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CONVERGENCE OF IMPLICIT ITERATION PROCESS FOR A COUNTABLE FAMILY OF CONTINUOUS PSEUDOCONTRACTIVE MAPPINGS

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Abstract: We study weak convergence of implicit iterations for a countable family of continuous pseudocontractive mappings and a nonexpansive mapping in Banach spaces. Moreover, necessary and sufficient conditions for strong convergence to a common fixed point of continuous hemicontractive mappings and a continuous quasi-nonexpansive mapping are given in real Banach spaces. The obtained results extend those announced by many authors.

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1. Introduction

Let E be a real Banach space and K a nonempty subset of E. Let J denote the normalized duality mapping from E into 2^{E^*} given by $J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}$, $\forall x \in E$, where E^* denotes the dual space of E and $\langle \cdot, \cdot\rangle$ denotes the duality pairing. If E is smooth or E^* is strictly convex, then J is single-valued.

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Throughout this paper, we denote the single-valued duality mapping by j and denote the set of fixed points of a nonlinear mapping $T : K \to E$ by

$$F(T) = \{x \in K : \; Tx = x\}.$$

Definition 1.1. A mapping T with domain $D(T)$ and range $R(T)$ in E is called
(i) *pseudocontractive* [2] if for all $x, y \in D(T)$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \leq \|x-y\|^2;$$

equivalently [2, 8], for all $x, y \in D(T)$ and for all $s > 0$,

$$\|x-y\| \leq \|x-y+s[(I-T)x-(I-T)y]\|; \tag{1.1}$$

(ii) *hemicontractive* if for all $x \in D(T),\; x^* \in F(T)$, there exists $j(x-x^*) \in J(x-x^*)$ such that

$$\langle Tx - x^*, j(x-x^*)\rangle \leq \|x-x^*\|^2;$$

(iii) λ*-strictly pseudocontractive in the terminology of Browder-Petryshyn* [1] if for all $x, y \in D(T)$, there exists $\lambda > 0$ and $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \leq \|x-y\|^2 - \lambda\|x-y-(Tx-Ty)\|^2;$$

(iv) *strongly pseudocontractive* if for all $x, y \in D(T)$, there exists $\lambda \in (0,1)$ and $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \leq \lambda\|x-y\|^2;$$

(v) *L-Lipschitzian* if for all $x, y \in D(T)$, there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x-y\|;$$

(vi) *nonexpansive* if for all $x, y \in D(T)$,

$$\|Tx - Ty\| \leq \|x-y\|;$$

(vii) *quasi-nonexpansive* if for all $x \in D(T),\; x^* \in F(T)$,

$$\|Tx - x^*\| \leq \|x-x^*\|.$$

Remark 1.2. It is obvious by definitions that:
(1) Every strictly pseudocontractive mapping is pseudocontractive.
(2) Every pseudocontractive mapping is hemicontractive.
(3) Every λ-strictly pseudocontractive mapping is $(\frac{1+\lambda}{\lambda})$-Lipschitzian; see [5].

Let $\{T_i\}_{i=1}^N$ be a finite family of nonlinear self-mappings on a subset K. Let $\{x_n\}$ be defined by $x_0 \in K$ and

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)T_n x_n, \quad n \geq 1 \tag{1.2}$$

where $\alpha_n \in (0,1)$ and $T_n = T_{n \bmod N}$. The implicit iteration (1.2) was introduced by Xu and Ori [18] for a finite family of nonexpansive mappings in a Hilbert space. To be more precise, they proved the following theorem:

Theorem 1.3. *[18] Let H be a real Hilbert space, K a nonempty, closed and convex subset of H, and $\{T_i\}_{i=1}^N$ a finite family of nonexpansive self-mappings on K such that $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $\{x_n\}$ be defined by (1.2). If $\lim_{n\to\infty} \alpha_n = 0$, then $\{x_n\}$ converges weakly to a common fixed point of $\{T_i\}_{i=1}^N$.*

Motivated by Xu and Ori [18]'s idea, Osilike [12] extended the above theorem from the class of nonexpansive mappings to the more general class of strictly pseudocontractive mappings. He proved the following theorem:

Theorem 1.4. *[12] Let H be a real Hilbert space, K a nonempty closed convex subset of H, and $\{T_i\}_{i=1}^N$ a finite family of strictly pseudocontractive self-mappings on K such that $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $\{x_n\}$ be defined by (1.2). If $\lim_{n\to\infty} \alpha_n = 0$, then $\{x_n\}$ converges weakly to a common fixed point of $\{T_i\}_{i=1}^N$.*

In 2006, Chen et al. [5] extended Osilike [12]'s result from Hilbert spaces to q-uniformly smooth and uniformly convex Banach spaces. They proved the following theorem:

Theorem 1.5. *[5] Let E be a real q-uniformly smooth Banach space which is also uniformly convex and satisfies Opial's condition. Let K be a nonempty, closed and convex subset of E, and $T_i : K \to K$, $i = 1, 2, ..., N$ be a finite family of strictly pseudocontractive self-mappings on K such that $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $\{x_n\}$ be defined by (1.2). If $0 < a \leq \alpha_n \leq b < 1$, then $\{x_n\}$ converges weakly to a common fixed point of $\{T_i\}_{i=1}^N$.*

Recently, Zhou [19] extended the results of Xu and Ori [18], Osilike [12] and Chen et al. [5] to the more general uniformly convex Banach spaces and the more general class of Lipschitzian pseudocontractive mappings; in particular, he proved the following theorem:

Theorem 1.6. *[19] Let E be a real uniformly convex Banach space with a Fréchet differentiable norm. Let K be a closed and convex subset of E, and $\{T_i\}_{i=1}^N$ be a finite family of Lipschitzian pseudocontractive self-mappings of K such that $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $\{x_n\}$ be defined by (1.2). If $\{\alpha_n\}$ is* *chosen so that* $\alpha_n \in (0,1)$ *with* $\limsup_{n\to\infty} \alpha_n < 1$, *then* $\{x_n\}$ *converges weakly to a common fixed point of* $\{T_i\}_{i=1}^N$.

Motivated and inspired by Xu and Ori [18], Osilike [12], Chen et al. [5] and Zhou [19], we consider the following implicit iteration: $x_0 \in K$ and

$$\begin{aligned} x_n &= \alpha_n y_{n-1} + (1-\alpha_n)T_n x_n, \\ y_{n-1} &= \beta_n x_{n-1} + (1-\beta_n)Sx_{n-1}, \quad n \geq 1 \end{aligned} \tag{1.3}$$

where $\alpha_n, \beta_n \in (0,1)$ and S, $\{T_n\}_{n=1}^\infty$ are nonlinear mappings on a closed and convex subset K of a real Banach space E.

In this paper, we prove strong convergence for the implicit iteration (1.3) in the frameworks of an arbitrary real Banach space. Then we prove weak convergence results in a uniformly convex Banach space which satisfies Opial's condition or has a Fréchet differentiable norm. The results obtained in this paper extend the results of Zhou [19] from a finite family Lipschitzian pseudocontractions to a countable family of continuous pseudocontractions. Consequently, our results also extend and improve the results of Xu and Ori [18], Osilike [12], Rafiq [14], Song [16], Chen et al. [5] and some others.

The implicit iteration process for nonlinear mappings in the framework of Hilbert spaces and Banach spaces has been studied by several authors; see also [3, 4, 7, 9].

We will use the notation:
- $\rightharpoonup$ for weak convergence and $\to$ for strong convergence.
- $\omega_\omega(x_n) = \{x : x_{n_i} \rightharpoonup x\}$ denotes the weak ω-limit set of $\{x_n\}$.
- $d(x, C) = \inf_{z\in C} \|x - z\|$.

2. Preliminaries

Let E be a real Banach space and $S(E) = \{x \in E : \|x\| = 1\}$. Then E is said to be *smooth* if

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in S(E)$. The norm of E is said to be *Fréchet differentiable* if for each $x \in S(E)$, the limit is attained uniformly for $y \in S(E)$.

A Banach space E is called *uniformly convex* if for each $\epsilon > 0$ there is a $\delta > 0$ such that for $x, y \in E$ with $\|x\|, \|y\| \leq 1$ and $\|x - y\| \geq \epsilon$, $\|x + y\| \leq 2(1-\delta)$ holds. A Banach space E is called *strictly convex* if $\|x + y\|/2 < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. In a strictly convex Banach E we have that if $\|x\| = \|y\| = \|\lambda x + (1-\lambda)y\|$ for $x, y \in E$ and $\lambda \in (0,1)$, then $x = y$. It is well known that a uniformly convex Banach space is strictly convex.

In the sequel, we shall need the following definitions and lemmas.

Definition 2.1. A Banach space E is said to satisfy *Opial's condition* [13], if whenever $\{x_n\}$ is a sequence in E which converge weakly to x as $n \to \infty$, then

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|, \quad \forall y \in E, \ x \neq y.$$

Lemma 2.2. *[19] Let E be a real uniformly convex Banach space, K a nonempty, closed and convex subset of E, and $T : K \to K$ be a continuous pseudocontractive mapping. Then, $I - T$ is demiclosed at zero, that is, for all sequence $\{x_n\} \subset K$ with $x_n \rightharpoonup p$ and $\|x_n - Tx_n\| \to 0$ it follows that $p = Tp$.*

Lemma 2.3. *[19] Let E be a smooth Banach space and K be a nonempty and convex subset of E. Given an integer $N \geq 1$, assume that for each $i \in \Lambda$, $S_i : K \to K$ is a λ_i-strictly pseudocontraction for some $0 \leq \lambda_i < 1$. Assume that $\{\mu_i\}_{i=1}^N$ is a positive sequence such that $\sum_{i=1}^N \mu_i = 1$. Then $\sum_{i=1}^N \mu_i S_i : K \to K$ is a λ-strictly pseudocontraction with $\lambda = \min\{\lambda_i : \ 1 \leq i \leq N\}$.*

Lemma 2.4. *[19] Let E be a smooth Banach space and K be a nonempty and convex subset of E. Given an integer $N \geq 1$, assume that $\{S_i\}_{i=1}^N : K \to K$ is a finite family of λ_i-strictly pseudocontraction for some $0 \leq \lambda_i < 1$ such that $F = \bigcap_{i=1}^N F(S_i) \neq \emptyset$. Assume that $\{\mu_i\}_{i=1}^N$ is a positive sequence such that $\sum_{i=1}^N \mu_i = 1$. Then $F(\sum_{i=1}^N \mu_i S_i) = F$.*

Lemma 2.5. *[15] Suppose that E is a uniformly convex Banach space and $0 < s \leq t_n \leq t < 1$ for all positive integers n. Also suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences of E such that $\limsup_{n\to\infty} \|x_n\| \leq r$, $\limsup_{n\to\infty} \|y_n\| \leq r$ and $\lim_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = r$ for some $r \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 2.6. *[6] Let E be a real Banach space and K a nonempty, closed and convex subset of E, and $T : K \to K$ a continuous strongly pseudocontractive mapping. Then T has a unique fixed point in K.*

Let K be a nonempty, closed and convex subset of a real Banach space E and T a continuous strongly pseudocontractive mapping of K. For every $u \in K$ and $t \in (0,1)$, the mapping $S_t : K \to K$ defined by

$$S_t x = tu + (1-t)Tx, \quad x \in K,$$

is a continuous and strongly pseudocontractive mapping; by utilizing Lemma

2.6, there exists a unique fixed point $x_t \in K$ of S_t such that

$$x_t = tu + (1-t)Tx_t, \quad t \in (0,1).$$

Lemma 2.7. *[17] Let E be a real uniformly convex Banach space with a Fréchet differentiable norm. Let K be a closed and convex subset of E, and $\{T_n\}_{n=1}^{\infty}$ be a family of Lipschitzian self-mappings on K such that $\sum_{n=1}^{\infty}(L_n - 1) < \infty$ and $F = \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. For arbitrary $x_1 \in K$, define $x_{n+1} = T_n x_n$ for all $n \geq 1$. Then for every $p, q \in F$, $\lim_{n\to\infty} \langle x_n, j(p-q)\rangle$ exists, in particular, for all $u, v \in \omega_\omega(x_n)$, and $p, q \in F$, $\langle u - v, j(p-q)\rangle = 0$.*

3. Convergence in Banach Spaces

In this section, we prove a strong convergence of an implicit iteration for continuous hemicontractive mappings and a continuous quasi-nonexpansive mapping in a real arbitrary Banach space.

To prove our main results, we need the following lemma:

Lemma 3.1. *Let E be a real Banach space and K a nonempty, closed and convex subset of E. Let S be a continuous quasi-nonexpansive self-mapping on K and $\{T_n\}_{n=1}^{\infty}$ a countable family of continuous hemicontractive self-mappings on K such that $F = \bigcap_{n=1}^{\infty} F(T_n) \cap F(S) \neq \emptyset$. Let $\{x_n\}$ be defined by (1.3) and let $\{\alpha_n\}, \{\beta_n\}$ be real sequences in $(0,1)$. Then:*

(i) $\lim_{n\to\infty} \|x_n - p\|$, $\lim_{n\to\infty} \|y_n - p\|$ exist and $\lim_{n\to\infty} \|x_n - p\| = \lim_{n\to\infty} \|y_n - p\|$ for all $p \in F$,

(ii) $\lim_{n\to\infty} d(x_n, F)$, $\lim_{n\to\infty} d(y_n, F)$ exist and $\lim_{n\to\infty} d(x_n, F) = \lim_{n\to\infty} d(y_n, F)$.

Proof. Let $p \in F$ and $n \geq 1$. Then there exists $j(x_n - p) \in J(x_n - p)$ such that

$$\begin{aligned}
\|x_n - p\|^2 &= \langle x_n - p, j(x_n - p)\rangle \\
&= \alpha_n \langle y_{n-1} - p, j(x_n - p)\rangle + (1-\alpha_n)\langle T_n x_n - p, j(x_n - p)\rangle \\
&\leq \alpha_n \|y_{n-1} - p\| \|x_n - p\| + (1-\alpha_n)\|x_n - p\|^2 \\
&\leq \alpha_n \big(\beta_n \|x_{n-1} - p\| + (1-\beta_n)\|Sx_{n-1} - p\|\big)\|x_n - p\| \\
&\quad + (1-\alpha_n)\|x_n - p\|^2 \\
&\leq \alpha_n \|x_{n-1} - p\| \|x_n - p\| + (1-\alpha_n)\|x_n - p\|^2.
\end{aligned}$$

Hence

$$\|x_n - p\| \leq \|y_{n-1} - p\| \leq \|x_{n-1} - p\|. \tag{3.1}$$

This implies that (i) holds. By taking the infimum over all $p \in F$ in (3.1), we also obtain

$$d(x_n, F) \leq d(y_{n-1}, F) \leq d(x_{n-1}, F). \tag{3.2}$$

This shows that $\lim_{n\to\infty} d(x_n, F)$ exists. Moreover, by taking the limit as $n \to \infty$ to (3.2), $\lim_{n\to\infty} d(y_n, F) = \lim_{n\to\infty} d(x_n, F)$. Thus, (i) and (ii) are proved. □

Now, we prove our result.

Theorem 3.2. *Let E be a real Banach space and K a nonempty, closed and convex subset of E. Let S be a continuous quasi-nonexpansive self-mapping on K and $\{T_n\}_{n=1}^{\infty}$ a countable family of continuous hemicontractive self-mappings on K such that $F = \bigcap_{n=1}^{\infty} F(T_n) \cap F(S) \neq \emptyset$. Let $\{x_n\}$ be defined by (1.3), and let $\{\alpha_n\}, \{\beta_n\}$ be real sequences in $(0, 1)$. Then, $\{x_n\}$ converges strongly to $x^* \in F$ if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$.*

Proof. Since the necessity is obvious, it suffices to show the sufficiency. Suppose $\liminf_{n\to\infty} d(x_n, F) = 0$. From Lemma 3.1 (ii) we have $\lim_{n\to\infty} d(x_n, F) = 0$. It follows from (3.1) that for $n, m \in \mathbb{N}$ and $p \in F$,

$$\|x_{n+m} - x_n\| \leq \|x_{n+m} - p\| + \|x_n - p\| \leq 2\|x_n - p\|.$$

Consequently,

$$\|x_{n+m} - x_n\| \leq 2d(x_n, F) \to 0,$$

as $n \to \infty$. Hence $\{x_n\}$ is a Cauchy sequence. By the completeness of E, we can assume that $\lim_{n\to\infty} x_n = x^*$ for some $x^* \in E$. Then

$$d(x^*, F) = \lim_{n\to\infty} d(x_n, F) = 0.$$

Hence $x_n \to x^* \in F$ as $n \to \infty$. This completes the proof. □

Remark 3.3. Theorem 3.2 improves and extends Theorem 2.3 of Chen et al.[5], Theorem 2.2 of Boonchari and Saejung [3], and Theorem 2 of Osilike [12].

4. Convergence in Uniformly Convex Banach Spaces

In this section, we prove weak convergence theorems of implicit iteration process for continuous pseudocontractive mappings and a nonexpansive mapping in a uniformly convex Banach space.

Let K be a subset of a Banach space E. Let $\{T_n\}$ and Γ be families of mappings on K such that $\emptyset \neq F(\Gamma) \subset \bigcap_{n=1}^{\infty} F(T_n)$. Then, a countable family of mappings $\{T_n\}$ is said to satisfy:
(i) *The NST-condition* [10] if for each bounded sequence $\{z_n\}$ in K,

$$\lim_{n\to\infty} \|z_n - T_n z_n\| = 0 \ \text{ implies } \ \lim_{n\to\infty} \|z_n - T z_n\| = 0 \ \ \forall T \in \Gamma.$$

(ii) *The NST*-condition* [11] if for each bounded sequence $\{z_n\}$ in K,

$$\lim_{n\to\infty} \|z_n - T_n z_n\| = \lim_{n\to\infty} \|z_n - z_{n+1}\| = 0 \ \text{ implies } \ \lim_{n\to\infty} \|z_n - T z_n\| = 0 \ \ \forall T \in \Gamma.$$

Remark 4.1. It follows directly that if $\{T_n\}$ satisfies the NST-condition, then $\{T_n\}$ satisfies the NST*-condition.

Using the NST-condition and the NST*-condition, we obtain the following:

Lemma 4.2. *Let E be a real uniformly convex Banach space and K a nonempty, closed and convex subset of E. Let S be a nonexpansive self-mapping on K and $\{T_n\}_{n=1}^{\infty}$ a countable family of continuous pseudocontractive self-mappings on K such that $F = \bigcap_{n=1}^{\infty} F(T_n) \cap F(S) \neq \emptyset$. Let Γ be any subclass of continuous pseudocontractive mappings such that $\emptyset \neq F(\Gamma) \subset \bigcap_{n=1}^{\infty} F(T_n)$. Let $\{x_n\}$ be defined by (1.3) and let $\{\alpha_n\}, \{\beta_n\}$ be real sequences with $0 < \alpha_n \leq a < 1$ and $0 < b \leq \beta_n \leq c < 1$ for some $a, b, c \in (0,1)$. If $\{T_n\}_{n=1}^{\infty}$ satisfies the NST*-condition, then*

$$\lim_{n\to\infty} \|x_n - Sx_n\| = \lim_{n\to\infty} \|x_n - Tx_n\| = 0, \quad \forall T \in \Gamma.$$

Proof. Let $p \in F$. Then, by Lemma 3.1 (i), we have

$$\lim_{n\to\infty} \|x_n - p\| = \lim_{n\to\infty} \|y_n - p\| = d,$$

for some $d \geq 0$. By using (1.1) and (3.1), we also have

$$\begin{aligned} \|x_n - p\| &\leq \left\| x_n - p + \frac{1-\alpha_n}{2\alpha_n}(x_n - T_n x_n) \right\| \\ &= \left\| x_n - p + \frac{1-\alpha_n}{2}(y_{n-1} - T_n x_n) \right\| \end{aligned}$$

$$\begin{aligned}
&= \left\|\alpha_n y_{n-1} + (1-\alpha_n)T_n x_n - p + \frac{1-\alpha_n}{2}(y_{n-1} - T_n x_n)\right\| \\
&= \left\|\frac{y_{n-1} + x_n}{2} - p\right\| \\
&= \left\|\frac{y_{n-1} - p}{2} + \frac{x_n - p}{2}\right\| \\
&\leq \frac{1}{2}\|y_{n-1} - p\| + \frac{1}{2}\|x_n - p\| \\
&\leq \frac{1}{2}\|x_{n-1} - p\| + \frac{1}{2}\|x_n - p\| \\
&\leq \|x_{n-1} - p\|,
\end{aligned}$$

which implies that

$$\lim_{n\to\infty} \left\|\frac{y_{n-1} - p}{2} + \frac{x_n - p}{2}\right\| = d.$$

By Lemma 2.5, we obtain

$$\lim_{n\to\infty} \|y_{n-1} - x_n\| = 0. \tag{4.1}$$

On the other hand, we also have

$$\begin{aligned}
\|x_n - p\| &\leq \|y_{n-1} - p\| \\
&= \|\beta_n(x_{n-1} - p) + (1-\beta_n)(Sx_{n-1} - p)\| \\
&\leq \|x_{n-1} - p\|.
\end{aligned}$$

Hence

$$\lim_{n\to\infty} \|\beta_n(x_{n-1} - p) + (1-\beta_n)(Sx_{n-1} - p)\| = d.$$

It is easy to see that $\limsup_{n\to\infty} \|Sx_{n-1} - p\| \leq d$. Since $0 < b \leq \beta_n \leq c < 1$, it follows from Lemma 2.5 that

$$\lim_{n\to\infty} \|x_n - Sx_n\| = 0. \tag{4.2}$$

Again by (1.3) we observe that

$$x_n - T_n x_n = \frac{\alpha_n}{1-\alpha_n}(y_{n-1} - x_n),$$

which implies

$$\|x_n - T_n x_n\| = \frac{\alpha_n}{1-\alpha_n}\|y_{n-1} - x_n\|.$$

From (4.1) and $0 < \alpha_n \le a < 1$, we obtain

$$\lim_{n\to\infty} \|x_n - T_n x_n\| = 0. \tag{4.3}$$

On the other hand, from (4.2), we also obtain

$$\|y_{n-1} - x_{n-1}\| = (1 - \beta_n)\|Sx_{n-1} - x_{n-1}\| \to 0, \tag{4.4}$$

as $n \to \infty$. So, by (4.1) and (4.4), we have

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0. \tag{4.5}$$

Since $\{T_n\}$ satisfies the NST*-condition, it follows from (4.3) and (4.5) that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0,$$

for all $T \in \Gamma$. This completes the proof. □

Now we prove our main results.

Theorem 4.3. *Let E be a real uniformly convex Banach space which satisfies Opial's condition and K a nonempty, closed and convex subset of E. Let S be a nonexpansive self-mapping on K and $\{T_n\}_{n=1}^{\infty}$ a countable family of continuous pseudocontractive self-mappings on K such that $F = \bigcap_{n=1}^{\infty} F(T_n) \cap F(S) \ne \emptyset$. Let Γ be any subclass of continuous pseudocontractive mappings such that $\emptyset \ne F(\Gamma) \subset \bigcap_{n=1}^{\infty} F(T_n)$. Let $\{\alpha_n\}, \{\beta_n\}$ be real sequences with $0 < \alpha_n \le a < 1$ and $0 < b \le \beta_n \le c < 1$ for some $a, b, c \in (0, 1)$. If $\{T_n\}_{n=1}^{\infty}$ satisfies the NST*-condition, then a sequence $\{x_n\}$ defined by (1.3) converges weakly to $x^* \in F$.*

Proof. By Lemma 2.2, we know that T is demiclosed at zero for all $T \in \Gamma$. It follows from Lemma 3.1 (i) and Lemma 4.2 that $\omega_\omega(x_n) \subset F(S) \cap F(\Gamma) \subset F$. Moreover, in a uniformly convex Banach space, Opial's condition ensures that $\omega_\omega(x_n)$ is a singleton. We thus complete the proof. □

Remark 4.4. If $S = I$, then Theorem 4.3 improves and extends Theorem 5 of Chen et al. [4] and Theorem 2.6 of Chen et al. [5] in several respects:
(i) From real q-uniformly smooth and uniformly convex Banach spaces to real uniformly convex Banach spaces.
(ii) From a finite family of strictly pseudocontractions to a countable family of continuous pseudocontractions.
(iii) Relax the restriction on $\{\alpha_n\}$ in Theorem 2.6 of [5].

Theorem 4.5. *Let E be a real uniformly convex Banach space with a Fréchet differentiable norm and K a nonempty, closed and convex subset of E. Let S be a nonexpansive self-mapping on K and $\{T_n\}_{n=1}^{\infty}$ a countable family of continuous pseudocontractive self-mappings on K such that $F = \bigcap_{n=1}^{\infty} F(T_n) \cap F(S) \neq \emptyset$. Let Γ be any subclass of continuous pseudocontractive mappings such that $\emptyset \neq F(\Gamma) \subset \bigcap_{n=1}^{\infty} F(T_n)$. Let $\{\alpha_n\}, \{\beta_n\}$ be real sequences with $0 < \alpha_n \leq a < 1$ and $0 < b \leq \beta_n \leq c < 1$ for some $a, b, c \in (0,1)$. If $\{T_n\}_{n=1}^{\infty}$ satisfies the NST*-condition, then a sequence $\{x_n\}$ defined by (1.3) converges weakly to $x^* \in F$.*

Proof. As shown in Theorem 4.3, we get that $\omega_\omega(x_n) \subset F$. So, it suffices to prove that $\omega_\omega(x_n)$ is a singleton. First, we show that the mapping $D_n : K \to K$ defined by $D_n x = \frac{1}{\alpha_{n+1}}\big(I - (1-\alpha_{n+1})T_{n+1}\big)x,\ x \in K$ is one-to-one for all $n \geq 0$. From (1.1), for each $x, y \in K$, we have

$$\begin{aligned} \|x-y\| &\leq \Big\|x - y + \frac{(1-\alpha_{n+1})}{\alpha_{n+1}}[(x - T_{n+1}x) - (y - T_{n+1}y)]\Big\| \\ &= \frac{1}{\alpha_{n+1}}\Big\|x - y - (1-\alpha_{n+1})T_{n+1}x + (1-\alpha_{n+1})T_{n+1}y\Big\| \\ &= \frac{1}{\alpha_{n+1}}\Big\|(I - (1-\alpha_{n+1})T_{n+1})x - (I - (1-\alpha_{n+1})T_{n+1})y\Big\| \\ &= \Big\|D_n x - D_n y\Big\|. \end{aligned} \tag{4.6}$$

Thus D_n is one-to-one for all $n \geq 0$. For each $n \geq 0$, let $C_n = D_n^{-1}$ and $Q_n = \beta_{n+1} I + (1-\beta_{n+1})S$. Then, we observe that (1.3) is equivalent to

$$x_{n+1} = C_n Q_n x_n, \quad \forall n \geq 0.$$

Next, we shall show that $C_n(K) \subset K$ for every $n \geq 0$. For $u \in K$, we know from Lemma 2.6 that the continuous and strongly pseudocontractive mapping $S_{n,u} : K \to K$ defined by

$$S_{n,u}x = \alpha_n u + (1-\alpha_n)T_n x \ \ \forall n \geq 1,\ x \in K$$

has a unique fixed point $p_n \in K$. So we have

$$p_n = S_{n,u}p_n = \alpha_n u + (1-\alpha_n)T_n p_n \ \ \forall n \geq 1.$$

This implies that

$$u = \alpha_n^{-1}\big(I - (1-\alpha_n)T_n\big)p_n.$$

Hence, we obtain that $K \subset \alpha_n^{-1}\big(I-(1-\alpha_n)T_n\big)(K)$ for all $n \geq 1$; consequently,

$$\alpha_n\big(I-(1-\alpha_n)T_n\big)^{-1}(K) \subset K \quad \forall n \geq 1.$$

This shows that $C_n(K) \subset K$ for all $n \geq 0$. To apply Lemma 2.7, we will show that C_nQ_n is nonexpansive. Since S is nonexpansive, Q_n is also nonexpansive. So it suffices to show that C_n is nonexpansive. From (4.6), we see that $D_n = C_n^{-1}$ is one-to-one for all $n \geq 0$. For each x', $y' \in K$ and $n \geq 0$, we set $x' = C_n^{-1}x$ and $y' = C_n^{-1}y$. Then

$$\|C_nx' - C_ny'\| \leq \|x' - y'\|, \quad \forall n \geq 0.$$

Thus C_n is nonexpansive for all $n \geq 0$. So is C_nQ_n. Moreover, $C_n^{-1}(K)$ is closed for all $n \geq 0$.

Next, we show that $\bigcap_{n=0}^{\infty} F(C_n) = \bigcap_{n=1}^{\infty} F(T_n)$. Let $z \in \bigcap_{n=0}^{\infty} F(C_n)$. Then

$$z = C_nz = \alpha_{n+1}\big(I-(1-\alpha_{n+1})T_{n+1}\big)^{-1}z,$$

this implies that

$$\frac{1}{\alpha_{n+1}}\big(I-(1-\alpha_{n+1})T_{n+1}\big)z = z.$$

Hence $z = T_nz$ for all $n \geq 1$. On the other hand, let $z \in \bigcap_{n=1}^{\infty} F(T_n)$. Then from (4.6) we have

$$\begin{aligned}
\|z - C_nz\| &\leq \big\|D_nz - D_nC_nz\big\| \\
&= \left\|\frac{1}{\alpha_{n+1}}(I-(1-\alpha_{n+1})T_{n+1})z - z\right\| \\
&= 0.
\end{aligned}$$

Thus $z = C_nz$ for all $n \geq 0$. Hence, $\bigcap_{n=0}^{\infty} F(C_n) = \bigcap_{n=1}^{\infty} F(T_n)$.

Next, we will show that $\bigcap_{n=0}^{\infty} F(C_nQ_n) = F$. $F \subset \bigcap_{n=0}^{\infty} F(C_nQ_n)$ is obvious. Let $z \in \bigcap_{n=0}^{\infty} F(C_nQ_n)$ and $p \in F \subset \bigcap_{n=0}^{\infty} F(C_n)$. Then,

$$\begin{aligned}
\|z - p\| &= \|C_nQ_nz - C_np\| \\
&\leq \|Q_nz - p\| \\
&= \|\beta_{n+1}z + (1-\beta_{n+1})Sz - p\| \\
&= \|\beta_{n+1}(z-p) + (1-\beta_{n+1})(Sz-p)\| \\
&\leq \beta_{n+1}\|z-p\| + (1-\beta_{n+1})\|Sz-p\| \\
&\leq \|z-p\|.
\end{aligned}$$

It follows that $\|z-p\| = \|Sz-p\| = \|\beta_{n+1}(z-p)+(1-\beta_{n+1})(Sz-p)\|$. Since E is strictly convex, $z = Sz$. We also have $z = C_nQ_nz = C_nz$; consequently, $\bigcap_{n=0}^{\infty} F(C_nQ_n) \subset F$. Hence, $\bigcap_{n=0}^{\infty} F(C_nQ_n) = F$.

Finally, we will show that $\omega_\omega(x_n)$ is a singleton. Suppose that $x^*, y^* \in \omega_\omega(x_n)$. By Lemma 2.2, we know that $x^*, y^* \in F = \bigcap_{n=0}^{\infty} F(C_nQ_n)$. By Lemma 2.7, we also get that $\lim_{n\to\infty} \langle x_n, j(x^*-y^*)\rangle$ exists. Suppose that $\{x_{n_k}\}$ and $\{x_{m_k}\}$ are subsequences of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x^*$ and $x_{m_k} \rightharpoonup y^*$. Then

$$\|x^*-y^*\|^2 = \langle x^*-y^*, j(x^*-y^*)\rangle = \lim_{k\to\infty} \langle x_{n_k}-x_{m_k}, j(x^*-y^*)\rangle = 0.$$

Hence $x^* = y^*$; consequently, $x_n \rightharpoonup x^* \in F$ as $n \to \infty$. This completes the proof. □

Remark 4.6. In Theorem 4.3 and Theorem 4.5, if $T_n : K \to K$ is defined by $T_nx = \gamma_nx + (1-\gamma_n)Tx, \quad x \in K$ where $T : K \to K$ is a continuous pseudocontraction, $0 < \gamma_n \leq d < 1$ for some $d \in (0,1)$. Then we see that $\bigcap_{n=1}^{\infty} F(T_n) = F(T)$, $\{T_n\}_{n=1}^{\infty}$ is a countable family of continuous pseudocontractions and satisfies the NST*-condition. Hence, a sequence $\{x_n\}$ defined by $x_0 \in K$ and

$$\begin{aligned} x_n &= \alpha_n y_{n-1} + (1-\alpha_n)\big(\gamma_n x_n + (1-\gamma_n)Tx_n\big), \\ y_{n-1} &= \beta_n x_{n-1} + (1-\beta_n)Sx_{n-1}, \quad n \geq 1 \end{aligned}$$

converges weakly to $x^* \in \bigcap_{n=1}^{\infty} F(T_n) \cap F(S) = F(T) \cap F(S)$.

Remark 4.7. If $S = I$, then Theorem 4.5 extends Theorem 3.1 of Zhou [19] from a finite family of Lipschitzian pseudocontractions to a countable family of continuous pseudocontractions.

Since every strictly pseudocontractive mapping is continuous pseudocontractive, we immediately obtain the following results.

Theorem 4.8. *Let E be a real uniformly convex Banach space and K a nonempty, closed and convex subset of E. Let S be a nonexpansive self-mapping on K and $\{T_n\}_{n=1}^{\infty}$ a countable family of strictly pseudocontractive self-mappings on K such that $F = \bigcap_{n=1}^{\infty} F(T_n) \cap F(S) \neq \emptyset$. Let Γ be any subclass of strictly pseudocontractive mappings such that $\emptyset \neq F(\Gamma) \subset \bigcap_{n=1}^{\infty} F(T_n)$. Let $\{x_n\}$ be defined by (1.3), and let $\{\alpha_n\}, \{\beta_n\}$ be real sequences with $0 < \alpha_n \leq a < 1$ and $0 < b \leq \beta_n \leq c < 1$ for some $a, b, c \in (0,1)$. If $\{T_n\}_{n=1}^{\infty}$ satisfies the NST*-condition, then the following statements hold:*

(i) *If E satisfies Opial's condition, then $\{x_n\}$ converges weakly to $x^* \in F$.*

(ii) *If E has a Fréchet differentiable norm, then $\{x_n\}$ converges weakly to $x^* \in F$.*

Theorem 4.9. *Let E be a real smooth and uniformly convex Banach space and K a nonempty, closed and convex subset of E. Let S be a nonexpansive self-mapping on K and $\{S_i\}_{i=1}^N$ a finite family of λ_i-strictly pseudocontractive self-mappings on K such that $F = \bigcap_{i=1}^N F(S_i) \cap F(S) \neq \emptyset$. Let $\{\alpha_n\}, \{\beta_n\}$ and $\{\mu_{n,i}\}_{i=1}^N$ be real sequences with $0 < \alpha_n \leq a < 1,\ 0 < b \leq \beta_n \leq c < 1,\ 0 < d \leq \mu_{n,i} < 1$ and $\sum_{i=1}^N \mu_{n,i} = 1$ for some $a, b, c, d \in (0,1)$. Let $\{x_n\}$ be defined by the following manner: $x_0 \in K$ and*

$$\begin{aligned} x_n &= \alpha_n y_{n-1} + (1-\alpha_n)\sum_{i=1}^N \mu_{n,i} S_i x_n, \\ y_{n-1} &= \beta_n x_{n-1} + (1-\beta_n) S x_{n-1}, \quad n \geq 1. \end{aligned}$$

Then the following statements hold:
(i) *If E satisfies Opial's condition, then $\{x_n\}$ converges weakly to $x^* \in F$.*
(ii) *If E has a Fréchet differentiable norm, then $\{x_n\}$ converges weakly to $x^* \in F$.*

Proof. For each $n \geq 1$, define $T_n x = \sum_{i=1}^N \mu_{n,i} S_i x,\ x \in K$. By Lemma 2.3 and Lemma 2.4, we see that $T_n : K \to K$ is a λ-strictly pseudocontractive mapping with $\lambda = \min\{\lambda_i : \ 1 \leq i \leq N\}$ and $\bigcap_{i=1}^N F(S_i) = \bigcap_{n=1}^\infty F(T_n)$.

Next, we will show that $\{T_n\}$ satisfies the NST*-condition. By Remark 4.1, it suffices to show that $\{T_n\}$ satisfies the NST-condition. Let $\{z_n\}$ be a bounded sequence in K such that $\lim_{n\to\infty} \|z_n - T_n z_n\| = 0$ and let $z \in \bigcap_{n=1}^\infty F(T_n)$. Then,

$$\begin{aligned} \|z_n - z\|^2 &= \langle z_n - z, j(z_n - z)\rangle \\ &= \langle z_n - T_n z_n, , j(z_n - z)\rangle + \langle T_n z_n - z, , j(z_n - z)\rangle \\ &\leq \|z_n - T_n z_n\|\|z_n - z\| + \sum_{i=1}^N \mu_{n,i}\langle S_i z_n - z, j(z_n - z)\rangle \\ &\leq \|z_n - T_n z_n\|\|z_n - z\| + \|z_n - z\|^2 - \lambda \sum_{i=1}^N \mu_{n,i}\|z_n - S_i z_n\|^2, \end{aligned}$$

which implies that

$$\lambda d \sum_{i=1}^N \|z_n - S_i z_n\|^2 \leq \lambda \sum_{i=1}^N \mu_{n,i}\|z_n - S_i z_n\|^2 \leq \|z_n - T_n z_n\|\|z_n - z\|.$$

Since $\{z_n\}$ is bounded and $\lim_{n\to\infty} \|z_n - T_n z_n\| = 0$,

$$\lim_{n\to\infty} \|z_n - S_i z_n\| = 0, \quad 1 \leq i \leq N.$$

Hence, $\{T_n\}$ satisfies the NST-condition. By Theorem 4.8, the statements (i) and (ii) hold. □

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