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CONE METRIC SPACES AND FIXED POINT THEOREMS OF GENERALIZED T-CONTRACTIVE MAPPINGS

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Abstract: The purpose of this paper is to establish the generalization of T-contractive type mappings on complete cone metric spaces see [2].

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1. Introduction

In [3] Huang and Zhang generalized the concept of metric spaces, replacing the set of real numbers by an ordered Banach space, there by they have defined the cone metric spaces. These authors also described the convergence of sequences in the cone metric spaces and introduce the corresponding notion of completeness. They have proved some fixed point theorems of contractive mappings on complete cone metric space with the assumption of normality of a cone. The results in [3] were generalized by Sh. Rezapour and R. Hamlbarani in [8] omitting the assumption of normality on the cone. Subsequently many authors have generalized the results of Huang and Zhang and have studied fixed point theorems for normal and non-normal cone.

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Recently, A. Beiranvand, S. Moradi, M. Omid and H. Pazandeh [1] introduced a new class of contractive mappings: T-contraction and T-contractive extending the Banach's contraction principle and the Edelstein's fixed point theorem, (see [6]) respectively. In sequel, J.R. Morales and E. Rojas [2] obtained sufficient conditions for the existence of a unique fixed point of T-contractive type mappings on complete cone metric spaces.

Our results extend and generalized some fixed point theorems of [2].

2. Preliminaries

We recall some definitions of cone metric spaces and some properties of theirs [3].

Definition 2.1. Let E be a real Banach space and P a subset of E. P is called a cone if and only if:

i) P is closed, non-empty, and $P \neq \{0\}$,

ii) $ax + by \in P$ for all $x, y \in P$ and non-negative real numbers a,b,

iii) $x \in P$ and $-x \in P \Rightarrow x = 0 \Leftrightarrow P \cap (-P) = \{o\}$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ on E with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x \ll y$ if $y - x \in$ int P, int P denotes the interior of P. The cone P is called normal if there is a number $K > 0$ such that, for all $x, y \in E, 0 \leq x \leq y$ implies $\| x \| \leq K \| y \|$.

The least positive number K satisfying the above is called the normal constant of P.

In the following, we always suppose E is a Banach space, P is a cone in E with $Int\ P \neq \phi\ and\ \leq$ is partial ordering with respect to P.

Definition 2.2. Let X be a non-empty set and $d : X \times X \to E$ a mapping such that:

(d1) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(d2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(d3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called cone metric on X, and (X, d) is called a cone metric space [3].

Example 2.1. (see [3]) Let $E = \mathbb{R}^2, P = \{(x, y) \in E : x, y \geq 0\} \subset R^2, X = R \ and \ d : X \times X \to E$ such that $d(x, y) = (| x - y |, \propto| x - y |)$, where $\propto \geq 0$ is a constant. Then (X, d) is cone metric space.

Definition 2.3. (see [3]) Let (X, d) be a cone metric space, $x \in X$, and $\{x_n\}_{n\geq1}$ a sequence in X. Then:

(i) $\{x_n\}_{n\geq1}$ converges to x, whenever for every $c \in E \ with \ o \ll c$, there is a natural number N such that $d(x_n, x) \ll c$ for all $n \geq N$. We denote this by $\lim_{n\to\infty} x_n = x \ or \ x_n \to x, (n \to \infty)$.

(ii) $\{x_n\}_{n\geq1}$ is said to Cauchy sequence if for every $c \in E$ with $o \ll c$, there is a natural number N such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$.

(iii) $(X, d)$is called a complete cone metric space if every Cauchy sequence in X is convergent.

Lemma 2.1. (see [3]) *Let (X, d) be a cone metric space, $P \subset E$ a normal cone with normal constant K. Let $\{x_n\}, \{y_n\}$be a sequences in X. and $x, y \in X$.*

(i) *$\{x_n\}$ converges to $x \in X$ if and only if* $\lim_{n\to\infty} d(x_n, x) = o;$.

(ii) *If $\{x_n\}$ converges to x and $\{x_n\}$ converges to y then $x = y$. That is the limit of $\{x_n\}$ is unique;*

(iii) *If $\{x_n\}$ is convergent then it is a Cauchy sequence.*

(iv) *$\{x_n\}$ is a Cauchy sequence if and only if* $\lim_{n,m\to\infty} d(x_n, x_m) = 0;$

(v) *If $x_n \to x$ and $\{y_n\}$is another sequence in X such that $y_n \to y$, then $d(x_n, y_n) \to d(x, y)$.*

Definition 2.4. Let (X, d) be a cone metric space. If for any sequence $\{x_n\}$in X, there is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that$\{x_{n_i}\}$ is convergent in X. Then X is called a sequentially compact cone metric space.

Definition 2.5. (see [2]) Let (X, d) be a cone metric space, P be a normal cone with normal constant K and. Let $T : X \to X$. Then:

(i) T is said to be continuous if $\lim_{n\to\infty} x_n = x$, implies that $\lim_{n\to\infty} Tx_n = Tx$ for every $\{x_n\}$ in X;

(ii) T is said to be sequentially convergent, if we have, for every sequence $\{y_n\}$, if $T\{y_n\}$ is convergent, then $\{y_n\}$ also is convergent.

(iii) T is said to be subsequentially convergent, if we have, for every sequence $\{y_n\}$, if $T\{y_n\}$ is convergent, then $\{y_n\}$ also is convergent.

Lemma 2.2. (see [2]) *If (X,d) be a sequence compact cone metric space, then every function $T : X \to X$ is subsequentially convergent and every continuous function $T : X \to X$ is sequentially convergent.*

Definition 2.6. (see [1]) Let (X,d) be a cone metric space and $T, S : X \to X$ two functions. A mapping S is said to be a T-contraction if there is $a \in [0,1)$ constant such that

$$d(TSx, TSy) \le ad(Tx, Ty), \tag{2.1}$$

for all $x, y \in X$.

Example 2.2. (see [2]) Let $E = (C_{[0,1]}, \mathbb{R})$,

$$P = \{\gamma \in E : \gamma \ge o\} \subset E, X = \mathbb{R}$$

and

$$d(x,y) = \mid x - y \mid e^t,$$

where $e^t \in E$. Then (X,d) is a cone metric space.

We consider the functions $T, S : X \to X$ defined by $Tx = e^{-x}$ and $Sx = 2x + 1$. Then:

(i) It is clear that S is not a contraction;

(ii) S is a T-contraction. In fact

$$\begin{aligned} d(TSx, TSy) &= \mid TSx - TSy \mid e^t \\ &= \frac{1}{e} \mid e^{-x} - e^{-y} \mid\mid e^{-x} - e^{-y} \mid e^t \\ &\le \frac{2}{e} \mid e^{-x} - e^{-y} \mid e^t = \frac{2}{e} d(Tx, Ty). \end{aligned}$$

3. Main Results

The following theorem is extends and improves Theorem 3.3 from [2].

Theorem 3.1. *Let (X,d)be a complete cone metric space, P be a normal cone with normal constant K, in addition let $T : X \to X$ be an one to one and continuous function and $R, S : X \to X$ be pair of T-contraction continuous functions. Then:*

(1) *for every* $x_o \in X$,

$$\lim_{n\to\infty} d(TR^{2n+1}x_{o,}TR^{2+2}x_o) = 0$$

and

$$\lim_{n\to\infty} d(TS^{2n+2}x_{o,}TS^{2n+3}x_o) = 0;$$

(2) *There is* $\vartheta \in X$ *such that*

$$\lim_{n\to\infty} TR^{2n+1}x_o = \vartheta = \lim_{n\to\infty} TS^{2n+2}x_o;$$

(3) *If* T *is subsequentially convergent, then* $(R^{2n+1}x_o)$ *and* $(S^{2n+2}x_o)$ *have a convergent subsequences;*

(4) *There is a unique common fixed point* $u \in X$ *such that* $Ru = u = Su$;

(5) *If* T *is sequentially convergent, then for each* $x_o \in X$ *the iterate sequences* $(R^{2n+1}x_o)$ *and* $(S^{2n+2}x_o)$ *converge to* u.

Proof. For every $x_1, x_2 \in X$,

$$\begin{aligned} d(Tx_1, Tx_2) \leq & d(Tx_{1,}TRx_1) + d(TRx_1, TRx_2) + d(TRx_2, Tx_2) \\ \leq & d(Tx_1, TRx_1) + ad(Tx_1, Tx_2) + d(TRx_2, Tx_2). \end{aligned}$$

So

$$d(Tx_1, Tx_2) \leq \frac{1}{1-a}[d(Tx_{1,}TRx_1) + d(TRx_2, Tx_2)]. \tag{3.1}$$

Now, choose $x_o \in X$ and define the Picard iteration associated to R, (X_{2n+1}) given by

$$x_{2n+2} = Rx_{2n+1} = R^{2n+1}x_o, \quad n = 0, 1, 2, \dots .$$

Similarly, associated to S, (x_{2n+2}) given by

$$x_{2n+3} = Sx_{2n+2} = S^{2n+2}x_o, \quad n = 0, 1, 2, \dots .$$

Now

$$d(Tx_{2n+1}, Tx_{2n+2}) = d(TR^{2n+1}x_{o,}TR^{2n+2}x_o) \leq ad(TR^{2n}x_o, TR^{2n+1}x_o),$$

hence

$$d(TR^{2n+1}x_{o,}TR^{2n+2}x_o) \leq a^{2n+1}d(Tx_o, TRx_o). \tag{3.2}$$

Similarly

$$d(TS^{2n+2}x_{o,}TS^{2n+3}x_o) \leq b^{2n+2}d(Tx_o, TSx_o). \tag{3.3}$$

Since P is a normal cone with normal constant K, from (3.2) we get

$$\| d(TR^{2n+1}x_{o,}TR^{2n+2}x_o) \| \leq a^{2n+1}K \| d(Tx_o, TRx_o) \|,$$

which implies

$$\lim_{n\to\infty} d(TR^{2n+1}x_o, TR^{2n+2}x_o) = 0. \tag{3.4}$$

Therefore, for $m, n \in N$ with $m > n$, by (3.1) and (3.2) we have

$$\begin{aligned} d(Tx_{2n+1}, Tx_{2m+1}) =& d(TR^{2n+1}x_{o,}TR^{2m+1}x_o) \\ \leq& \frac{1}{1-a}[d(TR^{2n+1}x_oTR^{2n+2}x_o) \\ &+ d(TR^{2m+2}x_o, TR^{2m+1}x_o)] \\ \leq& \frac{1}{1-a}[a^{2n+1}d(Tx_o, TRx_o) + a^{2m+1}d(Tx_o, TRx_o)]. \end{aligned}$$

Hence

$$d(TR^{2n+1}x_{o,}TR^{2m+1}x_o) \leq \frac{a^{2n+1} + a^{2m+1}}{1-a} d(Tx_o, TRx_o). \tag{3.5}$$

Taking norm to inequality above, we obtain that

$$\| d(TR^{2n+1}x_{o,}TR^{2m+1}x_o) \| \leq \frac{a^{2n+1} + a^{2m+1}}{1-a} K \| d(Tx_o, TRx_o) \| .$$

Consequently

$$\lim_{n,m\to\infty} d(TR^{2n+1}x_{o,}TR^{2m+1}x_o) = 0, \tag{3.6}$$

which prove (1). On the other hand, (3.6) implies that $(TR^{2n+1}x_o)$ is a Cauchy sequence in X. By the completeness of X, there is $\vartheta \in X$ such that

$$\lim_{n\to\infty} TR^{2n+1}x_o = \vartheta. \tag{3.7}$$

Proving in this way assertion (2). Now, if T is subsequentially convergent, then $(R^{2n+1}x_o)$ has a convergent subsequence. So, there exist $u \in X$ and $\{(2n+1)_i\}_{i=1}^{\infty}$ such that

$$\lim_{i\to\infty} R^{(2n+1)_i}x_o = u. \tag{3.8}$$

Since T is continuous we have

$$\lim_{i\to\infty} TR^{(2n+1)_i}x_o = Tu. \tag{3.9}$$

From equality(3.7) we conclude that

$$Tu = \vartheta. \tag{3.10}$$

Since R is continuous,(and also by using (3.8)) then

$$\lim_{i\to\infty} R^{(2n+1)_i+1}x_o = Ru,$$

as well as

$$\lim_{i\to\infty} TR^{(2n+1)_i+1}x_o = TRu, \tag{3.11}$$

Again by (3.7), the following equality holds

$$\lim_{i\to\infty} TR^{(2n+1)_i+1}x_o = \vartheta.$$

Hence, $TRu = \vartheta = Tu$. Since T is injective, then $Ru = u$. So R has a fixed point. Therefore assertion (3), is proved. On the other hand, since T is one to one and R is a T-contraction, R has a unique fixed point, i.e. conclusion (4).

Finally, if T is sequentially convergent, $(R^{2n+1}x_o)$is convergent to u, that is

$$\lim_{n\to\infty} R^{2n+1}x_o = u.$$

Proving in this way conclusion(5). Similarly, it can be proved that all five assertion for T-contraction function S.

Hence u is a unique common fixed point of R and S.

This completes the proof of the Theorem. □

Definition 3.1. (see [2]) Let (X,d) be a cone metric space and $T, S : X \to X$ two functions. A mapping S is said to be a T-contractive if for each $x, y \in X$ such that $Tx \neq Ty$ then

$$d(TSx, TSy) < d(x, y).$$

It is clear that every T-contraction function is T-contractive, but the converse is not true.

Example 3.1. (see [2]) (1) Let$E = (C_{[0,1]}, \mathbb{R})$, $P = \{\gamma \in E : \gamma \geq 0\} \subset E, X = [1, +\infty)$ and $d : X \times X \to E$ defined by $d(x, y) =| x - y | e^t$,where $e^t \in E$. Then (X, d) is a cone metric space.

Let $T, S : X \to X$ be two functions defined by $Tx = x$ and $Sx = \sqrt{x}$. Then:

(i) S is a T- contractive function;

(ii) S is not a T-contraction mapping.

(2) Let$E = (C_{[0,1]}, R), P = \{\gamma \in E : \gamma \geq 0\} \subset E, X = [0, \frac{1}{2}]$ and $d : X \times X \to E$ defined by $d(x,y) =\mid x - y \mid e^t$, where $e^t \in E$. Obviously (X,d) is a cone metric space and the function $S : X \to X$ defined by $Sx = \frac{x^2}{\sqrt{2}}$ is not contractive. If $T : X \to X$ is defined by $Tx = x^2$, then S is T-contractive, because

$$\begin{aligned} d(TSx, TSy) &= \mid TSx - TSy \mid e^t =\mid \frac{x^4}{2} - \frac{y^4}{2} \mid e^t \\ &= \frac{1}{2} \mid x^2 + y^2 \mid\mid Tx - Ty \mid e^t \\ &< \mid Tx - Ty \mid e^t \\ &= d(Tx, Ty). \end{aligned}$$

The next result extend the Theorem 2.9 of [3] and Theorem 3.13 of [2].

Theorem 3.2. *Let (X,d) be a compact cone metric space, P be a normal cone with nomal constant K, in addition let $T : X \to X$ is injective and continuous function and $R, S : X \to X$ be a pair of T-contractive mappings. Then:*

(1) R and S have a unique common fixed point;

(2) For any $x_o \in X$ the iterate sequences $(R^{2n+1}x_o)$ and $(S^{2n+2}x_o)$ converge to the common fixed point of R and S.

Proof. First, we are going to show that R and S are continuous functions. Let $\lim_{n\to\infty} x_{2n+1} = x$. We want to prove that $\lim_{n\to\infty} Rx_{2n+1} = Rx$. Since R is T-contractive, we get

$$d(TRx_{2n+1}, TRx) \leq d(Tx_{2n+1}, Tx).$$

So,

$$\| d(TRx_{2n+1}, TRx) \| \leq K \| d(Tx_{2n+1}, Tx) \| .$$

Now, since T is continuous, we have

$$\lim_{n\to\infty} \| d(TRx_{2n+1}, TRx) \| = 0.$$

Also

$$\lim_{n\to\infty} d(TRx_{2n+1}, TRx) = 0.$$

Therefore

$$\lim_{n\to\infty} TRx_{2n+1} = TRx. \tag{3.12}$$

Let $\{Rx_{(2n+1)_i}\}$ be an arbitrary convergent subsequence of (x_{2n+1}). There is a $y \in X$ such that

$$\lim_{i\to\infty} Rx_{(2n+1)_i} = y.$$

By the continuity of T we have,

$$\lim_{i\to\infty} TRx_{(2n+1)_i} = Ty. \tag{3.13}$$

By (3.12) and (3.13) we conclude that $TRx = Ty$. Since T is one to one then, $Rx = y$. Hence, every convergence subsequence of (Rx_{2n+1}) converge to Rx. From the fact X a compact cone metric space, we get the conclusion that R is a continuous function. Similarly we can show that S is also a continuous function.

Now, because of T and R are continuous functions, then the function $\gamma : X \to P$ defined by $\gamma(y) = d(TRy, Ty)$,for all $y \in X$, is continuous on X and from the compactness of X, the function γ attains its minimum, say at $x \in X$.

If $Rx \neq x$,then $\gamma(Rx) = d(TR^3x, TRx) < d(TRx, Tx) = \gamma(x)$ which is a contradiction. So $Rx = x$. Similarly we have $Sx = x$, proving part (1).

Choose $x_o \in X$and set $a_{2n+1} = d(TR^{2n+1}x_o, Tx)$.

Since $a_{2n+2} = d(TR^{2n+2}x_o, Tx) = d(TR^{2n+2}x_o, TRx) \leq d(TR^{2n+1}x_o, Tx) = a_{2n+1}$, then (a_{2n+1}) is a non increasing sequence of non negative real numbers and so it has a limit, say a, that is

$$a = \lim_{n\to\infty} a_{2n+1} \text{ or } \lim_{n\to\infty} d(TR^{2n+1}x_o, T_x) = a.$$

By compactness, $(TR^{2n+1}x_o)$ has a convergent subsequence $\{TR^{(2n+1)_i}x_o\}$, i.e.

$$\lim_{i\to\infty} TR^{(2n+1)_i}x_o = Z, \tag{3.14}$$

from the sequentially convergence of T, there exists $w \in X$ such that

$$\lim_{i\to\infty} R^{(2n+1)_i}x_o = \omega.$$

So

$$\lim_{i\to\infty} TR^{(2n+1)_i}x_o = T\omega. \tag{3.15}$$

By (3.14) and (3.15), $T\omega = Z$. Then $d(T\omega, Tx) = a$. Now we are going to show that $R\omega = x$. If $R\omega \neq x$, then

$$a = \lim_{n\to\infty} d(TR^{2n+1}x_o, Tx)$$

$$\begin{aligned} &= \lim_{i\to\infty} d(TR^{(2n+1)_i}x_o, Tx) \\ &= d(TRw, Tx) \\ &= d(TRw, TRx) \\ &< d(T\omega, Tx) = a, \end{aligned}$$

which is a contradiction. In this way, we get that $R\omega = x$ and hence

$$a = \lim_{i\to\infty} d(TR^{(2n+1)_i+1}x_o, Tx) = d(TR\omega, Tx) = 0.$$

Therefore, $\lim_{n\to\infty} TR^{2n+1}x_o = Tx$. Finally condition T sequentially convergent implies $\lim_{n\to\infty} R^{2n+1}x_o = x$. Similarly it can be established that $\lim_{n\to\infty} S^{2n+2}x_o = x$, which means that the iterate sequences $(R^{2n+1}x_o)$ and $(S^{2n+2}x_o)$ converge to the common fixed point of R and S. This completes the proof of the Theorem.

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