International Journal of Pure and Applied Mathematics
Volume 84 No. 5 2013, 651-667
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v84i5.16

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AN ABSTRACT ALGEBRAIC-TOPOLOGICAL APPROACH TO THE NOTIONS OF A FIRST AND A SECOND DUAL SPACE II

René Bartsch[1 §], Harry Poppe[2]

[1]Department of Mathematics
TU Darmstadt
Schlossgartenstr. 7, D-64289 Darmstadt, GERMANY

[2]Department of Mathematics
University of Rostock
D-18057 Rostock, GERMANY

Abstract: In our paper [2] we developed and studied a general duality system $(X, Y, X^d, X^{dd}, J : X \to X^{dd})$, where X^d is the first dual space of X with respect to Y, X^{dd} is the second dual space of X with respect to Y. J denotes the canonical map as is known from classical examples. In the present paper we continue the invstigations, started in [2].

AMS Subject Classification: 46A20, 46A40, 54H12
Key Words: first dual, second dual, Riesz space, non commutative algebra

1. Introduction

We start with the paper [2] and consider spaces X, Y for which some algebraic operations are defined and in some cases topologies are defined, too. In [2] was defined the dual space X^d of X with respect to a space Y, and a clear rule was given, how to construct the second dual space X^{dd} of X. We will use here the notations and results of [2] which we need.

Received: February 5, 2013

§Correspondence author

What is the aim of this paper? We subsume some further examples of representations under our general scheme developped in [2]: at first we consider dual spaces of topological spaces. Here we reproduce and extend some well-known results. Secondly, we embed the C^*-algebra $M_n(\mathbb{C})$ in a space of continuous mappings, where of course $M_n(\mathbb{C})$ is a noncommutative algebra. Finally we will define within the setting of our duality-approach [2] a new first and a second dual space of a Riesz-space (vector lattice).

2. The First and the Second Dual Space of a Topological Space

Let (X, τ) be a topological space. Now, in order to define a first dual space of X we chose another topological space (Y, σ), which we will not specify at this moment. The only morphisms we have here are the continuous functions from X to Y. According to [2], definition 3.2, we get:

Definition 2.1. $X^d = \{h : X \to Y \mid h \text{ is continuous }\}$ is the first dual space of X w.r.t. Y and we provide X^d with the pointwise topology τ_p. Thus $X^d = (C(X, Y), \tau_p))$.

Because there are no algebraic operations on X^d, it has no defect D, and consequently by definitions 4.1, 4.2 of [2] we find:

Definition 2.2. $X^{dd} = (X^d, \tau_p)^d = \{\psi : (X^d, \tau_p) \to Y \mid \psi \text{ is continuous}\}$ is called the second dual space of X w.r.t. Y, and we provide X^{dd} again with pointwise topology τ_p: $X^{dd} = C_p(C_p(X, Y), Y)$.

The canonical map from X to X^{dd} we denote by J, given as $J : X \to X^{dd} : Jx = \omega(x, .), \forall x \in X$, with ω being the evaluation map $\omega : X \times Y^X \to Y : \omega(x, h) := h(x)$.

By corollary 4.1 and theorem 4.1(2) of [2] we obtain

Proposition 2.3.

1. *$J(X) \subseteq X^{dd}$*
2. *$J : (X, \tau) \to (X^{dd}, \tau_p)$ is continuous.*

Remark 2.4. If (X, τ) is a Tychonoff space and $Y = \mathbb{R}$, this result is well known, see for instance the book of Archangel'skii, [1]. But in this special case one shows more: the canonical map is an embedding of (X, τ) onto $J(X) \subseteq C_p(C_p(X, \mathbb{R}), \mathbb{R})$.

Generalizing the notion of complete regularity we can prove a somewhat more general result. We consider two generalizations of the complete regularity of a topological space (X, τ) w.r.t. another topological space (Y, σ).

Of course, the usual notion of separation of points from closed sets easily can be generalized:

Definition 2.5. Let $(X, \tau), (Y, \sigma)$ be topological spaces. We say that $C(X, Y)$ *separates in* X *the points from closed sets* iff

$$\forall x \in X, A \subseteq X, A \text{ closed }, x \notin A : \exists f \in C(X, Y) : f(x) \notin \overline{f(A)}.$$

For every topological space (X, τ) holds, that $C(X, X)$ separates in X the points from closed sets, by the identity map.

Moreover, for every topological space (X, τ) holds, that $C(X, Y)$ separates the points in X from closed sets, if we chose the Sierpinski-space for (Y, σ).

So, separation of points from closed sets in (X, τ) w.r.t. (Y, σ) generally not even implies R_0 for X; but if (Y, σ) is T_3, then it implies T_3 for (X, τ) immediately.

We still consider another possibility:

Definition 2.6. If (X, τ) and (Y, σ) are topological spaces, then X is called $(Y; a, b)$-*completely regular* iff in Y there exists distinct points $a \neq b$ in Y such that

1. there exists an open neighbourhood W of a, that doesn't contain b, and
2. if $G \subseteq X$ is open and $x \in G$ then there exists $f \in C(X, Y)$ with $f(x) = a$ and $f(X \setminus G) \subseteq \{b\}$.

Note again, that every topological space (X, τ) is $(Y; 0, 1)$-completely regular for the Sierpinski-space $Y = \{0, 1\}$ with open $\{0\}$.

The properties defined above are not independent.

Lemma 2.7. *If (X, τ) and (Y, σ) are topological spaces and X is $(Y; a, b)$-completely regular, then $C(X, Y)$ separates in X the points from the closed sets.*

Proof. Let $A \subseteq X$ be closed and $x \in X$, $x \notin A$ be given. By assumption we find $f \in C(X, Y)$ with $f(x) = a$ and $f(A) \subseteq \{b\}$; there exists an open neighbourhood W of a with $b \notin W$. Hence $W \cap f(A) = \emptyset$ implying $f(x) \notin \overline{f(A)}$. □

Theorem 2.8. *Let $(X, \tau), (Y, \sigma)$ be topological spaces, we consider the canonical map $J : X \to X^{dd} = C_p(C_p(X, Y), Y)$. Then holds*

1. *J is continuous.*

If in addition $C(X,Y)$ separates the points in X from the closed sets, then furthermore hold

2. *If (X,τ) is T_1, then J is injective.*

3. *J is an open map from (X,τ) onto the subspace $(J(X),\tau_p)$.*

Proof. 1. comes from proposition 2.3.

2. Since (X,τ) is T_1, the singletons are closed and hence by assumption follows that $C(X,Y)$ separates the points of X, too. But then J is injective by [2], 4.8.

3. Let $\emptyset \neq G \in \tau$ and $z \in J(G)$. So we have $\exists x \in G : z = J(x) = \omega(x,.)$. By assumption we find $f \in C(X,Y)$ s.t. $f(x) \notin \overline{f(X \setminus G)}$; but then there exists an open set $W \subseteq Y$ with $f(x) \in W$ and $W \cap f(X \setminus G) = \emptyset$. Since $f \in C(X,Y) = X^d$, $(\{f\},W) := \{\nu \in Y^{(X^d)} | \ \nu(\{f\}) \subseteq W\}$ is a τ_p-open set in $Y^{(X^d)}$, but $J(X) \subseteq Y^{(X^d)}$, yielding that $(\{f\},W) \cap J(X)$ is τ_p-open in $J(X)$ and obviously $J(x) \in (\{f\},W) \cap J(X)$.

Finally we show $(\{f\},W) \cap J(X) \subseteq J(G)$.

For all $h \in (\{f\},W) \cap J(X)$ we have $h = \omega(y,.)$ for some $y \in X$, and $h(\{f\}) = \{f(y)\} \subseteq W$, implying $f(y) \in W$. Now, assuming $y \notin G$ we would get $f(y) \in f(X \setminus G)$, implying $f(y) \notin W$; hence we find $y \in G$, and so $h = \omega(y,.) \in J(G)$. □

Corollary 2.9. *Let $(X,\tau),(Y,\sigma)$ be topological spaces, (X,τ) a T_1-space, and let $C(X,Y)$ separate the points in X from the closed sets. Then $J : (X,\tau) \to (J(X),\tau_p) \subseteq (X^{dd},\tau_p)$ is an embeding.*

Proof. By theorem 2.8 we know that J is continuous, injective and open. □

Now we can show the corresponding theorem using the complete $(Y;a,b)$-regularity of X.

Theorem 2.10. *Let $(X,\tau),(Y,\sigma)$ be topological spaces. If (X,τ) is $(Y;a,b)$-completely regular, then hold:*

1. *If (X,τ) is a T_0-space, $J : X \to X^{dd}$ is injective.*

2. *J is an open map from X onto $(J(X),\tau_p)$.*

Proof. 1. Let $x, y \in X$, $x \neq y$. For instance we find an open set $G \in \tau$ with $x \in G$ and $y \notin G$. Since X is $(Y; a, b)$-completely regular, there exists $g \in C(X, Y)$ such that $g(x) = a$ and $g(X \setminus G) \subseteq \{b\}$, showing that $g(x) = a \neq b = g(y)$; hence $\omega(x, g) \neq \omega(y, g)$, implying $J(x) = \omega(x, .) \neq \omega(y, .) = J(y)$.

2. follows from theorem 2.8 by lemma 2.7. □

Remark 2.11. The notion of $(Y; a, b)$-complete regularity was defined in [7]. Generalizing a result of R. Arens, in [7] was shown (see theorem 2.35): Let $(X, \tau), (Y, \sigma)$ be topological spaces, X regular and $(Y; a, b)$-completely regular. If for $C(X, Y)$ one of the following conditions

1. $(C(X, Y), c - \lim)$ is a pretopological convergence space, where $c - \lim$ is the convergence structure of continuous convergence.

2. For $C(X, Y)$ there exists a smallest conjoining topology.

is fulfilled, then X is locally compact.

Before we prove some corollaries of theorems 2.8 and 2.10, we want to provide another useful result.

Proposition 2.12. *Let $D = \{0, 1\}$ denote the two-point set with discrete topology. If (X, τ) is a zero-dimensional topological space, then $C(X, D)$ separates in X the points from the closed sets.*

Proof. For subsets $M \subseteq X$ we denote by χ_M the characteristic function of M in X. We have $C(X, D) = \{\chi_M \mid M \subseteq X, M \text{ is open and closed}\}$.

Now, let $A \subseteq X$ be closed and $x \in X \setminus A$; $X \setminus A \in \underline{U}(x) \Longrightarrow \exists B \in \underline{U}(x) :$ $B \subseteq X \setminus A, B$ is open-closed; $X \setminus B$ is open-closed, too, and $A \subseteq X \setminus B$; hence $\chi_{X \setminus B}(A) = \{1\} = \overline{\{1\}}$ and $x \notin X \setminus B$ implying $\chi_{X \setminus B}(x) = 0$, thus $\chi_{X \setminus B}(x) \notin \overline{\chi_{X \setminus B}(A)}$ whereas $\chi_{X \setminus B} \in C(X, D)$. □

Let X be a T_0-space and let $Y = \mathbb{R}$ with Euclidian topology, let $a, b \in \mathbb{R}, a \neq b$, then X is $(\mathbb{R}; a, b)$ (or $(\mathbb{R}; 0, 1)$) -completely regular iff X is completely regular.

Hence we get by proposition 2.3 and theorem 2.10 the well-known result (see [1] or [6]):

Corollary 2.13. *Let (X, τ) be a Tychonoff-space; let as usual $C_p(X) := (C(X, \mathbb{R}), \tau_p)$. Then $J : X \to X^{dd} = C_p(C_p(X))$ is injective, continuous and an open map onto $J(X)$, thus it is an embedding into X^{dd}.*

Let $\mathbb{C}$ denote the complex numbers with Euclidian topology. If (X,τ) is a Tychonoff-space, then (X,τ) is $(\mathbb{C};0,1)$-completely regular: $\forall G \in \tau : \forall x \in G : \exists f \in C(X,\mathbb{R}) : f(x) = 0 \wedge f(X \setminus G) \subseteq \{1\}$; but $f \in C(X,\mathbb{C})$, too, and of course we find an open set $W \subseteq \mathbb{C}$ such that $0 \in W$ and $1 \notin W$.

Corollary 2.14. *If (X,τ) is a Tychonoff-space, then*

$$J : X \to C_p(C_p(X,\mathbb{C}),\mathbb{C})$$

is injective, continuous and an open map onto $J(X)$.

Let $Y = \{0,1\}$ with open sets $\emptyset, \{0\}, Y$ be the Sierpinski-space; let (X,τ) be a topological space, $G \in \tau$ and $x \in G$; then the characteristic funktion $\chi_{X\setminus G} : X \to Y$ is continuous and $\chi_{X\setminus G}(x) = 0, \chi_{X\setminus G}(X \setminus G) \subseteq \{1\}$, showing that (X,τ) is $(Y;0,1)$-completely regular and we get by proposition 2.3 and theorem 2.10:

Corollary 2.15. *If (X,τ) is a T_0-space, then $J : C \to C_p(C_p(X,Y),Y) = X^{dd}$ is injective, continuous and an open map onto $J(X)$.*

This result was proved in [4].

Corollary 2.16. *If (X,τ) is a zero-dimensional T_0-space and D is the two-point discrete space, then*
$J : X \to X^{dd} = C_p(C_p(X,D),D)$
is injective, continuous and an open map onto $J(X) \subseteq X^{dd}$.

Proof. Each zero-dimensional space is T_3, yielding by T_0 that (X,τ) is T_1. Now proposition 2.12 and theorem 2.8 apply. □

3. The C^*-algebra $M_n(\mathbb{C})$

Proposition 3.1. *Let X,Y be C^*-algebras with units; let $h : X \to Y$ be an algebra-*-homomorphism. Then holds:*

1. $\forall x \in X : ||h(x)|| \leq ||x||$.

   Hence h is continuous and $||h|| := \sup_{||x||\leq 1} ||h(x)|| \leq 1$.

2. *The range $h(X) := \{h(x)|\ x \in X\}$ of h is a C^*-subalgebra of Y; that means especially that $h(X)$ is closed in Y w.r.t. the norm-topology $\tau_{||.||}$ in Y.(See [3].)*

Corollary 3.2. *With the assumptions of proposition 3.1 we get the equivalence of the following:*

1. *h is injective.*
2. *$ker(h) = \{0\}$.*
3. *$\forall x \in X : ||h(x)|| = ||x||$, hence h is an isometric isomorphism from X onto $h(X) \subseteq Y$.*

Proof. (1)⇔(2) and (3)⇒(1) are obvious; we show (1)⇒(3): by 3.1(2) we know that $h(X)$ is a C^*-subalgebra of Y; since h is injective, h^{-1} exists uniquely and $\forall x \in X : h^{-1}(h(x)) = x$, hence h^{-1} maps $h(X)$ onto X; h^{-1} is an algebra homomorphism. (Linearity and multiplicativity are trivial by injectivity and the homomorphy of h, so we show only that $h^{-1}(h(x)^*) = (h^{-1}(h(x)))^*$ holds: $h^{-1}(h(x)^*) = h^{-1}(h(x^*)) = x^* = (h^{-1}(h(x)))^*$). Now by 3.1(1) we find $\forall x \in X : ||x|| = ||h^{-1}(h(x))|| \leq ||h(x)|| \leq ||x|| \implies ||x|| \leq ||h(x)|| \leq ||x||$. □

Proposition 3.3. *Let X, Y be $\mathbb{C}$ -Banach algebras with units. Let $L(X, Y)$ be the set of all linear and continuous maps from X to Y; then $X^d \subseteq L(X, Y)$ and for $L(X, Y)$ we consider the operator norm: $||h|| := \sup_{||x|| \leq 1} ||h(x)||$ and we restrict this norm to X^d. Let $A \subseteq X^d$ with $A \neq \emptyset$ and $A \neq \{\underline{0}\}$ be given, where $\underline{0}$ here denotes the zero-homomorphism in X^d. Now let hold:*

1. *$\forall h \in A : ||h|| \leq 1$*
2. *$J(X) \subseteq (C_b((A, \tau_p), Y), ||.||_{\sup})$.*

   Here are: τ_p the pointwise topology, $C_b((A, \tau_p), Y)$ the space of bounded functions from A to the metric space Y; $||.||_{\sup}$ is the supremum-norm (Tchebycheff-norm).

Then hold:

3. *$\forall x \in X : ||Jx||_{\sup} \leq ||x||$,*
4. *J is uniformly continuous and hence continuous, and*
5. *$J(X)$ separates the points of A.*

Proof. (3) $||Jx||_{\sup} = \sup_{h \in A} ||(Jx)(h)|| = \sup_{h \in A} ||\omega(x, h)|| = \sup_{h \in A} ||h(x)|| \leq \sup_{h \in A}(||h|| \cdot ||x||) \leq \sup_{h \in A} ||x|| = ||x||$.

(4) From theorem 4.1. of [2] we know that J is linear; hence by (3) we get $\forall x, y \in X : ||Jx - Jy|| = ||J(x - y)|| \leq ||x - y||$.

(5) Let $f, g \in A$ with $f \neq g$, i.e. $\exists x_0 \in X : f(x_0) \neq g(x_0)$. So, $Jx_0 = \omega(x_0, .)$ separates f, g. □

We want to test the possibility to represent $M_n(\mathbb{C})$ by a space of continuous mappings. By $M_n(\mathbb{C})$, we mean the family of all $n \times n$-matrices with complex entries and with n strictly greater than 1. This becomes a $\mathbb{C}$-vector space by the usual addition of matrices and scalar multiplication with elements of $\mathbb{C}$. By the usual multiplication of matrices it becomes an algebra with unit.

Now, for $\mathbb{C}^n$ we can define the Euclidian norm $||(z_1, ..., z_n)|| := \sqrt{\sum_{i=1}^{n} |z_i|^2}$. Hence, we can consider $L(\mathbb{C}^n, \mathbb{C}^n)$. But we see at once, that the vector spaces $L(\mathbb{C}^n, \mathbb{C}^n)$ and $M_n(\mathbb{C})$ are isomorphic: For $h \in L(\mathbb{C}^n, \mathbb{C}^n)$ there exists a unique matrix $A_h \in M_n(\mathbb{C}) : \forall x \in \mathbb{C}^n : h(x) = A_h \cdot x^T$; but for each $B \in M_n(\mathbb{C})$ the function $g_B : \mathbb{C}^n \to \mathbb{C}^n : g_B(x) := B \cdot x^T$ is clearly linear and - since $\mathbb{C}^n$ with euclidian norm is a finite dimensional normed space - g_B is continuous, too; thus $g_B \in L(\mathbb{C}^n, \mathbb{C}^n)$.

For $L(\mathbb{C}^n, \mathbb{C}^n)$ we can define the operator norm:
$\forall h \in L(\mathbb{C}^n, \mathbb{C}^n) : ||h|| := \sup_{||x|| \leq 1} ||h(x)||$; but now we can carry over this norm to $M_n(\mathbb{C})$: $\forall A \in M_n(\mathbb{C}) : ||A|| := \sup_{||x|| \leq 1} ||A \cdot x^T||$, where $x \in \mathbb{C}^n$.

As is well known, to compute $||A||$ for $A \in M_n(\mathbb{C})$, we can use the fact, that this norm coincide with the spectral norm: let A^* the conjugate transpose of A (if $A = (a_{ij})$ then $A^* = (\overline{a_{ji}})$), then the spectral norm of A is defined as the square root of the largest eigenvalue of the positive semidefinit matrix A^*A.

Remark 3.4. With this norm, $M_n(\mathbb{C})$ is a Banach algebra, as we know. It is a C^*-algebra, too, where $A \to A^*$ is the involution.

First dual spaces of $M_n(\mathbb{C})$:

(1) Using definition 1.3 from [2], we set $X = M_n(\mathbb{C})$ and $Y = \mathbb{C}$; hence $M_n(\mathbb{C})^d = \{h : M_n(\mathbb{C}) \to \mathbb{C} \mid h \text{ is linear, multiplicative and involutory}\} = \{h : M_n(\mathbb{C}) | \; h \text{ is an algebra (ring) homomorphism, } h(A^*) = \overline{h(A)}, \text{ and } h \text{ is continuous } \}$.

Lemma 3.5. *Let $\underline{0} \in M_n(\mathbb{C})^d$ be the zero map. Then $M_n(\mathbb{C})^d = \{\underline{0}\}$.*

Proof. For $n = 2$ we have $E := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$.

Now assume $\exists h \in M_n(\mathbb{C})^d : h \neq \underline{0}$. Since h is a ring homomorphism with $h \neq \underline{0}$, we find $h(E) = 1$ and $1 = h(E) = h\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} h\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$, thus $h\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \neq 0$.

Otherwise,
$h\begin{pmatrix}-1 & 1\\ 1 & 1\end{pmatrix} = h\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix} h\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix} - h\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix} h\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix} = 0$ because the multiplication is commutative in the range space $\mathbb{C}$. So, we get a contradiction, yielding $M_n(\mathbb{C}) = \{\underline{0}\}$ here.

For $n = 3$ observe that the matrix $A := \begin{pmatrix}1 & 0 & 1\\ 0 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}$ yields an invertible $B := AA^T - A^TA = \begin{pmatrix}1 & 0 & -1\\ 0 & -1 & 0\\ -1 & 0 & 0\end{pmatrix}$ leading to the same contradiction as above, when used at the place of $\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix}$.

Now, by building block matrices, the assumption will follow for all $n > 1$. □

(2) As we have seen, the problem in the foregoing case was the commutativity of the multiplication in $\mathbb{C}$. Now, the multiplication in $Y = M_n(\mathbb{C})$ is non-commutative, hence we chose here $Y = X = M_n(\mathbb{C})$.

Thus, $M_n(\mathbb{C})^d = \{h : M_n(\mathbb{C}) \to M_n(\mathbb{C})|\ h$ is linear, multiplicative and involutory $\} = \{h : M_n(\mathbb{C}) \to M_n(\mathbb{C})|\ h$ is linear, multiplicative, involutory and continuous $\}$. (The continuity follows from the fact, that $X = M_n(\mathbb{C})$ is finite-dimensional as $\mathbb{C}$-algebra.)

Here we have the advantage, that the identity $\mathbb{1} : M_n(\mathbb{C}) \to M_n(\mathbb{C}) : \mathbb{1}(A) := A$ belongs to $M_n(\mathbb{C})^d$. But we find still more elements of $M_n(\mathbb{C})^d$.

Proposition 3.6. *(a) Let $U \in M_n(\mathbb{C})$ be invertible; for the map $h_U : M_n(\mathbb{C}) \to M_n(\mathbb{C}) : h_U(A) := UAU^{-1}$ hold:*

i. h_U is linear,

ii. h_U is multiplicative,

iii. h_U is bijective,

iv. $h_U(E_n) = E_n$ where E_n means the unit matrix in $X = Y = M_n(\mathbb{C})$.

v. h_U is continuous.

(b) If U is an unitary matrix, then additionally holds

vi. $\forall A \in M_n(\mathbb{C}) : h_U(A^*) = (h_U(A))^*$.

Proof. (i), (ii), (iii), (iv) and (vi) are straightforward matrix calculations, (v) again follows from the fact, that $M_n(\mathbb{C})$ is a finite dimensional normed space. □

By definition of $M_n(\mathbb{C})^d$ we get

$$M_n(\mathbb{C})^d \subseteq L((M_n(\mathbb{C}), ||.||), \quad (M_n(\mathbb{C}), ||.||)) = L(M_n(\mathbb{C}), M_n(\mathbb{C})),$$

and for $L(M_n(\mathbb{C}), M_n(\mathbb{C}))$ we consider the operator norm

$$||.|| : \forall h \in L(M_n(\mathbb{C}), M_n(\mathbb{C})) : ||h|| = \sup_{||A||\leq 1} ||h(A)|| = \sup_{||A||=1} ||h(A)||$$

and we restrict $||.||$ to $M_n(\mathbb{C})^d$. For instance, $\mathbb{1} \in M_n(\mathbb{C})^d$: $||\mathbb{1}|| = \sup_{||A||=1} ||\mathbb{1}(A)|| = 1$.

Lemma 3.7. *The algebraic operations on $Y = M_n(\mathbb{C})$ (addition and multiplication of matrices, scalar multiplication, involution) are continuous w.r.t. the operator norm.*

Proof. For the finite dimensional Banach space $(M_n(\mathbb{C}), ||.||)$ the vector space operations are contiuous, in arbitrary normed algebra, hence in $M_n(\mathbb{C})$ too, the multiplication is continuous. (And is easy to compute by the obvious submultiplicativity of the norm.) Now, we have $\forall A \in M_n(\mathbb{C}) : ||A|| = ||A^*||$. Let a sequence (A_i) in $M_n(\mathbb{C})$ be given with $A_i \to 0$ w.r.t. the norm topology $\tau_{||.||}$, just meaning $||A_i|| \to 0$, hence $||A_i^* - 0|| = ||A_i^*|| = ||A_i|| \overset{\tau_{||.||}}{\to} 0$; thus the involution is continuous on $M_n(\mathbb{C})$, too. □

Proposition 3.8. *(a) $M_n(\mathbb{C})^d$ has a zero-homomorphism $h_0 : \forall A \in X = M_n(\mathbb{C}) : h_0(A) = Z_n \in M_n(\mathbb{C}) = Y$, where Z_n means the zero-matrix.*

(b) $\forall h \in M_n(\mathbb{C})^d : ||h|| \leq 1$.

(c) Since $M_n(\mathbb{C})^d \subseteq M_n(\mathbb{C})^{M_n(\mathbb{C})}$, using the algebraic operations in $Y = M_n(\mathbb{C})$ by pointwise definition, we can carry over these operations to $M_n(\mathbb{C})^d$. Endowed with these pointwise operations, $M_n(\mathbb{C})^d$ is not an algebra.

Proof. (a) is evident, since Z_n is a zero-element of $Y = M_n(\mathbb{C})$.

(b) follows from proposition 3.1(a).

(c) $M_n(\mathbb{C})^d$ is no vector space: $\mathbb{1} \in M_n(\mathbb{C})^d$ and $||\mathbb{1}|| = 1$; hence $||2\cdot\mathbb{1}|| = 2 > 1$, implying $2 \cdot \mathbb{1} \notin M_n(\mathbb{C})^d$ by (b). □

By (c) and the definitions of an abstract second dual space in [2], definition 4.2., and by [2], corollary 4.1. we obtain:

Corollary 3.9. *The second dual space of* $X = M_n(\mathbb{C})$ *w.r.t.* $Y = M_n(\mathbb{C})$ *is*

$$M_n(\mathbb{C})^{dd} = C\left(\left(M_n(\mathbb{C})^d, \tau_p\right), (M_n(\mathbb{C}), ||.||)\right)$$

and for the canonical map J *holds* $J(M_n(\mathbb{C})) \subseteq M_n(\mathbb{C})^{dd}$.

Proposition 3.10. *(a)* $(M_n(\mathbb{C})^d, \tau_p)$ *is a Hausdorff and compact topological space.*

(b) The zero-homomorphism $\underline{0} \in M_n(\mathbb{C})^d$ *is an isolated point of* $M_n(\mathbb{C})^d$ *in* $\left(M_n(\mathbb{C})^{M_n(\mathbb{C})}, \tau_p\right)$.

(c) $M_n(\mathbb{C})^d \setminus \{\underline{0}\}$ *is a Hausdorff compact space, too.*

Proof. (a) $Y = M_n(\mathbb{C})$ is a finite-dimensional normed space and by lemma 3.7 the algebraic operations in Y are $\tau_{||.||}$-continuous; especially, Y is Hausdorff. Then by the generalized Alaoglu theorem in [8], corollary 3.3., $(M_n(\mathbb{C})^d, \tau_p) = \left(\{h \in M_n(\mathbb{C})^{M_n(\mathbb{C})} | \ h \text{ is a continuous algebra homomorphism and } ||h|| \leq 1\}, \tau_p\right)$ is a compact Hausdorff topological space.

(b) By proposition 3.10 there exists a $h \in M_n(\mathbb{C})^d$ with $h \neq \underline{0}$ and $h(E_n) = E_n$, where E_n is the unit-matrix in $M_n(\mathbb{C})$. Then [2], lemma 4.3., yields the assertion.

(c) follows from (a) and (b). □

Notation: Following the arguments of [2], lemma 4.2 and Section 4.3 ("Redefinition of X^d") we consider $M_n(\mathbb{C})^d \setminus \{\underline{0}\}$ as new dual space of $X = M_n(\mathbb{C})$ and denote this space again by the symbol $M_n(\mathbb{C})^d$, from now on.

Theorem 3.11. *Let*

$$C_b\left((M_n(\mathbb{C}), \tau_p), (M_n(\mathbb{C}), ||.||)\right)$$

be the space of all bounded continuous functions from

$(M_n(\mathbb{C}), \tau_p)$ *into* $(M_n(\mathbb{C}), ||.||)$ *and let this space be endowed with the supremum-norm. Then hold:*

(a) *The redefined* $(M_n(\mathbb{C})^d, \tau_p)$ *is a compact Hausdorff topological space.*

(b) $M_n(\mathbb{C})^{dd} = C_b\left((M_n(\mathbb{C}), \tau_p), (M_n(\mathbb{C}), ||.||)\right)$ *and*

$$\left(C_b\left((M_n(\mathbb{C}), \tau_p), (M_n(\mathbb{C}), ||.||)\right), ||.||_{\sup}\right)$$

is a C^**-algebra with unit* $\underline{1}$*:* $\forall h \in M_n(\mathbb{C})^d : \underline{1}(h) := E_n \in Y = M_n(\mathbb{C})$*.*

(c) $J : (M_n(\mathbb{C}), ||.||) \to M_n(\mathbb{C})^{dd}$ *is an isomorphism, i.e. an injective algebra homomorphism from* $M_n(\mathbb{C})$ *onto the* C^**-subalgebra* $J(M_n(\mathbb{C})) \subseteq \left(C_b\left((M_n(\mathbb{C}), \tau_p), (M_n(\mathbb{C}), ||.||)\right), ||.||_{\sup}\right)$*.*

(d) J *is is an isometric map, i.e.* $\forall A \in M_n(\mathbb{C}) : ||J(A)||_{\sup} = ||A||$*.*

Proof. (a) is just 3.10(c).

(b) Since $(M_n(\mathbb{C}), \tau_p)$ is compact and Hausdorff, we get

$$\begin{aligned} M_n(\mathbb{C})^{dd} &= C\left((M_n(\mathbb{C}), \tau_p), (M_n(\mathbb{C}), ||.||)\right) \\ &= C_b\left((M_n(\mathbb{C}), \tau_p), (M_n(\mathbb{C}), ||.||)\right) \end{aligned}$$

and in [2], proposition 4.2., was proved that the space

$$C_b\left((M_n(\mathbb{C}), \tau_p), (M_n(\mathbb{C}), ||.||)\right)$$

is a C^*-algebra and a space of continuous functions, which has a natural unit.

(c) By the homomorphy theorem in [2], theorem 4.1., we get that $J : (M_n(\mathbb{C}), ||.||) \to M_n(\mathbb{C})^{dd}$ is an algebra homomorphism; but then by 3.1(b) $J(M_n(\mathbb{C}))$ is a C^*-subalgebra of $(M_n(\mathbb{C})^{dd}, ||.||_{\sup})$.

Now, $\mathbb{1} \in M_n(\mathbb{C})^d$ implies, that $M_n(\mathbb{C})^d$ separates the points of $X = M_n(\mathbb{C})$ and hence by [2], proposition 4.5., J is injective, yielding by corollary 3.2, that hold: $\forall A \in M_n(\mathbb{C}) : ||J(A)||_{\sup} = ||A||$. So, (d) is proved. $\square$

4. Representing Non-Commutative C^*-Algebras

Let X be a (nontrivial) non-commutative C^*-algebra with unit e; we want to define the dual space X^d (in the sense of our approach), the second dual space X^{dd} and to prove a representation theorem.

To define X^d we set $Y = (M_n(\mathbb{C}), ||.||)$.

Definition 4.1. $X^d := \{h : X \to M_n(\mathbb{C})|\ h$ is a continuous, linear ring-homomorphism with $h(x^*) = h(x)^*\}$ is called the first dual space of X w.r.t. $Y = M_n(\mathbb{C})$.

Remark 4.2. By 3.1(a) we know, that it is enough to say, that each h is an algebra homomorphism onto it's image. Now we can follow the procedure of the case $X = M_n(\mathbb{C})$ and $Y = M_n(\mathbb{C})$. Again we have here $X^d \subseteq L(X, Y) = L(X, M_n(\mathbb{C}))$; hence we can define the operator norm for X^d.

Then by 3.1(a) we get: $\forall h \in X^d : ||h|| \leq 1$; thus $X^d = \{h : X \to M_n(\mathbb{C})|\ h$ is continuous algebra homomorphism with $||h|| \leq 1\}$.

X^d has a zero-element $\underline{0}$ and we assume, that $X^d \neq \{\underline{0}\}$.

Proposition 4.3.

(1) (X^d, τ_p) *is a compact Hausdorff topological space.*

(2) *The zero homomorphism* $\underline{0} \in X^d$ *is an isolated point of* X^d *in* $(M_n(\mathbb{C})^X, \tau_p)$.

(3) $X^d \setminus \{\underline{0}\}$ *is a compact Hausdorff topological space, too.*

Proof. (a) We know: $Y = (M_n(\mathbb{C}), ||.||)$ is Hausdorff, finite dimensional and all concerned algebraic operations on $M_n(\mathbb{C})$ are $\tau_{||.||}$-continuous. Thus using again the generalized Alaoglu theorem [8], corollary 3.3., we get (X^d, τ_p) being compact and Hausdorff.

(b) Sinve $X^d \neq \{\underline{0}\}$ by assumtion we can prove (b) quite analogously to part (b) of proposition 3.10 - and (c) will follow immediately. □

Hence again we define $X^d \setminus \{\underline{0}\}$ as the new dual space and **redefine** from this point on the symbol X^d to denote this new dual.

Thus $X^d = \{h : X \to M_n(\mathbb{C})|\ h$ is a continuous algebra homomorphism and $h \neq \underline{0}\}$.

4.1. Definition of the Second Dual Space

Again we have $X^d \subseteq M_n(\mathbb{C})^X$ and by using the algebraic operations on $Y = M_n(\mathbb{C})$, we can pointwise define similar those operations on X^d, too. In general

X^d then will not be a vector space, as we know from the case $X = M_n(\mathbb{C})$. In fact, whenever there exists an $h \in X^d$ with $h \neq \underline{0}$, then follows $\exists x \in X$: $h(x) \neq 0$, implying $x \neq 0$, so $||\frac{1}{||x||}h(x)|| > 0$, yielding $\exists n \in \mathbb{N} : \frac{1}{n} < ||\frac{1}{||x||}h(x)||$. From the assumption, that X^d is a vector space, we get $nh \in X^d$, but we have $1 < ||\frac{n}{||x||}h(x)|| = ||(nh)(\frac{x}{||x||})|| \leq ||nh|| \leq 1$ - a contradiction. So, X^d is not an algebra if it is nontrivial.

By the definition of an abstract second dual space in [2], definition 4.2., and by [2], corollary 4.1., we obtain:

Proposition 4.4. *The second dual space of X w.r.t. $Y = M_n(\mathbb{C})$ is*

$$X^{dd} := C\left((X^d, \tau_p), \left(M_n(\mathbb{C}), ||.||\right)\right)$$

endowed with supremum-norm, and for the canonical map J holds $J(X) \subseteq X^{dd}$.

Theorem 4.5. *Let X be a non-commutative C^*-algebra with unit; we assume $\exists h \in X^d : h \neq \underline{0}$. Then hold:*

(1) (The redefined) (X^d, τ_p) is a Hausdorff and compact topological space.

(2)

$$\left(C_b\left((X^d, \tau_p), (M_n(\mathbb{C}), ||.||)\right), ||.||_{\sup}\right)$$

is a (non-commutative) C^*-algebra with unit, the canonical map

$$J : X \to C\left((X^d, \tau_p), (M_n(\mathbb{C}), ||.||)\right)$$

is an algebra homomorphism and $J(X)$ is a C^*-subalgebra of X^{dd}.

(3) $\forall x \in X : ||Jx||_{\sup} \leq ||x||$.

(4) J is uniformly continuous and hence continuous.

(5) $J(X)$ separates the points of X^d.

Proof. (a) comes from 4.3(a), (c).

(b) Since (X^d, τ_p) is compact and Hausdorff, X^{dd} equals the space of bounded continuous mappings, and by [2], proposition 4.2., this space is a C^*-algebra. The homomorphy theorem [2], theorem 4.1., shows that $J : X \to X^{dd}$ is an algebra-homomorphism. By 3.1(b) we know, that $J(X)$ is a C^*-subalgebra of X^{dd}.

Assertions (c), (d) and (e) we obtain from proposition 3.3. □

5. A New Dual Space of a Vector Lattice

Let X, Y be vector lattices (Riesz spaces), let $L(X,Y) := \{f : X \to Y |\ f$ is linear$\}$ and $L_b(X,Y) := \{f \in L(X,Y)|\ f$ is order-bounded$\}$, where f is order-bounded means that images of order-bounded subsets of X under f are order-bounded in Y. For basic facts on ordered vector spaces, Riesz spaces, Banach lattices and mappings between such spaces see for instance [5], [9].

$X^{\sim} = L_b(X, I\!R)$ is called the dual space of X. This notion has the advantage that $L_b(X, I\!R)$ again is a vector lattice. Hence one can consider $X^{\sim\sim} = (X^{\sim})^{\sim}$ as the second dual space of X. But a functional $f \in L_b(X, I\!R)$ is not a homomorphism in the sense of [2], definition 2.2. The (algebraic) operations in a Riesz space X are the vector space (linear) operations and the lattice operations $\wedge$, $\vee$.

Indeed in Riesz space theory $f \in L(X,Y)$ such that $\forall x, y \in X : f(x \vee y) = f(x) \vee f(y), f(x \wedge y) = f(x) \wedge f(y)$ is called a lattice homomorphism. But here it is enough to use either $f(x \vee y) = f(x) \vee f(y)$ $(= \max\{f(x), f(y)\})$ or $f(x \wedge y) = f(x) \wedge f(y)$.

Now, setting $Y = I\!R$, we will define the dual space X^d of a Riesz space X according to definition 3.2. of [2].

Definition 5.1. Let X be a Riesz space.

$$X^d = \{h : X \to I\!R|\ h \text{ is a linear lattice homomorphism }\}$$

is called the dual space of X.

Remark 5.2. In X^d we can define pointwise linear operations and a partial order $h_1 \leq h_2$. But as we show by an example, in general X^d with pointwise operations is not a vector space again, and thus cannot be a Riesz space.

Example 5.3. For a Tychonoff space X, let $C(X) = C(X, I\!R)$. It is well-known that $C(X)$ with pointwise operations and order is an ordered vector space, even a Riesz space, where $\forall f, g \in C(X) : \forall x \in X : (f \vee g)(x) := \max\{f(x), g(x)\}$, $(f \wedge g)(x) := \min\{f(x), g(x)\}$.

$I\!R$ with natural order and topology (norm) is a Banach lattice, and $\forall a, b \in I\!R : a \vee b = \max\{a, b\}$.

Now by definition 5.1 $(C(X))^d = \{\phi : C(X) \to I\!R|\ \phi$ linear and $\forall f, g \in C(X) : \phi(f \vee g) = \phi(f) \vee \phi(g)\}$.

$(C(X))^d$ has many elements: for all $x \in X$ we consider the $x-$point evaluation (Dirac functional) $\omega(x, .) : C(X) \to I\!R : \forall h \in C(X) : \omega(x, .)(h) = \omega(x, h) = h(x)$.

We find $\forall x \in X : \omega(x,.) \in (C(X))^d$, because each $\omega(x,.)$ is linear by remark 1.1 of [2] and furthermore $\forall f,g \in C(X) : \omega(x,.)(f \vee g) = (f \vee g)(x) = \max\{f(x), g(x)\} = f(x) \vee g(x) = \omega(x,.)(f) \vee \omega(x,.)(g)$.

Now the question arises: does $x, y \in X$ imply $\omega(x,.) + \omega(y,.) \in (C(X))^d$?

Let us assume $\forall x, y \in X : \omega(x,.) + \omega(y,.) \in (C(X))^d$. Then for all $f, g \in C(X)$ we get (in $I\!R$):

$\boldsymbol{(f(x) \vee g(x)) + (f(y) \vee g(y))} = \omega(x, f \vee g) + \omega(y, f \vee g) =$

$(\omega(x,.) + \omega(y,.))(f \vee g) = (\omega(x,.) + \omega(y,.))(f) \vee (\omega(x,.) + \omega(y,.))(g) \boldsymbol{=}$ $\boldsymbol{(f(x) + f(y)) \vee (g(x) + g(y))}$.

Let $X = [0,1]$ with Euclidian topology, $x = 0$, $y = 1$. We assume that $\omega(0,.) + \omega(1,.) \in (C([0,1]))^d$ and let $f, g \in C([0,1])$ be given by $f(z) := 1 - 2z$, $g(z) := 0$ for all $z \in [0,1]$. The result above yields $\mathbf{1} \boldsymbol{=} (f(0) \vee g(0)) + (f(1) \vee g(1)) = (f(0) + f(1)) \vee (g(0) + g(1)) \boldsymbol{=} \mathbf{0}$ - a contradiction. Hence $\omega(0,.) + \omega(1,.) \notin (C([0,1]))^d$.

We want to define for a Riesz space X the second dual space with respect to our dual space X^d.

As we have seen by the example, in general X^d has the defect D according to [2], definition 4.1. Hence by definition 4.2. of [2] we get the second dual space X^{dd} of X as

$$X^{dd} = \left(C\left((X^d, \tau_p), (I\!R, \tau_{|.|}) \right), \mu \right)$$

where τ_p, μ denote the applicable pointwise topologies and $\tau_{|.|}$ the Euclidian topology on $I\!R$.

We know that with pointwise defined linear operations and pointwise order the space of continuous functions $C((X^d, \tau_p), I\!R)$ is a Riesz space.

Now we consider the canonical map

$$J : X \to X^{dd} : J(x) = \omega(x,.)$$

(By lemma 4.1 of [2] holds $J(X) \subseteq X^{dd}$.)

Applying the homomorphy theorem 4.1 of [2], we get:

Theorem 5.4. *Let X be a nontrivial Riesz space. Then hold*

(1) *$J : X \to X^{dd}$ is linear and a lattice homomorphism.*

(2) *If X has a compatible topology τ, then $J : (X, \tau) \to (X^{dd}, \tau_p)$ is continuous.*

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