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AN IMPROVED BOUND ON GEOMETRIC APPROXIMATION BY w-FUNCTIONS

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Abstract: In this paper, we improve a result of geometric approximation in [2] to be more appropriate for any $q \in (0,1)$.

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1. Introduction

Let X be a non-negative integer-valued random variable with probability function $p(x) > 0$ for every x in the support of X, $\mathcal{S}(x)$. For $x_0 \in \mathcal{S}(x)$, let $\mathbb{F}(x_0) = \sum_{x=0}^{x_0} p(x)$ denote the distribution function of X at x_0 and μ and σ^2 $(0 < \sigma^2 < \infty)$ denote the mean and variance of X, respectively. It is well known that the distribution of X can be approximated by some discrete distributions if their parameters are satisfied under certain conditions. Let $\mathbb{G}_p(x_0) = \sum_{x=0}^{x_0} pq^x$ denote the geometric distribution function with parameter $p = 1 - q$ at x_0. If

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we expect the distribution of X to be closer to the geometric distribution than other distributions, then it is reasonable to estimate $\mathbb{F}(x_0)$ by $\mathbb{G}_p(x_0)$. In this case, Teerapabolarn [2] gave uniform and non-uniform bounds for the difference $|\mathbb{F}(x_0) - \mathbb{G}_p(x_0)|$ for $x_0 \in \mathbb{N} \cup \{0\}$ and $\mu = \frac{q}{p}$ as follows:

$$|\mathbb{F}(x_0) - \mathbb{G}_p(x_0)| \le \sum_{x \in \mathcal{S}(x)} \left| q - \frac{\sigma^2 w(x) p}{x+1} \right| p(x) \tag{1.1}$$

and if $0 < q \le \frac{1}{2}$ then

$$|\mathbb{F}(x_0) - \mathbb{G}_p(x_0)| \le \frac{1}{x_0+1} \sum_{x \in \mathcal{S}(x)} \left| (x+1)q - \sigma^2 w(x) p \right| p(x), \tag{1.2}$$

where w is a function associated with the non-negative integer-valued random variable X defined as follows:

$$w(x) = \frac{1}{\sigma^2} \left\{ \mu + \frac{\sigma^2 w(x-1) p(x-1)}{p(x)} - x \right\}, \ x \in \mathcal{S}(x) \setminus \{0\} \tag{1.3}$$

and $w(0) = \frac{\mu}{\sigma^2}$. It is observed that the bound in (1.2) can be applied for $q \in \left(0, \frac{1}{2}\right]$. In this study, we are interest to improve this bound to be more appropriate for any $q \in (0,1)$.

2. Method

We will improve our main result by using the same methodology as in [2], which consists of Stein's method and w-functions. For w-functions, it follows from Majsnerowska [1] that mentioned in (1.3). For Stein's method, following [2], Stein's equation of the geometric distribution function with parameter $p \in (0,1)$ is of the form

$$h_{x_0}(x) - \mathbb{G}_p(x_0) = q(1+x) g_{x_0}(x+1) - x g_{x_0}(x), \tag{2.1}$$

where $x_0, x \in \mathbb{N} \cup \{0\}$, $h_{x_0}(x) = 1$ if $x \le x_0$ and $h_{x_0}(x) = 0$ if $x > x_0$ and

$$g_{x_0}(x) = \begin{cases} \dfrac{1}{x} \displaystyle\sum_{k=0}^{x-1} q^k \sum_{j=x_0+1}^{\infty} p q^{j-x} & \text{if } x \le x_0, \\ \dfrac{1}{x} \displaystyle\sum_{k=0}^{x_0} q^k & \text{if } x > x_0, \\ 0 & \text{if } x = 0. \end{cases} \tag{2.2}$$

Let $\Delta g_{x_0}(x) = g_{x_0}(x+1) - g_{x_0}(x)$. Then, by (2.2) and for $x \geq 1$, we have that

$$\Delta g_{x_0}(x) = \begin{cases} \displaystyle\sum_{k=x_0+1}^{\infty} q^k \left[\frac{1}{(x+1)q^{x+1}} \sum_{j=0}^{x} pq^j - \frac{1}{xq^x} \sum_{j=0}^{x-1} pq^j \right] & \text{if } x \leq x_0, \\ \displaystyle\sum_{k=0}^{x_0} q^k \left[-\frac{1}{x(x+1)} \right] & \text{if } x > x_0. \end{cases} \tag{2.3}$$

Lemma 2.1. *For $x, x_0 \in \mathbb{N}$, $\Delta g_{x_0}(x+1) - \Delta g_{x_0}(x) > 0$ for every $x \leq x_0 - 1$.*

Proof. Let $\Delta^2 g_{x_0}(x) = \Delta g_{x_0}(x+1) - \Delta g_{x_0}(x)$. From (2.3), we have

$$\begin{aligned} \Delta^2 g_{x_0}(x) = &\sum_{k=x_0+1}^{\infty} q^k \left\{ \left[\frac{1}{(x+2)q^{x+2}} \sum_{j=0}^{x+1} pq^j - \frac{1}{(x+1)q^{x+1}} \sum_{j=0}^{x} pq^j \right] \right. \\ &\left. - \left[\frac{1}{(x+1)q^{x+1}} \sum_{j=0}^{x} pq^j - \frac{1}{xq^x} \sum_{j=0}^{x-1} pq^j \right] \right\} \\ = &\sum_{k=x_0+1}^{\infty} q^k \left\{ \frac{(x+1)p - (q - q^{x+2})}{(x+1)(x+2)q^{x+2}} - \frac{xp - (q - q^{x+1})}{x(x+1)q^{x+1}} \right\} \\ = &\sum_{k=x_0+1}^{\infty} q^k \left\{ \frac{x^2p^2 + xp^2 - 2xpq + 2q^2 - 2q^{x+2}}{x(x+1)(x+2)q^{x+2}} \right\}. \end{aligned} \tag{2.4}$$

Let $\delta(x) = x^2p^2 + xp^2 - 2xpq + 2q^2 - 2q^{x+2}$. We have

$$\begin{aligned} \delta(x+1) - \delta(x) &= 2xp^2 + 2p^2 - 2pq + 2q^{x+2} - 2q^{x+3} \\ &= 2(x+1)p^2 - 2(q - q^{x+2}) \\ &= 2(x+1)p^2 - 2p^2 \sum_{k=1}^{x+1} q^k \\ &> 0, \end{aligned}$$

which implies that $\delta(x+1) > \delta(x)$. Because $\delta(1) = 2p^3 > 0$, by mathematical induction, we obtain $\delta(x) > 0$ for every $x \in \mathbb{N}$. From this fact and (2.4), it follows that $\Delta^2 g_{x_0}(x) > 0$. Hence, the proof is complete. □

Lemma 2.2. *For $x_0 \in \mathbb{N} \cup \{0\}$ and $x > 0$, we have the following:*

$$|\Delta g_{x_0}(x)| \leq \begin{cases} \frac{1}{2} & \textit{if } x_0 = 0, \\ \frac{1-q^{x_0+1}}{(x_0+1)p} \max\left\{p, \frac{1}{x_0+2}\right\} & \textit{if } x_0 > 0. \end{cases} \tag{2.5}$$

Proof. It is clear that $\Delta g_0(x) \leq \frac{1}{2}$ for $x_0 = 0$. For $x \leq x_0$ and $x_0 > 0$, it follows from Lemma 2.1 that

$$0 < \Delta g_{x_0}(x) \leq \Delta g_{x_0}(x_0) = \frac{x_0 - \sum_{k=1}^{x_0} q^k}{x_0(x_0+1)} \leq \frac{1 - q^{x_0+1}}{x_0+1}$$

and for $x > x_0 > 0$, we have

$$0 < -\Delta g_{x_0}(x) = \frac{\sum_{k=0}^{x_0} q^k}{x(x+1)} \leq \frac{\sum_{k=0}^{x_0} q^k}{(x_0+1)(x_0+2)} \leq \frac{1 - q^{x_0+1}}{(x_0+1)p(x_0+2)},$$

which yields $|\Delta g_{x_0}(x)| \leq \frac{1-q^{x_0+1}}{(x_0+1)p} \max\left\{p, \frac{1}{x_0+2}\right\}$. Hence, (2.5) is obtained. □

3. Result

The following theorem presents an improvement of the result in (1.2).

Theorem 3.1. *For $x_0 \in \mathcal{S}(x)$ and $\mu = \frac{q}{p}$, we have the following:*

$$|\mathbb{F}(x_0) - \mathbb{G}_p(x_0)| \leq E|(X+1)q - \sigma^2 w(X)p|\Delta(x_0), \tag{3.1}$$

where $\Delta(0) = \frac{1}{2}$ and $\Delta(x_0) = \frac{1-q^{x_0+1}}{(x_0+1)p} \max\left\{p, \frac{1}{x_0+2}\right\}$ when $x_0 > 0$.

Proof. Using the same arguments detailed as in the proof of Theorem 2.1 in [2], we obtain

$$|\mathbb{F}(x_0) - \mathbb{G}_p(x_0)| \leq E\{|(X+1)q - \sigma^2 w(X)p||\Delta g_{x_0}(X)|\}.$$

Hence, with Lemma 2.2, the theorem is easily obtained. □

The following corollary is a consequence of Theorem 3.1.

Corollary 3.1. *If $(x+1)q/p - \sigma^2 w(x) > / < 0$ for every $x \in \mathcal{S}(x)$, then*

$$|\mathbb{F}(x_0) - \mathbb{G}_p(x_0)| \leq |\mu^2 + \mu - \sigma^2| p\Delta(x_0), \tag{3.2}$$

where $x_0 \in \mathcal{S}(x)$.

Remark. The bound in Theorem 3.1 is sharper than the bound in (1.2) and is appropriate for any $q \in (0, 1)$.

4. Examples

We use the results in the Theorems 3.1 and Corollary 3.1 to give three examples concerning the beta-geometric, Pólya and Poisson distributions.

4.1. Let X be the beta-geometric random variable with parameters α and β. Then its probability function is of the form

$$p(x) = \frac{\alpha\Gamma(\beta+x)\Gamma(\alpha+\beta)}{\Gamma(\beta)\Gamma(\alpha+\beta+x+1)}, \ x = 0, 1, ...$$

and the mean and variance of X are $\mu = \frac{\beta}{\alpha-1}$ and $\sigma^2 = \frac{\alpha\beta(\alpha+\beta-1)}{(\alpha-2)(\alpha-1)^2}$, respectively, where $\alpha > 2$. Let $\mathbb{BG}_{\alpha,\beta}(x_0)$ denote the beta-geometric distribution function at $x_0 \in \mathbb{N}\cup\{0\}$. If $p = \frac{\alpha-1}{\alpha+\beta-1}$, then for $\alpha > 2$, we have

$$|\mathbb{BG}_{\alpha,\beta}(x_0) - \mathbb{G}_p(x_0)| \leq \begin{cases} \frac{\beta}{(\alpha-2)(\alpha-1)} & \text{if } x_0 = 0, \\ \frac{2\beta(1-q^{x_0+1})}{(\alpha-2)(\alpha-1)(x_0+1)p} \max\left\{p, \frac{1}{x_0+2}\right\} & \text{if } x_0 > 0. \end{cases}$$

4.2. Let X be the Pólya random variable with parameters m and d. Then its probability function is of the form

$$p(x) = \frac{\binom{d+m-x-2}{m-x}}{\binom{d+m-1}{m}}, \ x = 0, ..., m.$$

The mean and variance of X are $\mu = \frac{m}{d}$ and $\sigma^2 = \frac{m(d+m)(d-1)}{d^2(d+1)}$, respectively. Let $\mathbb{PY}_{m,d}(x_0)$ denote the Pólya distribution function at $x_0 \in \{0, ..., m\}$. If $p = \frac{d}{d+m}$, then

$$|\mathbb{PY}_{m,d}(x_0) - \mathbb{G}_p(x_0)| \leq \begin{cases} \frac{m}{d(d+1)} & \text{if } x_0 = 0, \\ \frac{2m(1-q^{x_0+1})}{d(d+1)(x_0+1)p} \max\left\{p, \frac{1}{x_0+2}\right\} & \text{if } x_0 = 1, ..., m. \end{cases}$$

4.3. Let X be the Poisson random variable with mean λ. Then its probability function is of the form

$$p(x) = \frac{e^{-\lambda}\lambda^x}{x!}, \ x = 0, 1, ...$$

The mean and variance of X are $\mu = \sigma^2 = \lambda$. Let $\mathbb{P}_\lambda(x_0)$ denote the Poisson distribution function at $x_0 \in \mathbb{N}\cup\{0\}$. If $p = \frac{1}{1+\lambda}$, then

$$|\mathbb{P}_\lambda(x_0) - \mathbb{G}_p(x_0)| \leq \begin{cases} \frac{2\lambda^2}{(\lambda+1)} & \text{if } x_0 = 0, \\ \frac{\lambda^2(1-q^{x_0+1})}{(\lambda+1)(x_0+1)p} \max\left\{p, \frac{1}{x_0+2}\right\} & \text{if } x_0 > 0. \end{cases}$$

5. Conclusion

The non-uniform bound in the Theorems 3.1, was improved by Stein's method and w-functions, provides a new general criteria for measuring the accuracy of the geometric approximation for non-negative integer-valued random variable. It is sharper than the bound in (1.2). In addition, this bound is more appropriate for any $q \in (0, 1)$.

References

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