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ON APPLICATIONS OF DIFFERENTIAL SUBORDINATION

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Abstract: Using a generalized differential operator we define certain subclasses of analytic functions and study about their inclusion relationships using differential subordination.

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1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) := z + \sum_{k=2}^{\infty} a_k z^k, \qquad a_k \geq 0 \tag{1.1}$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. Let $S, S^*(\alpha)$ and $K(\alpha)$ denote the subclasses of $\mathcal{A}$ consisting of functions that are univalent, starlike of order α and convexlike of order α respectively. Also $S^*(0) = S^*$ and $K(0) = K$ are the class of starlike and convex functions defined on U respectively. For two functions $f(z)$ given by (1.1) and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$,

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the Hadamard product or convolution of f and g is denoted by $(f*g)(z)$, defined as

$$(f*g)(z) := z + \sum_{k=2}^{\infty} a_k b_k z^k.$$

For complex numbers $\alpha_1, \alpha_2, \ldots, \alpha_q$ and $\beta_1, \beta_2, \ldots, \beta_s; (\beta_j \in \mathbb{C} \setminus \mathcal{Z}_0^-; \mathcal{Z}_0^- = \{0, -1, -2, \ldots\}$ for $j = 1, 2, .., s)$, we define the generalized hypergeometric function as

$$ {}_qF_s(\alpha_1, \alpha_2, \ldots, \alpha_q; \beta_1, \beta_2, \ldots, \beta_s; z) := \sum_{k=0}^{\infty} \frac{(\alpha_1)_k (\alpha_2)_k \ldots (\alpha_q)_k z^k}{(\beta_1)_k (\beta_2)_k \ldots (\beta_s)_k k!},$$

$$(q \leq s+1; q, s \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}; z \in U)$$

where $\mathbb{N}$ denotes the set of all positive integers and $(x)_k$ is the Pochhammer symbol defined in terms of gamma function, as

$$(x)_k = \frac{\Gamma(x+k)}{\Gamma(x)} = \begin{cases} 1 & \text{if} \quad k = 0 \\ x(x+1)\ldots(x+k-1) & \text{if} \quad k \in \mathbb{N}. \end{cases}$$

Corresponding to the function $g_{q,s}(\alpha_1, \beta_1; z)$, defined by

$$g_{q,s}(\alpha_1, \beta_1; z) := z\, {}_qF_s(\alpha_1, \alpha_2, \ldots, \alpha_q; \beta_1, \beta_2, \ldots, \beta_s; z),$$

recently in [9] an operator $\mathcal{D}^m_{\lambda,\mu}(\alpha_1, \beta_1) f(z) : \mathcal{A} \to \mathcal{A}$ has been defined by

$$\begin{aligned}
\mathcal{D}^0_{\lambda,\mu}(\alpha_1, \beta_1) f(z) &:= f(z) * g_{q,s}(\alpha_1, \beta_1; z), \\
\mathcal{D}^1_{\lambda,\mu}(\alpha_1, \beta_1) f(z) &:= (1-\lambda+\mu)(f(z) * g_{q,s}(\alpha_1, \beta_1; z)) \\
&\quad +(\lambda-\mu) z (f(z) * g_{q,s}(\alpha_1, \beta_1; z))' \\
&\quad +\lambda\mu z^2 (f(z) * g_{q,s}(\alpha_1, \beta_1; z)^{''}, \\
\mathcal{D}^m_{\lambda,\mu}(\alpha_1, \beta_1) f(z) &:= \mathcal{D}^1_{\lambda,\mu}(\mathcal{D}^{m-1}_{\lambda,\mu}(\alpha_1, \beta_1) f(z)),
\end{aligned}$$

where $0 \leq \mu \leq \lambda \leq 1$ and $m \in \mathbb{N}_0$. By using the above definition, we can find that

$$\begin{aligned}
\mathcal{D}^m_{\lambda,\mu}(\alpha_1, \beta_1) f(z) &= z \\
&+ \sum_{k=2}^{\infty} [1 + (k-1)(\lambda - \mu + k\mu\lambda)]^m \frac{(\alpha_1)_{k-1} (\alpha_2)_{k-1} \ldots (\alpha_q)_{k-1}}{(\beta_1)_{k-1} (\beta_2)_{k-1} \ldots (\beta_s)_{k-1} (k-1)!} a_k z^k.
\end{aligned}$$

For brevity, let us take

$$B_k = \frac{(\alpha_1)_{k-1} (\alpha_2)_{k-1} \ldots (\alpha_q)_{k-1}}{(\beta_1)_{k-1} (\beta_2)_{k-1} \ldots (\beta_s)_{k-1} (k-1)!}.$$

Hence we have

$$\mathcal{D}^m_{\lambda,\mu}(\alpha_1,\beta_1)f(z) = z + \sum_{k=2}^{\infty}[1+(k-1)(\lambda-\mu+k\mu\lambda)]^m B_k a_k z^k.$$

For suitable values of $\alpha_i's, \beta_j's, q, s, \lambda$ and μ we can deduce several operators [1, 6, 14]as a special case of this operator. Also a simple computation shows that

$$(1-\gamma)\mathcal{D}^m_{\lambda,\mu}(\alpha_1,\beta_1)f(z)+\gamma z[\mathcal{D}^m_{\lambda,\mu}(\alpha_1,\beta_1)f(z)]' = \gamma\alpha_1\mathcal{D}^m_{\lambda,\mu}(\alpha_1+1,\beta_1)f(z) \quad (1.2)$$
$$-(\gamma\alpha_1-1)\mathcal{D}^m_{\lambda,\mu}(\alpha_1,\beta_1)f(z).$$

Let $f(z)$ and $g(z)$ be analytic in the unit disc U. Then $f(z)$ is said to be subordinate to $g(z)$ in U, if there exists a Schwarz function $w(z)$, analytic in U with $w(0)=0$ and $|w(z)|<1$ for $z\in U$, such that $f(z)=g(w(z))$. We denote it as $f\prec g$. Further if $g(z)$ is univalent then we write $f\prec g$ if $f(0)=g(0)$ and $f(U)\subset g(U)$.

Definition 1.1. Let $h(z)$ be an analytic convex univalent function in U with $h(0)=1$ and $\Re\{h(z)\}>0$ for $z\in U$. Let $A(\alpha_1,\beta_1,\gamma,\lambda,\mu,m,h)$ denote the subclass of $\mathcal{A}$ consisting of functions $f(z)$ which satisfy the condition

$$\frac{z[\mathcal{D}^m_{\lambda,\mu}(\alpha_1,\beta_1)f(z)]'+\gamma z^2[\mathcal{D}^m_{\lambda,\mu}(\alpha_1,\beta_1)f(z)]''}{(1-\gamma)\mathcal{D}^m_{\lambda,\mu}(\alpha_1,\beta_1)f(z)+\gamma z[\mathcal{D}^m_{\lambda,\mu}(\alpha_1,\beta_1)f(z)]'} \prec h(z)$$

for some $\gamma (0\le\gamma\le 1)$ and for all $z\in U$.

Definition 1.2. Let $h(z)$ be an analytic convex univalent function in U with $h(0)=1$ and $\Re\{h(z)\}>0$ for $z\in U$. Let $B(\alpha_1,\beta_1,\gamma,\lambda,\mu,m,h)$ denote the subclass of $\mathcal{A}$ consisting of functions $f(z)$ which satisfy the condition

$$(1-\gamma)\frac{\mathcal{D}^m_{\lambda,\mu}(\alpha_1,\beta_1)f(z)}{z}+\gamma[\mathcal{D}^m_{\lambda,\mu}(\alpha_1,\beta_1)f(z)]' \prec h(z)$$

for some $\gamma (0\le\gamma\le 1)$ and for all $z\in U$.

Definition 1.3. Let $h(z)$ be an analytic convex univalent function in U with $h(0)=1$ and $\Re\{h(z)\}>0$ for $z\in U$. Let $C(\alpha_1,\beta_1,\gamma,\lambda,\mu,m,h)$ denote the subclass of $\mathcal{A}$ consisting of functions $f(z)$ which satisfy the condition

$$[\mathcal{D}^m_{\lambda,\mu}(\alpha_1,\beta_1)f(z)]'+\gamma z[\mathcal{D}^m_{\lambda,\mu}(\alpha_1,\beta_1)f(z)]'' \prec h(z)$$

for some $\gamma (0\le\gamma\le 1)$ and for all $z\in U$.

Note that special cases of $A(\alpha_1, \beta_1, \gamma, \lambda, \mu, m, h), B(\alpha_1, \beta_1, \gamma, \lambda, \mu, m, h)$ and $C(\alpha_1, \beta_1, \gamma, \lambda, \mu, m, h)$ include $S^*, K, S^*(\alpha)$, $K(\alpha)$ and many other subclasses of $\mathcal{A}$ which were defined earlier in [11, 12, 13]. By specializing the parameters we get the corresponding classes containing Hohlov operator, Ruscheweyh operator, fractional calculus operator, Sălăgean derivative operator, Libera-Bernardi-Livingston integral operator, Dziok-Srivatsava operator and the operator studied in [15].

2. Preliminaries

To prove our main results we need the following lemmas.

Lemma 2.1. *[10, p. 81] Let h be analytic, univalent and convex in U with $h(0) = 1$ and $\Re\{\beta h(z) + \gamma\} > 0, (\beta, \gamma \in \mathbb{C}, z \in U)$. If p is analytic in U with $p(0) = h(0)$, then*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z)$$

implies that

$$p(z) \prec h(z) \qquad (z \in U).$$

Lemma 2.2. *[10, p. 71] Let h be analytic, univalent and convex in U with $h(0) = 1$. Also let p be analytic in U with $p(0) = h(0)$. If*

$$p(z) + \frac{zp'(z)}{\gamma} \prec h(z)$$

then

$$p(z) \prec q(z) \prec h(z),$$

where

$$q(z) = \frac{\gamma}{z^\gamma} \int_0^z t^{\gamma-1} h(t)dt \qquad (z \in U, \Re\{\gamma\} \geq 0; \gamma \neq 0\}.$$

3. Inclusion Relations

Theorem 3.1. *For $\alpha_1 \geq 1$,*

$$A(\alpha_1 + 1, \beta_1, \gamma, \lambda, \mu, m, h) \subset A(\alpha_1, \beta_1, \gamma, \lambda, \mu, m, h).$$

Proof. Let $f(z) \in A(\alpha_1 + 1, \beta_1, \gamma, \lambda, \mu, m, h)$ and let

$$p(z) := \frac{z[\mathcal{D}^m_{\lambda,\mu}(\alpha_1, \beta_1)f(z)]' + \gamma z^2[\mathcal{D}^m_{\lambda,\mu}(\alpha_1, \beta_1)f(z)]''}{(1-\gamma)\mathcal{D}^m_{\lambda,\mu}(\alpha_1, \beta_1)f(z) + \gamma z[\mathcal{D}^m_{\lambda,\mu}(\alpha_1, \beta_1)f(z)]'}.$$

By differentiating (1.2) we have,

$$p(z) + (a-1) = \frac{\gamma\alpha_1 z[\mathcal{D}^m_{\lambda,\mu}(\alpha_1 + 1, \beta_1)f(z)]' + (1-\gamma)\alpha_1\mathcal{D}^m_{\lambda,\mu}(\alpha_1, \beta_1)f(z)}{(1-\gamma)\mathcal{D}^m_{\lambda,\mu}(\alpha_1, \beta_1)f(z) + \gamma z[\mathcal{D}^m_{\lambda,\mu}(\alpha_1, \beta_1)f(z)]'}.$$

Taking logarithmic differentiation on both sides we get

$$\begin{aligned} p(z) &+ \frac{zp'(z)}{p(z) + (\alpha_1 - 1)} \\ &= \frac{z[\mathcal{D}^m_{\lambda,\mu}(\alpha_1 + 1, \beta_1)f(z)]' + \gamma z^2[\mathcal{D}^m_{\lambda,\mu}(\alpha_1 + 1, \beta_1)f(z)]''}{(1-\gamma)\mathcal{D}^m_{\lambda,\mu}(\alpha_1 + 1, \beta_1)f(z) + \gamma z[\mathcal{D}^m_{\lambda,\mu}(\alpha_1 + 1, \beta_1)f(z)]'}. \end{aligned}$$

As $f(z) \in A(\alpha_1 + 1, \beta_1, \gamma, \lambda, \mu, m, h)$ we have

$$p(z) + \frac{zp'(z)}{p(z) + (\alpha_1 - 1)} \prec h(z).$$

It follows from Lemma 2.1 that

$$p(z) \prec h(z)$$

for $\alpha_1 \geq 1$. Thus $f \in A(\alpha_1, \beta_1, \gamma, \lambda, \mu, m, h)$. □

Theorem 3.2. *If $f \in A(\alpha_1, \beta_1, \gamma, \lambda, \mu, m, h)$ for $\alpha_1 \geq 1$, then $F_c(f) \in A(\alpha_1, \beta_1, \gamma, \lambda, \mu, m, h)$, where F_c is the integral operator defined by*

$$F_c(f) = F_c(f)(z) := \frac{c+1}{z^c}\int_0^z t^{c-1}f(t)dt \qquad (c \geq 0). \tag{3.1}$$

Proof. Let $f \in A(\alpha_1, \beta_1, \gamma, \lambda, \mu, m, h)$ and

$$p(z) := \frac{z[\mathcal{D}^m_{\lambda,\mu}(\alpha_1, \beta_1)F_c(f(z))]' + \gamma z^2[\mathcal{D}^m_{\lambda,\mu}(\alpha_1, \beta_1)F_c(f(z))]''}{(1-\gamma)\mathcal{D}^m_{\lambda,\mu}(\alpha_1, \beta_1)F_c(f(z)) + \gamma z[\mathcal{D}^m_{\lambda,\mu}(\alpha_1, \beta_1)F_c(f(z))]'}.$$

A simple computation using (3.1) yields that

$$z(F_c(f)(z))' + cF_c(f)(z) = (c+1)f(z)$$

and so

$$\mathcal{D}^m_{\lambda,\mu}(\alpha_1,\beta_1)(zF_c(f(z))') + c\mathcal{D}^m_{\lambda,\mu}(\alpha_1,\beta_1)F_c(f)(z) = (c+1)\mathcal{D}^m_{\lambda,\mu}(\alpha_1,\beta_1)f(z).$$

By making use of the identity

$$z[\mathcal{D}^m_{\lambda,\mu}(\alpha_1,\beta_1)F_c(f(z))]' = \mathcal{D}^m_{\lambda,\mu}(\alpha_1,\beta_1)(zF_c(f(z))')$$

we get

$$z[\mathcal{D}^m_{\lambda,\mu}(\alpha_1,\beta_1)F_c(f(z))]' + c\mathcal{D}^m_{\lambda,\mu}(\alpha_1,\beta_1)F_c(f(z) = (c+1)\mathcal{D}^m_{\lambda,\mu}(\alpha_1,\beta_1)f(z) \quad (3.2)$$

Differentiating (3.2), we have

$$\begin{aligned} &p(z)+c \\ &= (c+1)\left(\frac{(1-\gamma)\mathcal{D}^m_{\lambda,\mu}(\alpha_1,\beta_1)f(z) + \gamma z[\mathcal{D}^m_{\lambda,\mu}(\alpha_1,\beta_1)f(z)]'}{(1-\gamma)\mathcal{D}^m_{\lambda,\mu}(\alpha_1,\beta_1)F_c(f(z)) + \gamma z[\mathcal{D}^m_{\lambda,\mu}(\alpha_1,\beta_1)F_c(f(z))]'}\right). \end{aligned} \quad (3.3)$$

Taking logarithmic differentiation of (3.3), we get

$$p(z) + \frac{zp'(z)}{p(z)+c} = \frac{z[\mathcal{D}^m_{\lambda,\mu}(\alpha_1,\beta_1)f(z)]' + \gamma z^2[\mathcal{D}^m_{\lambda,\mu}(\alpha_1,\beta_1)f(z)]''}{(1-\gamma)\mathcal{D}^m_{\lambda,\mu}(\alpha_1,\beta_1)f(z) + \gamma z[\mathcal{D}^m_{\lambda,\mu}(\alpha_1,\beta_1)f(z)]'}. \quad (3.4)$$

By applying Lemma 2.1 in (3.3), it follows that

$$p(z) \prec h(z) \qquad (z \in U).$$

Hence $F_c(f) \in A(\alpha_1,\beta_1,\gamma,\lambda,\mu,m,h)$. □

Special cases of Theroems 3.1 and 3.2 include the results which were given in [2, 11, 12, 13]. Interestingly for $q = 2, s = 1, m = 0, \alpha_1 = \beta_1 = \alpha_2 = 1, h(z) = \frac{1+z}{1-z}$ and $\gamma = 0$ in Theorem 3.1 we obtain $K \subset S^*$.

Theorem 3.3. *$f \in A(\alpha_1,\beta_1,\gamma,\lambda,\mu,m,h)$ if and only if $\gamma zf' + (1-\gamma)f \in A(\alpha_1,\beta_1,0,\lambda,\mu,m,h)$.*

Proof. Let $f \in A(\alpha_1,\beta_1,\gamma,\lambda,\mu,m,h)$ and $g(z) = \gamma zf' + (1-\gamma)f$. Using the definition of $\mathcal{D}^m_{\lambda,\mu}(\alpha_1,\beta_1)f(z)$ and a property of the Hadamard product, we find that $g \in A(\alpha_1,\beta_1,0,\lambda,\mu,m,h)$. Converse is obvious. □

Special cases of the Theorem 3.3 includes results which were in [11, 12].

Theorem 3.4. *If $f \in A(\alpha_1,\beta_1,\gamma,\lambda,\mu,m,h)$ then $\gamma f + (1-\gamma)\int_0^z \frac{f(t)}{t}dt \in A(\alpha_1,\beta_1,1,\lambda,\mu,m,h)$.*

Proof. Let $f \in A(\alpha_1, \beta_1, \gamma, \lambda, \mu, m, h)$ then from Theorem 3.3

$$g(z) = zf' + (1 - \gamma)f \in A(\alpha_1, \beta_1, 0, \lambda, \mu, m, h).$$

It can be easily seen that $f \in A(\alpha_1, \beta_1, 1, \lambda, \mu, m, h)$ if and only if $zf' \in A(\alpha_1, \beta_1, 0, \lambda, \mu, m, h)$. Applying this result for $g(z)$, we see that

$$\gamma f + (1 - \gamma) \int_0^z \frac{f(t)}{t} dt \in A(\alpha_1, \beta_1, 1, \lambda, \mu, m, h).$$

□

Theorem 3.5.

$$B(\alpha_1 + 1, \beta_1, \gamma, \lambda, \mu, m, h) \subset B(\alpha_1, \beta_1, \gamma, \lambda, \mu, m, h).$$

Proof. Let $f \in B(\alpha_1 + 1, \beta_1, \gamma, \lambda, \mu, m, h)$ and

$$p(z) = (1 - \gamma)\frac{\mathcal{D}^m_{\lambda,\mu}(\alpha_1, \beta_1)f(z)}{z} + \gamma[\mathcal{D}^m_{\lambda,\mu}(\alpha_1, \beta_1)f(z)]'.$$

Taking $\gamma = 1$ in (1.2) we get

$$z[\mathcal{D}^m_{\lambda,\mu}(\alpha_1, \beta_1)f(z)]' = \alpha_1\mathcal{D}^m_{\lambda,\mu}(\alpha_1 + 1, \beta_1)f(z) - (\alpha_1 - 1)\mathcal{D}^m_{\lambda,\mu}(\alpha_1, \beta_1)f(z). \quad (3.5)$$

Using (3.5) and the differentiation of (3.5), we get

$$p(z) + \frac{zp'(z)}{\alpha_1} = (1 - \gamma)\frac{\mathcal{D}^m_{\lambda,\mu}(\alpha_1 + 1, \beta_1)f(z)}{z} + \gamma[\mathcal{D}^m_{\lambda,\mu}(\alpha_1 + 1, \beta_1)f(z)]'. \quad (3.6)$$

By applying Lemma 2.2 in (3.6), we obtain

$$p(z) \prec q(z) \qquad (z \in U).$$

Hence the result follows. □

Theorem 3.6. *If $f \in B(\alpha_1, \beta_1, \gamma, \lambda, \mu, m, h)$, then*

$$F_c(f) \in B(\alpha_1, \beta_1, \gamma, \lambda, \mu, m, h).$$

Proof. Assume $f \in B(\alpha_1, \beta_1, \gamma, \lambda, \mu, m, h)$ and

$$p(z) = (1 - \gamma)\frac{\mathcal{D}^m_{\lambda,\mu}(\alpha_1, \beta_1)F_c(f(z))}{z} + \gamma[\mathcal{D}^m_{\lambda,\mu}(\alpha_1, \beta_1)F_c(f(z))]'.$$

Differentiating (3.2) we have

$$p(z)+\frac{zp'(z)}{c+1}=(1-\gamma)\frac{\mathcal{D}^m_{\lambda,\mu}(\alpha_1,\beta_1)f(z)}{z}+\gamma[\mathcal{D}^m_{\lambda,\mu}(\alpha_1,\beta_1)f(z)]'. \tag{3.7}$$

By applying Lemma 2.2 in (3.7) we get

$$p(z)\prec h(z)$$

and hence

$$F_c(f)\in B(\alpha_1,\beta_1,\gamma,\lambda,\mu,m,h).$$

□

Theorem 3.7. *$C(\alpha_1+1,\beta_1,\gamma,\lambda,\mu,m,h)\subset C(\alpha_1,\beta_1,\gamma,\lambda,\mu,m,h)$.*

Proof. Let $f\in C(\alpha_1+1,\beta_1,\gamma,\lambda,\mu,m,h)$ and

$$p(z)=[\mathcal{D}^m_{\lambda,\mu}(\alpha_1,\beta_1)f(z)]'+\gamma z[\mathcal{D}^m_{\lambda,\mu}(\alpha_1,\beta_1)f(z)]''.$$

Differentiating (3.5), we have

$$p(z)+\frac{zp'(z)}{\alpha_1}=[\mathcal{D}^m_{\lambda,\mu}(\alpha_1+1,\beta_1)f(z)]'+\gamma z[\mathcal{D}^m_{\lambda,\mu}(\alpha_1+1,\beta_1)f(z)]''. \tag{3.8}$$

Applying Lemma 2.2 in (3.8), we get

$$p(z)\prec h(z)\qquad (z\in U)$$

and the result now follows. □

Theorem 3.8. *If $f\in C(\alpha_1,\beta_1,\gamma,\lambda,\mu,m,h)$, then*

$$F_c(f)\in C(\alpha_1,\beta_1,\gamma,\lambda,\mu,m,h).$$

Proof. Let $f\in C(\alpha_1,\beta_1,\gamma,\lambda,\mu,m,h)$ and

$$p(z)=[\mathcal{D}^m_{\lambda,\mu}(\alpha_1,\beta_1)F_c(f(z))]'+\gamma z[\mathcal{D}^m_{\lambda,\mu}(\alpha_1,\beta_1)F_c(f(z))]''.$$

Differentiating (3.2) we get

$$p(z)+\frac{zp'(z)}{c+1}=[\mathcal{D}^m_{\lambda,\mu}(\alpha_1,\beta_1)f(z)]'+\gamma z[\mathcal{D}^m_{\lambda,\mu}(\alpha_1,\beta_1)f(z)]''. \tag{3.9}$$

A simple application of Lemma 2.2 will give the desired result. □

Theorem 3.9. *$f \in C(\alpha_1, \beta_1, \gamma, \lambda, \mu, m, h)$ if and only if*

$$zf' \in B(\alpha_1, \beta_1, \gamma, \lambda, \mu, m, h).$$

Proof. Using the equality

$$z[\mathcal{D}^m_{\lambda,\mu}(\alpha_1, \beta_1)f(z)]' = \mathcal{D}^m_{\lambda,\mu}(\alpha_1, \beta_1)(zf'(z)).$$

We see that

$$(1-\gamma)\frac{\mathcal{D}^m_{\lambda,\mu}(\alpha_1, \beta_1)zf'(z)}{z} + \gamma[\mathcal{D}^m_{\lambda,\mu}(\alpha_1, \beta_1)zf'(z)]'$$
$$= [\mathcal{D}^m_{\lambda,\mu}(\alpha_1, \beta_1)f(z)]' + \gamma z[\mathcal{D}^m_{\lambda,\mu}(\alpha_1, \beta_1)f(z)]''$$

which implies the required result. □

Theorem 3.10. $C(\alpha_1, \beta_1, \gamma, \lambda, \mu, m, h) \subset B(\alpha_1, \beta_1, \gamma, \lambda, \mu, m, h)$.

Proof. Let $f \in C(\alpha_1, \beta_1, \gamma, \lambda, \mu, m, h)$ and

$$p(z) = (1-\gamma)\frac{\mathcal{D}^m_{\lambda,\mu}(\alpha_1, \beta_1)f(z)}{z} + \gamma[\mathcal{D}^m_{\lambda,\mu}(\alpha_1, \beta_1)f(z)]'.$$

Hence

$$p(z) + zp'(z) = [\mathcal{D}^m_{\lambda,\mu}(\alpha_1, \beta_1)f(z)]' + \gamma z[\mathcal{D}^m_{\lambda,\mu}(\alpha_1, \beta_1)f(z)]''$$

and the result follows as an application of Lemma 2.2. □

Theorem 3.11. *For $\gamma > \delta \geq 0$,*

$$B(\alpha_1, \beta_1, \gamma, \lambda, \mu, m, h) \subset B(\alpha_1, \beta_1, \delta, \lambda, \mu, m, h).$$

Proof. Let $f \in B(\alpha_1, \beta_1, \gamma, \lambda, \mu, m, h)$ and $p(z) = \frac{\mathcal{D}^m_{\lambda,\mu}(\alpha_1, \beta_1)f(z)}{z}$.
When $\delta = 0$, we have

$$p(z) + zp'(z) = (1-\gamma)\frac{\mathcal{D}^m_{\lambda,\mu}(\alpha_1, \beta_1)f(z)}{z} + \gamma[\mathcal{D}^m_{\lambda,\mu}(\alpha_1, \beta_1)f(z)]' \qquad (3.10)$$

Hence the result follows as an application of Lemma 2.2 in (3.10), when $\delta = 0$. Suppose $\delta \neq 0$. Since $f \in B(\alpha_1, \beta_1, \gamma, \lambda, \mu, m, h)$, we have

$$(1-\gamma)\frac{\mathcal{D}^m_{\lambda,\mu}(\alpha_1, \beta_1)f(z)}{z} + \gamma[\mathcal{D}^m_{\lambda,\mu}(\alpha_1, \beta_1)f(z)]' \in h(U) \qquad (z \in U).$$

But $\frac{\mathcal{D}^m_{\lambda,\mu}(\alpha_1,\beta_1)f(z)}{z} \in h(U)$ and $h(U)$ is convex. Also

$$(1-\delta)\frac{\mathcal{D}^m_{\lambda,\mu}(\alpha_1,\beta_1)f(z)}{z}+\delta[\mathcal{D}^m_{\lambda,\mu}(\alpha_1,\beta_1)f(z)]' = (1-\frac{\delta}{\gamma})\frac{\mathcal{D}^m_{\lambda,\mu}(\alpha_1,\beta_1)f(z)}{z}$$
$$+\frac{\delta}{\gamma}\left[(1-\gamma)\frac{\mathcal{D}^m_{\lambda,\mu}(\alpha_1,\beta_1)f(z)}{z}\gamma[\mathcal{D}^m_{\lambda,\mu}(\alpha_1,\beta_1)f(z)]'\right].$$

Therefore we have

$$(1-\delta)\frac{\mathcal{D}^m_{\lambda,\mu}(\alpha_1,\beta_1)f(z)}{z}+\delta[\mathcal{D}^m_{\lambda,\mu}(\alpha_1,\beta_1)f(z)]' \in h(U).$$

Hence the result follows. □

Theorem 3.12. *For $\gamma > \delta \geq 0$,*

$$C(\alpha_1,\beta_1,\gamma,\lambda,\mu,m,h) \subset C(\alpha_1,\beta_1,\delta,\lambda,\mu,m,h).$$

Proof. Let $f(z) \in C(\alpha_1,\beta_1,\gamma,\lambda,\mu,m,h)$ and $p(z) = [\mathcal{D}^m_{\lambda,\mu}(\alpha_1,\beta_1)f(z)]'$. When $\delta = 0$, we have

$$p(z)+\gamma z p'(z) = [\mathcal{D}^m_{\lambda,\mu}(\alpha_1,\beta_1)f(z)]' + \gamma z[\mathcal{D}^m_{\lambda,\mu}(\alpha_1,\beta_1)f(z)]''.$$

Hence the result follows as an application of Lemma 2.2, when $\delta = 0$.

Suppose $\delta \neq 0$. Then

$$[\mathcal{D}^m_{\lambda,\mu}(\alpha_1,\beta_1)f(z)]' + \gamma z[\mathcal{D}^m_{\lambda,\mu}(\alpha_1,\beta_1)f(z)]'' \in h(U) \qquad (z \in U).$$

Note that

$$\begin{aligned}&[\mathcal{D}^m_{\lambda,\mu}(\alpha_1,\beta_1)f(z)]' + \delta z[\mathcal{D}^m_{\lambda,\mu}(\alpha_1,\beta_1)f(z)]''\\ &= (1-\frac{\delta}{\gamma})[\mathcal{D}^m_{\lambda,\mu}(\alpha_1,\beta_1)f(z)]' + \frac{\delta}{\gamma}\left[\mathcal{D}^m_{\lambda,\mu}(\alpha_1,\beta_1)f(z)]' + \gamma z[\mathcal{D}^m_{\lambda,\mu}(\alpha_1,\beta_1)f(z)\right]''.\end{aligned}$$

As $h(U)$ is convex and $\frac{\delta}{\gamma} < 1$ we have

$$[\mathcal{D}^m_{\lambda,\mu}(\alpha_1,\beta_1)f(z)]' + \delta z[\mathcal{D}^m_{\lambda,\mu}(\alpha_1,\beta_1)f(z)]'' \in h(U) \qquad (z \in U)$$

and hence the result follows. □

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