-\mathbb{P}_\lambda(w_0))}{p_\lambda(w_0+1)} \min\left\{\frac{1}{w_0+1}, \frac{1}{k}\right\}. \tag{2.3}$$

Proof. The first bound of (2.3) is directly obtained from [4]. Thus, we shall show that the second bound of (2.3) holds. For $k \le w \le w_0$, we have

$$f_{w_0}(w) \le \frac{\mathbb{P}_\lambda(w_0)(1-\mathbb{P}_\lambda(w_0))}{p_\lambda(w_0+1)(w_0+1)} \le \frac{\mathbb{P}_\lambda(w_0)(1-\mathbb{P}_\lambda(w_0))}{p_\lambda(w_0+1)(k+1)}. \tag{2.4}$$

For $w > w_0$ and $w \ge k$, we obtain

$$\begin{aligned} f_{w_0}(w) &= \mathbb{P}_\lambda(w_0)(w-1)!\sum_{j=w}^{\infty}\frac{\lambda^{j-w}}{j!} \\ &= \mathbb{P}_\lambda(w_0)\left\{\frac{1}{w}+\frac{\lambda}{w(w+1)}+\frac{\lambda^2}{w(w+1)(w+2)}+\cdots\right\} \\ &\le \frac{\mathbb{P}_\lambda(w_0)(w_0+1)!}{w}\left\{\frac{1}{(w_0+1)!}+\frac{\lambda}{(w_0+2)!}+\frac{\lambda^2}{(w_0+3)!}+\cdots\right\} \\ &\le \frac{\mathbb{P}_\lambda(w_0)(w_0+1)!}{k}\sum_{j=w_0+1}^{\infty}\frac{\lambda^{j-(w_0+1)}}{j!} \\ &= \frac{\mathbb{P}_\lambda(w_0)(1-\mathbb{P}_\lambda(w_0))}{p_\lambda(w_0+1)k}. \end{aligned} \tag{2.5}$$

Hence, from (2.4) and (2.5), the second bound of (2.3) holds. □

3. Result

The following theorem presents a new non-uniform bound on the distance $|\mathbb{P}\mathbb{W}(w_0) - \mathbb{P}_\lambda(w_0)|$, which can be obtained by using the Stein-Chen method.

Theorem 3.1. *For $w_0 \in \mathbb{N}\cup\{0\}$, if $\lambda = \sum_{i=1}^{n} q_i$ then we have the following:*

$$|\mathbb{P}\mathbb{W}(w_0) - \mathbb{P}_\lambda(w_0)| \le \frac{\mathbb{P}_\lambda(w_0)(1-\mathbb{P}_\lambda(w_0))}{p_\lambda(w_0+1)}\sum_{i=1}^{n}\min\left\{\frac{1}{p_i(w_0+1)},1\right\}q_i^2. \tag{3.1}$$

Proof. Teerapabolarn[3] showed that

$$|\mathbb{P}\mathbb{W}(w_0) - \mathbb{P}_\lambda(w_0)| \le \sum_{i=1}^{n}\sum_{k\ge 2}(k-1)p_i q_i^k \sup_{w\ge k} f_{w_0}(w).$$

With Lemma 2.1, we have that

$$|\mathbb{P}\mathbb{W}(w_0) - \mathbb{P}_\lambda(w_0)| \le \frac{\mathbb{P}_\lambda(w_0)(1-\mathbb{P}_\lambda(w_0))}{p_\lambda(w_0+1)}\sum_{i=1}^{n}\sum_{k\ge 2}(k-1)p_i q_i^k \min\left\{\frac{1}{w_0+1},\frac{1}{k}\right\}$$

$$\le \frac{\mathbb{P}_\lambda(w_0)(1-\mathbb{P}_\lambda(w_0))}{p_\lambda(w_0+1)} \sum_{i=1}^{n} \min\left\{\frac{1}{p_i(w_0+1)}, 1\right\} q_i^2.$$

Hence, the theorem is proved. □

Remark. It is seen that $\frac{\mathbb{P}_\lambda(w_0)(1-\mathbb{P}_\lambda(w_0))}{p_\lambda(w_0+1)} \le \mathbb{P}_\lambda(w_0)\lambda^{-1}(e^\lambda-1) < \lambda^{-1}(e^\lambda-1)$ for every $w_0 \in \mathbb{N}\cup\{0\}$. Therefore, the bound in (3.1) is sharper than the bound in (1.2).

4. Conclusion

The non-uniform bound in the Theorem 3.1 was obtained by improving the bound in (1.2) using the Stein-Chen method. It provides a new general criteria for measuring the accuracy of Poisson approximation to the distribution function of a sum of independently distributed geometric random variables with Poisson mean $\lambda = \sum_{i=1}^{n} q_i$.

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