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CHROMATIC EQUIVALENCE OF A FAMILY OF K_4-HOMEOMORPHS WITH GIRTH 9

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Abstract: For a graph G, let $P(G, \lambda)$ denote the chromatic polynomial of G. Two graphs G and H are chromatically equivalent (or simply $\chi-$equivalent), denoted by $G \sim H$, if $P(G, \lambda) = P(H, \lambda)$. A graph G is chromatically unique (or simply $\chi-$unique) if for any graph H such as $H \sim G$, we have $H \cong G$, i.e, H is isomorphic to G. A K_4-homeomorph is a subdivision of the complete graph K_4. In this paper, we determine when two K_4-homeomorphs of the form $K_4(2, 3, 4, d, e, f)$ and $K_4(1, 2, 6, d', e', f')$ are chromatically equivalent. The result obtained can be extended in the study of chromatic equivalence classes of $K_4(2, 3, 4, d, e, f)$ and chromatic uniqueness of K_4-homeomorphs with girth 9.

AMS Subject Classification: 05C15
Key Words: chromatic polynomial, chromatic equivalence, K_4-homeomorphs

1. Introduction

All graphs considered here are simple graphs. For such a graph G, let $P(G, \lambda)$ denote the chromatic polynomial of G. Two graphs G and H are chromatically equivalent (or simply $\chi-$equivalent), denoted by $G \sim H$, if $P(G, \lambda) = P(H, \lambda)$.

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A graph G is chromatically unique (or simply $\chi-$unique) if for any graph H such as $H \sim G$, we have $H \cong G$, i.e, H is isomorphic to G.

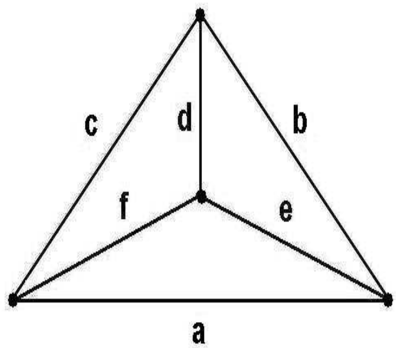

Figure 1: $K_4(a,b,c,d,e,f)$

A K_4-homeomorph is a subdivision of the complete graph K_4. Such a homeomorph is denoted by $K_4(a,b,c,d,e,f)$ if the six edges of K_4 are replaced by the six paths of length a,b,c,d,e,f, respectively, as shown in Figure 1. So far, the chromaticity of K_4-homeomorphs with girth g, where $3 \leq g \leq 7$ has been studied by many authors (see [2,6,7,8]). Recently, Zhao et al. [9] studied the chromaticity of one family of K_4-homeomorphs with girth 8, i.e., $K_4(2,3,3,d,e,f)$. In [10], Shi has solved completedly the chromaticity of K_4-homeomorphs with girth 8. When referring to the chromaticity of K_4-homeomorphs with girth 9, we know that ten types of K_4-homeomorphs need to be solved, i.e.,

$$K_4(1,2,6,d,e,f),\ K_4(1,3,5,d,e,f),\ K_4(1,4,4,d,e,f),\ K_4(2,2,5,d,e,f),$$

$$K_4(2,3,4,d,e,f),\ K_4(1,2,c,2,e,4),\ K_4(1,2,c,4,e,2),\ K_4(1,2,c,3,e,3),$$

$$K_4(1,3,c,2,e,3) \text{ and } K_4(2,2,c,2,e,3).$$

In this paper, we consider one family of K_4-homeomorphs with girth 9, i.e., $K_4(2,3,4,d,e,f)$.

Hasni et al. [3,4] characterized chromatically equivalence pairs of K_4-homeomorphs, $K_4(1,3,5,d,e,f)$ with $K_4(1,3,5,d',e',f')$; and $K_4(1,4,4,d,e,f)$

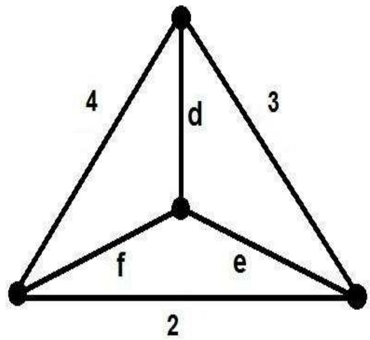

Figure 2: $K_4(2,3,4,d,e,f)$

with $K_4(1,4,4,\ d',e',f')$. In this paper, we shall discuss a chromatically equivalence pair of K_4-homeomorphs, $K_4(2,3,4,d,e,f)$ (as shown in Figure 2) and $K_4(1,2,6,d',e',f')$. Our main aim is to provide a result which can be extended to the study of the chromatic equivalence classes of $K_4(2,3,4,d,e,f)$. Such results are an indispensable tool in the study of the chromatic uniqueness of K_4-homeomorphs with girth 9.

2. Preliminary Results

In this section, we give some known results used in the sequel.

Lemma 2.1. *Assume that G and H are $\chi-$equivalent. Then*

(1) $|V(G)| = |V(H)|$, $|E(G)| = |E(H)|$ *(see [5]);*

(2) *G and H has the same girth and same number of cycles with length equal to their girth (see [12]);*

(3) *If G is a K_4-homeomorph, then H must itself be a K_4-homeomorph (see [1]);*

(4) *Let $G = K_4(a,b,c,d,e,f)$ and $H = K_4(a',b',c',d',e',f')$, then*

(i) *min* $(a,b,c,d,e,f) = $ *min* (a',b',c',d',e',f') *and the number of times that this minimum occurs in the list* $\{a,b,c,d,e,f\}$ *is equal to the number of times that this minimum occurs in the list* $\{a',b',c',d',e',f'\}$ *(see[11]);*

(ii) *if* $\{a,b,c,d,e,f\} = \{a',b',c',d',e',f'\}$ *as multisets, then* $H \cong G$ *(see [6]).*

Theorem 2.1. *(Hasni et al. [3]) Let* $K_4(1,3,5,d,e,f)$ *and* $K_4(1,3,5,d',e',f')$ *be chromatically equivalent, then*

$$K_4(1,3,5,i,i+6,i+1) \sim K_4(1,3,5,i+2,i,i+5),$$
$$K_4(1,3,5,i,i+1,i+4) \sim K_4(1,3,5,i+2,i+3,i).$$

Theorem 2.2. *(Hasni et al. [4]) Let* $K_4(1,4,4,d,e,f)$ *and* $K_4(1,4,4,d',e',f')$ *be chromatically equivalent, then*

$$K_4(1,4,4,i,i+1,i+5) \sim K_4(1,4,4,i+2,i,i+4).$$

Theorem 2.3. *(Zhao et al. [9])* K_4*-homeomorph* $K_4(2,3,3,d,e,f)$ *with girth* 8 *is not* χ*-unique if and only if it is isomorphic to* $K_4(2,3,3,1,6,\delta)$ $(\delta \geq 6)$, $K_4(2,3,3,1,\beta,\beta+2)$ $(\beta \geq 4)$, *or* $K_4(2,3,3,1,5,6)$.

3. Main Results

In this section, we present our main result.

Theorem 3.1. *If* G *is in the type of* $K_4(2,3,4,d,e,f)$, *and* H *is in the type of* $K_4(1,2,6,d',e',f')$, *then* $G \sim H$ *if* G *is isomorphic to* $K_4(2,3,4,1,7,f)$, $K_4(2,3,4,5,1,6)$, $K_4(2,3,4,7,1,5)$ *or* $K_4(2,3,4,4,1,6)$, *or* $K_4(2,3,4,1,7,6)$, *or* $K_4(2,3,4,1,5,8)$, *where* $f \geq 7$.

Proof. Let G and H be two graphs such that $G \cong K_4(2,3,4,d,e,f)$ and $H \cong K_4(1,2,6,d',e',f')$. Since the girth of G is 9, there is at most 1 among d,e and f.

Let

$$Q(K_4(a,b,c,d,e,f)) \;=\; -(s+1)(s^a+s^b+s^c+s^d+s^e+s^f)+s^{a+d}+s^{b+f}+$$

$$s^{c+e} + s^{a+b+e} + s^{b+d+c} + s^{a+c+f} + s^{d+e+f}.$$

Let $s = 1 - \lambda$ and x is the number of edges in G. From [11], we have the chromatic polynomial of K_4-homeomorphs $K_4(a,b,c,d,e,f)$ is as follows:

$$P(K_4(a,b,c,d,e,f) \\ = (-1)^{x-1}\frac{s}{(s-1)^2}\Big[(s^2+3s+2) + Q(K_4(a,b,c,d,e,f)) - s^{x-1})\Big].$$

Hence $P(G) = P(H)$ if and only if $Q(G) = Q(H)$. We solve the equation $Q(G) = Q(H)$ to get all solutions. Let the lowest remaining power and the highest remaining power be denoted by l.r.p. and h.r.p., respectively.

$$\begin{aligned} Q(G) &= -(s+1)(s^2+s^3+s^4+s^d+s^e+s^f) + s^{2+d} + s^{3+f} + \\ &\quad s^{4+e} + s^{5+e} + s^{7+d} + s^{6+f} + s^{d+e+f}. \\ Q(H) &= -(s+1)(s+s^2+s^6+s^{d'}+s^{e'}+s^{f'}) + s^{1+d'} + s^{2+f'} + \\ &\quad s^{6+e'} + s^{3+e'} + s^{8+d'} + s^{7+f'} + s^{d'+e'+f'}. \end{aligned}$$

We can assume $e \le f$. Since $K_4(2,3,4,d,e,f)$ has exactly 1 path of length 1 and $e \le f$, we have min $\{d,e,f\} = \{d,e\} = 1$. From Lemma 2.1 (1),

$$d + e + f = d' + e' + f' \tag{1}$$

There are two cases to be considered.

Case 1. min $\{d,e\} = d = 1$. We obtain the following after simplification.

$Q_1(G) = -2s^4 - s^5 - s^{e+1} - s^{f+1} - s^e - s^f + s^{3+f} + s^{4+e} + s^{5+e} + s^8 + s^{6+f}$,
$Q_1(H) = -s^7 - s^{e'+1} - s^{f'+1} - s^6 - s^{d'} - s^{e'} - s^{f'} + s^{2+f'} + s^{6+e'} + s^{3+e'} + s^{8+d'} + s^{7+f'}$.

After comparing the h.r.p in $Q_1(G)$ and the h.r.p in $Q_1(H)$, we have the h.r.p in $Q_1(G)$ is $6+f$. Considering the h.r.p in $Q_1(G)$ and the h.r.p in $Q_1(H)$, we know that there are three cases to be considered.

Case 1.1 max $\{6+e', 8+d', 7+f'\} = 6+e' = 6+f$. So $e' = f$. From $Q_1(G)$ and $Q_1(H)$, we obtain the following after simplification.

$Q_2(G) = -2s^4 - s^5 - s^{e+1} - s^e + s^{4+e} + s^{5+e} + s^8$,
$Q_2(H) = -s^7 - s^{f'+1} - s^6 - s^{d'} - s^{f'} + s^{2+f'} + s^{8+d'} + s^{7+f'}$.

Consider the $-2s^4$ in $Q_2(G)$. Since $-2s^4$ cannot be cancelled by the terms in $Q_2(G)$, there are two terms in $Q_2(H)$ which are equal to $-s^4$. So $f'+1 = d = 4$ or $f' = d' = 4$.

If $f' + 1 = d' = 4$, from Equa (1), we get $e = 6$. So $Q_2(G) \neq Q_2(H)$, a contradiction.

If $f' = d' = 4$, from Equa (1), we get $e = 7$. In this case, we obtain a solution where G is isomorphic to $K_4(2,3,4,1,7,f)$ and H isomorphic to $K_4(1,2,6,4,f,4)$, i.e., $K_4(2,3,4,1,7,f) \sim K_4(1,2,6,4,f,4)$.

Case 1.2 max $\{6+e', 8+d', 7+f'\} = 7+f' = 6+f$. So $1+f' = f$. From $Q_1(G)$ and $Q_1(H)$, we obtain the following after simplification.

$$Q_3(G) = -2s^4 - s^5 - s^{e+1} - s^{f+1} - s^e + s^{3+f} + s^{4+e} + s^{5+e} + s^8,$$
$$Q_3(H) = -s^7 - s^{e'+1} - s^6 - s^{d'} - s^{e'} - s^{f'} + s^{2+f'} + s^{6+e'} + s^{3+e'} + s^{8+d'}.$$

Assume $6+e' < 6+f = 7+f'$ since $6+e' = 6+f$ has been discussed in Case 1.1. As the term $8+d' \leq 7+f'$, the term $s^{2+f'}$ cannot be cancelled by any negative terms in $Q_3(H)$, then none of the terms in $Q_3(H)$ are equal to the term $-s^{f+1}$ in $Q_3(G)$ by noting $f+1 = f'+2$. Therefore, $2s^{2+f'}(\text{or}-2s^{f+1}) \in Q_3(G)$. We also get $4+e = 8 = 2+f'$. Thus, $e = 4$, $f' = 6$ and $f = 7$. Then $-3s^4 \in Q_3(G)$, but $-3s^4 \notin Q_3(G)$, a contradition.

Case 1.3 max $\{6+e', 8+d', 7+f'\} = 8+d' = 6+f$. After discussing the case $6+e' = 6+f$ in Case 1.1, then we suppose that $6+e' < 6+f$. From $Q_1(G)$ and $Q_1(H)$, we obtain the following after simplification.

$$Q_4(G) = -2s^4 - s^5 - s^{e+1} - s^{f+1} - s^e - s^f + s^{3+f} + s^{4+e} + s^{5+e} + s^8,$$
$$Q_4(H) = -s^7 - s^{e'+1} - s^{f'+1} - s^6 - s^{d'} - s^{e'} - s^{f'} + s^{2+f'} + s^{6+e'} + s^{3+e'} + s^{7+f'}.$$

Comparing the l.r.p in $Q_4(G)$ and the l.r.p in $Q_4(H)$, we have $d' = e' = 4$ or $d' = f' = 4$ or $e' = f' = 4$.

Case 1.3.1 If $d' = e' = 4$, we obtain the following after simplification.

$$Q_5(G) = -s^{e+1} - s^{f+1} - s^e - s^f + s^{3+f} + s^{4+e} + s^{5+e} + s^8,$$
$$Q_5(H) = -s^7 - s^{f'+1} - s^6 - s^{f'} + s^{2+f'} + s^{10} + s^7 + s^{7+f'}.$$

Comparing the h.r.p in $Q_5(G)$ and the h.r.p in $Q_5(H)$, we have $7+f' = 3+f$ or $7+f' = 5+e$.

If $7+f' = 3+f$, from Equa (1), we get $e = 3$. Then $Q_5(G) \neq Q_5(H)$, a contradiction.

If $7+f' = 5+e$, from Equa (1), we get $f = 5$. Then $Q_5(G) \neq Q_5(H)$, a contradiction.

Case 1.3.2 If $d' = f' = 4$, we obtain the following after simplification.

$Q_6(G) = -s^{e+1} - s^{f+1} - s^e - s^f + s^{3+f} + s^{4+e} + s^{5+e} + s^8$,
$Q_6(H) = -s^7 - s^{e'+1} - s^{e'} + s^{6+e'} + s^{3+e'} + s^{11}$.

Comparing the h.r.p in $Q_6(G)$ and the h.r.p in $Q_6(H)$, we have $6+e' = 3+f$ or $6 + e' = 5 + e$.

If $6 + e' = 3 + f$, from Equa (1), we get $e = 4$. Then $Q_6(G) \neq Q_6(H)$, a contradiction.
If $6 + e' = 5 + e$, from Equa (1), we get $f = 6$. Then $e' = 6$ and $e = 7$. So we obtain a solution where $G \cong K_4(2, 3, 4, 1, 7, 6)$ and $H \cong K_4(1, 2, 6, 4, 6, 4)$.

Case 1.3.3 If $e' = f' = 4$, from Equa (1), we get $e = 5$. We obtain the following after simplification.

$Q_7(G) = -s^{f+1} - s^f + s^{3+f} + s^9 + s^8$, $Q_7(H) = -s^{d'} + s^6 + s^{11}$.

It is clear that $d' = 6$ and $f = 8$. So we obtain a solution where $G \cong K_4(2, 3, 4, 1, 5, 8)$ and $H \cong K_4(1, 2, 6, 6, 4, 4)$.

Case 2. $\min\{d, e\} = e = 1$. Since $d + e \geq 6$, $e + f \geq 7$, we have $d \geq 5$ and $f \geq 6$. We obtain the following after simplification.

$Q_8(G) = -s^3 - 2s^4 - s^{d+1} - s^{f+1} - s^d - s^f + s^{2+d} + s^{3+f} + s^6 + s^{7+d} + s^{6+f}$,
$Q_8(H) = -s^7 - s^{e'+1} - s^{f'+1} - s^6 - s^{d'} - s^{e'} - s^{f'} + s^{2+f'} + s^{6+e'} + s^{3+e'} + s^{8+d'} + s^{7+f'}$.

Consider the l.r.p in $Q_8(G)$ and the l.r.p in $Q_8(H)$, we have $\min\{d', e', f'\} = 3$.

Case 2.1 $d' = 3$. From $Q_8(G)$ and $Q_8(H)$, we obtain the following after simplification.

$Q_9(G) = -2s^4 - s^{d+1} - s^{f+1} - s^d - s^f + s^{2+d} + s^{3+f} + s^6 + s^{7+d} + s^{6+f}$,
$Q_9(H) = -s^7 - s^{e'+1} - s^{f'+1} - s^6 - s^{e'} - s^{f'} + s^{2+f'} + s^{6+e'} + s^{3+e'} + s^{11} + s^{7+f'}$.

Consider the $-2s^4$ in $Q_9(G)$. Since $Q_9(G) = Q_9(H)$, there are two terms in $Q_9(H)$ which are equal to $-s^4$. So we have $e' = f' = 4$ or $e' = f' + 1 = 4$ or $f' = e' + 1 = 4$.

If $e' = f' = 4$, then we obtain the following after simplification.
$Q_{10}(G) = -s^{d+1} - s^{f+1} - s^d - s^f + s^{2+d} + s^{3+f} + s^{7+d} + s^{6+f}$,
$Q_{10}(H) = -s^7 - s^5 - s^5 - s^6 + s^{10} + s^7 + s^{11} + s^{11}$.

Consider the h.r.p in $Q_{10}(G)$ and the h.r.p in $Q_{10}(H)$. If $7+d=11$, then $d=4$ which contradicts $d\geq 5$. If $6+f=11$, then $f=5$ which contradicts $f\geq 6$.

If $e'=f'+1=4$, then we obtain the following after simplification.
$Q_{11}(G)=-s^{d+1}-s^{f+1}-s^{d}-s^{f}+s^{2+d}+s^{3+f}+s^{6}+s^{7+d}+s^{6+f}$,
$Q_{11}(H)=-s^{7}-s^{5}-s^{6}-s^{3}+s^{5}+s^{10}+s^{7}+s^{11}+s^{10}$.

Note that $d\geq 5$. Then $-s^{3}\in Q_{11}(H)$ but not in $Q_{11}(G)$, a contradiction.
If $f'=e'+1=4$, then we obtain the following after simplification.
$Q_{12}(G)=-s^{d+1}-s^{f+1}-s^{d}-s^{f}+s^{2+d}+s^{3+f}+s^{6}+s^{7+d}+s^{6+f}$,
$Q_{12}(H)=-s^{7}-s^{5}-s^{3}-s^{3}+s^{9}+s^{6}+s^{11}+s^{11}$.

Then $-2s^{3}\in Q_{12}(H)$ but not in $Q_{12}(G)$, a contradiction.
Case 2.2 $e'=3$. From $Q_{8}(G)$ and $Q_{8}(H)$, we obtain the following after simplification.

$Q_{13}(G)=-s^{4}-s^{d+1}-s^{f+1}-s^{d}-s^{f}+s^{2+d}+s^{3+f}+s^{7+d}+s^{6+f}$,
$Q_{13}(H)=-s^{7}-s^{f'+1}-s^{6}-s^{d'}-s^{f'}+s^{2+f'}+s^{9}+s^{8+d'}+s^{7+f'}$.

Consider the l.r.p in $Q_{13}(G)$ and the l.r.p in $Q_{13}(H)$. So we have $d'=4$ or $f'=4$.

Case 2.2.1 $d'=4$. We obtain the following after simplification.

$Q_{14}(G)=-s^{d+1}-s^{f+1}-s^{d}-s^{f}+s^{2+d}+s^{3+f}+s^{7+d}+s^{6+f}$,
$Q_{14}(H)=-s^{7}-s^{f'+1}-s^{6}-s^{f'}+s^{2+f'}+s^{9}+s^{12}+s^{7+f'}$.

Consider the h.r.p in $Q_{14}(G)$ and the h.r.p in $Q_{14}(H)$. So we have $7+f'=7+d$ or $7+f'=6+f$.

If $7+f'=7+d$, from Equa (1), we get $f=6$. We have $Q_{14}(G)=Q_{14}(H)$. Thus, $G\cong H$.
If $7+f'=6+f$, from Equa (1), we get $d=5$. We obtain the following after simplification.

$Q_{15}(G)=-s^{5}+s^{7}+s^{3+f}$, $Q_{15}(H)=-s^{7}-s^{f'}+s^{2+f'}+s^{9}$.

It is easy to see that $f'=5$ and $f=6$. Thus we obtain a solution where G is isomorphic to $K_4(2,3,4,5,1,6)$ and H is isomorphic to $K_4(1,2,6,4,3,5)$.

Case 2.2.2 $f'=4$. We obtain the following after simplification.

$Q_{16}(G)=-s^{d+1}-s^{f+1}-s^{d}-s^{f}+s^{2+d}+s^{3+f}+s^{7+d}+s^{6+f}$,
$Q_{16}(H)=-s^{7}-s^{5}-s^{d'}+s^{9}+s^{8+d'}+s^{11}$.

Consider the h.r.p in $Q_{16}(G)$ and the h.r.p in $Q_{16}(H)$. So we have $8+d'=7+d$ or $8+d'=6+f$.

If $8+d' = 7+d$, from Equa (1), we get $f = 5$. We obtain the following after simplification.

$Q_{17}(G) = -s^{d+1} - s^6 - s^d + s^{2+d} + s^8$, $Q_{17}(H) = -s^7 - s^{d'} + s^9$.

It is easy to see that $d' = 6$ and $d = 7$. Thus we obtain a solution where G is isomorphic to $K_4(2,3,4,7,1,5)$ and H is isomorphic to $K_4(1,2,6,6,3,4)$.

If $8+d' = 6+f$, from Equa (1), we get $d = 4$. We obtain the following after simplification.

$Q_{18}(G) = -s^{f+1} - s^4 - s^f + s^6 + s^{3+f} + s^{6+f}$, $Q_{18}(H) = -s^7 - s^{d'} + s^9 + s^{12}$.

It is easy to see that $d' = 4$ and $f = 6$. Thus we obtain $G \cong H$.

Case 2.3 $f' = 3$. From $Q_8(G)$ and $Q_8(H)$, we obtain the following after simplification.

$Q_{19}(G) = -s^4 - s^{d+1} - s^{f+1} - s^d - s^f + s^{2+d} + s^{3+f} + s^6 + s^{7+d} + s^{6+f}$,
$Q_{19}(H) = -s^7 - s^{e'+1} - s^6 - s^{d'} - s^{e'} + s^5 + s^{6+e'} + s^{3+e'} + s^{8+d'} + s^{11}$.

Consider the l.r.p in $Q_{19}(G)$ and the l.r.p in $Q_{19}(H)$. So we have $d' = 4$ or $e' = 4$.

Case 2.3.1 $d' = 4$. We obtain the following after simplification.

$Q_{20}(G) = -s^{d+1} - s^{f+1} - s^d - s^f + s^{2+d} + s^{3+f} + s^6 + s^{7+d} + s^{6+f}$,
$Q_{20}(H) = -s^7 - s^{e'+1} - s^6 - s^{e'} + s^5 + s^{6+e'} + s^{3+e'} + s^{12} + s^{10}$.

Consider the h.r.p in $Q_{20}(G)$ and the h.r.p in $Q_{20}(H)$. We obtain $7+d = 6+e'$ or $6+f = 6+e'$.

If $6+f = 6+e'$, from Equa (1), we get $d = 6$. Then $Q_{20}(G) \neq Q_{20}(H)$, a contradiction.
If $7+d = 6+e'$, from Equa (1), we get $7 = 6$. Then $Q_{20}(G) \neq Q_{20}(H)$, a contradiction.

Case 2.3.2 $e' = 4$. We obtain the following after simplification.

$Q_{21}(G) = -s^{d+1} - s^{f+1} - s^d - s^f + s^{2+d} + s^{3+f} + s^6 + s^{7+d} + s^{6+f}$,
$Q_{21}(H) = -s^6 - s^{d'} + s^{10} + s^{8+d'} + s^{10}$.

Consider the h.r.p in $Q_{21}(G)$ and the h.r.p in $Q_{21}(H)$. We obtain $7+d = 8+d'$ or $6+f = 8+d'$.

If $7+d = 8+d'$, from Equa (1), we get $f = 5$. Then $Q_{21}(G) \neq Q_{21}(H)$, a contradiction.

If $6+f=8+d'$, from Equa (1), we get $d=4$. Then $Q_{21}(G)\neq Q_{21}(H)$, a contradiction.

So far, we have solved the equation $Q(G)=Q(H)$ and obtained the solutions as follows:

$K_4(2,3,4,1,7,a)\sim K_4(1,2,6,4,a,4)$,

$K_4(2,3,4,5,1,6)\sim K_4(1,2,6,4,3,5)$,

$K_4(2,3,4,7,1,5)\sim K_4(1,2,6,6,3,4)$,

$K_4(2,3,4,1,7,6)\sim K_4(1,2,6,4,6,4)$,

$K_4(2,3,4,1,5,8)\sim K_4(1,2,6,6,4,4)$,

where $a\geq 4$.

The proof is completed. □

We close the paper with the following open problems.

Problem 1. Study the chromatic equivalence of the graph G is in the type of $K_4(2,3,4,d,e,f)$ and H is in the type of the following graphs:

$$K_4(1,3,5,d,e,f),\ K_4(1,4,4,d,e,f),\ K_4(2,2,5,d,e,f),\ K_4(2,3,4,d,e,f),$$

$$K_4(1,2,c,2,e,4),\ K_4(1,2,c,4,e,2),\ K_4(1,2,c,3,e,3),\ K_4(1,3,c,2,e,3)$$

and $K_4(2,2,c,2,e,3)$.

Problem 2. Study the chromatic uniqueness of the graph $K_4(2,3,4,d,e,f)$, where $d+e\geq 6$, $d+f\geq 5$ and $e+f\geq 7$.

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