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NEW IDENTITIES FOR THE COMMON FACTORS OF BALANCING AND LUCAS-BALANCING NUMBERS

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Abstract: Balancing numbers n and balancers r are originally defined as the solution of the Diophantine equation $1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r)$. If n is a balancing number, then $8n^2 + 1$ is a perfect square. Further, If n is a balancing number then the positive square root of $8n^2+1$ is called a Lucas-balancing number. These numbers can be generated by the linear recurrences $B_{n+1} = 6B_n - B_{n-1}$ and $C_{n+1} = 6C_n - C_{n-1}$ where B_n and C_n are respectively denoted by the n^{th} balancing number and n^{th} Lucas-balancing number. In this study, we establish some new identities for the common factors of both balancing and Lucas-balancing numbers.

AMS Subject Classification: 11B39, 11B83
Key Words: balancing numbers, Lucas-balancing numbers, recurrence relation

1. Introduction

Behera and Panda [1] recently introduced a number sequence called balancing numbers defined in the following way: A positive integer n is called a balancing number with balancer r, if it is the solution of the Diophantine equation $1 + 2 + \ldots + (n-1) = (n+1) + (n+2) + \ldots + (n+r)$. They also proved that the

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recurrence relation for balancing numbers is

$$B_{n+1} = 6B_n - B_{n-1}, \quad n > 2, \tag{1.1}$$

where B_n is the n^{th} balancing number with $B_1 = 1$ and $B_2 = 6$.

It is well known that (see [1]), n is a balancing number if and only if n^2 is a triangular number, that is $8n^2 + 1$ is a perfect square. In [10], Lucas-balancing numbers are defined as follows: If n is a balancing number, $C_n = \sqrt{8n^2 + 1}$ is called a Lucas-balancing number. The recurrence relation for Lucas-balancing numbers is same as that of balancing numbers, that is

$$C_{n+1} = 6C_n - C_{n-1}, \quad n > 2, \tag{1.2}$$

where C_n is the n^{th} Lucas-balancing number with $C_1 = 3$ and $C_2 = 17$. Liptai [4], showed that the only balancing number in the sequence of Fibonacci numbers is 1. In [11] and [12], Ray obtain nice product formulas for both balancing and Lucas-balancing numbers. Panda and Ray [8], link balancing numbers with Pell and associated Pell numbers. They shown that balancing numbers are indeed the product of Pell and associated Pell numbers. Many interesting properties and important identities are available in the literature. Interested readers can follow [2, 3, 5, 6, 7, 13, 14].

The closed form of both balancing and Lucas-balancing numbers are respectively given by

$$B_n = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2} \tag{1.3}$$

and

$$C_n = \frac{\lambda_1^n + \lambda_2^n}{2} \tag{1.4}$$

for $n \geq 1$ with $\lambda_1 = 3 + \sqrt{8}$, $\lambda_2 = 3 - \sqrt{8}$. These relations (1.3) and (1.4) are popularly known as Binet's formulas for balancing and Lucas-balancing numbers. In this paper, we obtain some new identities for the common factors of these numbers.

2. New Identities for the Common Factors of Balancing and Lucas-Balancing Numbers

In this section, we present some new identities for the common factors of both balancing and Lucas-balancing numbers with the help of Binet's formula. It is clear that

$$\lambda_1 + \lambda_2 = 6, \quad \lambda_1 - \lambda_2 = 2\sqrt{8}, \quad \lambda_1\lambda_2 = 1. \tag{2.1}$$

Theorem 2.1. *For $n \geq 1$, the following identity is valid:*

$$B_{4n} - 6 = 2B_{2n-1}C_{2n+1}.$$

Proof. By (2.1), we obtain

$$\begin{aligned} 2B_{2n-1}C_{2n+1} &= 2\frac{\lambda_1^{2n-1} - \lambda_2^{2n-1}}{\lambda_1 - \lambda_2}\frac{\lambda_1^{2n+1} + \lambda_2^{2n+1}}{2} \\ &= \frac{\lambda_1^{4n} - \lambda_2^{4n}}{\lambda_1 - \lambda_2} - \frac{\lambda_1^{2} - \lambda_2^{2}}{\lambda_1 - \lambda_2} \\ &= B_{4n} - 6 \end{aligned}$$

which finishes the proof. □

Theorem 2.2. *For $n \geq 1$, the following identity is valid:*

$$B_{4n+1} + 1 = 2B_{2n+1}C_{2n}.$$

Proof. By (2.1), we get

$$\begin{aligned} 2B_{2n+1}C_{2n} &= 2\frac{\lambda_1^{2n+1} - \lambda_2^{2n+1}}{\lambda_1 - \lambda_2}\frac{\lambda_1^{2n} + \lambda_2^{2n}}{2} \\ &= \frac{\lambda_1^{4n+1} - \lambda_2^{4n+1}}{\lambda_1 - \lambda_2} + \frac{(\lambda_1\lambda_2)^n\lambda_1 - (\lambda_1\lambda_2)^n\lambda_2}{\lambda_1 - \lambda_2} \\ &= B_{4n+1} + 1 \end{aligned}$$

which is the end of the proof. □

Theorem 2.3. *For $n \geq 1$, the following identity is valid:*

$$B_{4n+2} + 6 = 2B_{2n+2}C_{2n}.$$

Proof. By (2.1), we have

$$\begin{aligned} 2B_{2n+2}C_{2n} &= 2\frac{\lambda_1^{2n+2} - \lambda_2^{2n+2}}{\lambda_1 - \lambda_2}\frac{\lambda_1^{2n} + \lambda_2^{2n}}{2} \\ &= \frac{\lambda_1^{4n+2} - \lambda_2^{4n+2}}{\lambda_1 - \lambda_2} + (\lambda_1\lambda_2)^{2n}\frac{\lambda_1^{2} - \lambda_2^{2}}{\lambda_1 - \lambda_2} \\ &= B_{4n+2} + 6 \end{aligned}$$

which is the end of the proof. □

By the same way, we have the following result.

Theorem 2.4. *For $n \geq 1$, the following identity is valid:*

$$B_{4n+3} - 1 = 2B_{2n+1}C_{2n+2}.$$

The following lemma is already established in[8]. For the sake of simplicity we present the proof again.

Lemma 2.5. *For $n \geq 1$, the following identity is valid:*

$$B_{2n} = 2B_nC_n.$$

Proof. By (2.1), we get

$$\begin{aligned} 2B_nC_n &= 2\frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}\frac{\lambda_1^n + \lambda_2^n}{2} \\ &= \frac{\lambda_1^{2n} - \lambda_2^{2n}}{\lambda_1 - \lambda_2} \\ &= B_{2n} \end{aligned}$$

which completes the proof. □

Lemma 2.6. *For $n \geq 1$, the following identity is valid:*

$$B_{4n+1} - 1 = 2B_{2n}C_{2n+1}.$$

Proof. By (2.1), we have

$$\begin{aligned} 2B_{2n}C_{2n+1} &= 2\frac{\lambda_1^{2n} - \lambda_2^{2n}}{\lambda_1 - \lambda_2}\frac{\lambda_1^{2n+1} + \lambda_2^{2n+1}}{2} \\ &= \frac{\lambda_1^{4n+1} - \lambda_2^{4n+1}}{\lambda_1 - \lambda_2} - (\lambda_1\lambda_2)^{2n}\frac{\lambda_1 - \lambda_2}{\lambda_1 - \lambda_2} \\ &= B_{4n+1} - 1 \end{aligned}$$

which is the end of the proof. □

By virtue of Lemma 2.5 and Lemma 2.6, we have the following result.

Corollary 2.7. *For $n \geq 1$, we have*

$$B_{4n+1} - 1 = 2B_nC_nC_{2n+1}.$$

Lemma 2.8. *For $n \geq 1$, the following identity is valid:*

$$C_{4n+1} - 3 = 16B_{2n}B_{2n+1}.$$

Proof. Since $(\lambda_1 - \lambda_2)^2 = 32$, we get

$$\begin{aligned}16B_{2n}B_{2n+1} &= 16\frac{\lambda_1^{2n} - \lambda_2^{2n}}{\lambda_1 - \lambda_2}\frac{\lambda_1^{2n+1} - \lambda_2^{2n+1}}{\lambda_1 - \lambda_2} \\ &= \frac{\lambda_1^{4n+1} + \lambda_2^{4n+1}}{2} - (\lambda_1\lambda_2)^{2n}\frac{\lambda_1 + \lambda_2}{2} \\ &= C_{4n+1} - 3\end{aligned}$$

which ends the proof. □

Since $B_{2n} = 2B_nC_n$, the following identity is valid for $n \geq 1$:

Theorem 2.9.

$$C_{4n+1} - 3 = 2C_{2n}C_{2n+1}.$$

Theorem 2.10. *For $n \geq 1$, the following identity is valid:*

$$C_{4n+1} + 3 = 2C_{2n}C_{2n+1}.$$

Proof. By (2.1), we have

$$\begin{aligned}2C_{2n}C_{2n+1} &= 2\frac{\lambda_1^{2n} + \lambda_2^{2n}}{2}\frac{\lambda_1^{2n+1} + \lambda_2^{2n+1}}{2} \\ &= \frac{\lambda_1^{4n+1} + \lambda_2^{4n+1}}{2} + (\lambda_1\lambda_2)^{2n}\frac{\lambda_1 + \lambda_2}{2} \\ &= C_{4n+1} + 3\end{aligned}$$

which is the end of the proof. □

Lemma 2.11. *For $n \geq 1$, the following identity is valid:*

$$B_{4n+3} + 1 = 2B_{2n+2}C_{2n+1}.$$

Proof. By using (2.1), we have

$$\begin{aligned}2B_{2n+2}C_{2n+1} &= 2\frac{\lambda_1^{2n+2} - \lambda_2^{2n+2}}{\lambda_1 - \lambda_2}\frac{\lambda_1^{2n+1} + \lambda_2^{2n+1}}{2} \\ &= \frac{\lambda_1^{4n+2} - \lambda_2^{4n+2}}{\lambda_1 - \lambda_2} + (\lambda_1\lambda_2)^{2n+1} \\ &= B_{4n+3} + 1\end{aligned}$$

which completes the proof. □

Theorem 2.12. *For $n \geq 1$, we have*

$$B_{4n+3} + 1 = 4B_{n+1}C_{n+1}C_{2n+1}.$$

Proof. Substituting $B_{2n+2} = 2B_{n+1}C_{n+1}$ from Lemma 2.5 into Lemma 2.11, we obtain the desired result. □

Theorem 2.13. *For $n \geq 1$, the following identity is valid:*

$$C_{4n+3} + 3 = 2C_{2n+1}C_{2n+2}.$$

Proof. By (2.1), we have

$$\begin{aligned} 2C_{2n+1}C_{2n+2} &= 2\frac{\lambda_1^{2n+1} + \lambda_2^{2n+1}}{2}\frac{\lambda_1^{2n+2} + \lambda_2^{2n+2}}{2} \\ &= \frac{\lambda_1^{4n+3} + \lambda_2^{4n+3}}{2} + (\lambda_1\lambda_2)^{2n+1}\frac{\lambda_1 + \lambda_2}{2} \\ &= C_{4n+3} + 3 \end{aligned}$$

which completes the proof. □

Theorem 2.14. *For $n \geq 1$, the following identity is valid:*

$$C_{4n+3} - 3 = 16B_{2n+1}B_{2n+2}.$$

Proof. Since $(\lambda_1 - \lambda_2)^2 = 32$, we obtain

$$\begin{aligned} 16B_{2n+1}B_{2n+2} &= 2\frac{\lambda_1^{2n+1} - \lambda_2^{2n+1}}{\lambda_1 - \lambda_2)}\frac{\lambda_1^{2n+2} - \lambda_2^{2n+2}}{\lambda_1 - \lambda_2} \\ &= \frac{\lambda_1^{4n+3} + \lambda_2^{4n+3}}{2} - (\lambda_1\lambda_2)^{2n+1}\frac{\lambda_1 + \lambda_2}{2} \\ &= C_{4n+3} - 3 \end{aligned}$$

which ends the proof. □

The following corollary is an immediate consequence of Theorem 2.14.

Corollary 2.15. *For $n \geq 1$, we have*

$$C_{4n+3} - 3 = 32B_{n+1}C_{n+1}B_{2n+1}.$$

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