International Journal of Pure and Applied Mathematics
Volume 85 No. 4 2013, 745-750
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v85i4.10
ijpam.eu

DOMINATION ON LEXICOGRAPHICAL PRODUCT OF COMPLETE GRAPHS

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Abstract: Let $\gamma(G)$ and $\gamma'(G)$ be the domination number and edge domination number, respectively. The lexicographical Product $G_1 \bullet G_2$ of graph of G_1 and G_2 has vertex set $V(G_1 \bullet G_2) = V(G_1) \times V(G_2)$ and edge set $E(G_1 \bullet G_2) = \{(u_1v_1)(u_2v_2)|\ [u_1u_2 \in E(G_1)] \cup [u_1 = u_2 \text{ and } v_1v_2 \in E(G_2)]\}$. In this paper, we determined generalization of domination and edge domination number on lexicographical product of complete graphs and any simple graph.

AMS Subject Classification: 05C69, 05C70, 05C76
Key Words: lexicographical product, domination number, edge domination number

1. Introduction

In this paper, graphs must be simple graphs which can be the trivial graph. Let G_1 and G_2 be graphs. The lexicographical product of graph G_1 and G_2, denote by $G_1 \bullet G_2$, is the graph with $V(G_1 \bullet G_2) = V(G_1) \times V(G_2)$ and $E(G_1 \bullet G_2) = \{(u_1v_1)(u_2v_2)|[u_1u_2 \in E(G_1)] \cup [u_1 = u_2 \text{ and } v_1v_2 \in E(G_2)]\}$. There are some properties about lexicographical product of graph. We recall these here.

Received: February 9, 2013

Proposition 1. *Let $H = G_1 \bullet G_2 = (V(H), E(H))$, then:*

(i) $|V(H)| = |V(G_1)||V(G_2)|$;

(ii) $|E(H)| = |V(G_1)||V(G_2)|^2 + |V(G_1)||E(G_2)|$;

(iii) for every $(u, v) \in V(H), d_H((u, v)) = 2|V(G_2)| + d_{G_2}(v)$.

Theorem 2. *Let G_1 and G_2 be connected graphs, The graph $H = G_1 \bullet G_2$ is connected if and only if G_1 is connected .*

Next we get that general form of graph of lexicographical product of K_n and a simple graph.

Proposition 3. *Let G be connected graph order m, the graph of $K_n \bullet G$ is*

$$[(\bigcup_{i=1}^{n-1} H_i)] \cup \bigcup_{i=1}^{n} R_i; \quad H_i = \bigcup_{j=i+1}^{n} H_{ij}$$

where $V(H_i)=W_i \cup W_j$; $W_i = \{(i, 1), ..., (i, m)\}$; $E(H_{ij}) = \{(i, v)(j, v)/v \in V(G)\}$ *and* $V(R_i) = W_i; E(R_i) = \{(i, u)(i, v)/uv \in E(G)\}$ *Moreover, H_{ij} isomorphic to complete bipartite graph $K_{|V(G)|,|V(G)|}$ and R_i isomorphic to G.*

Example

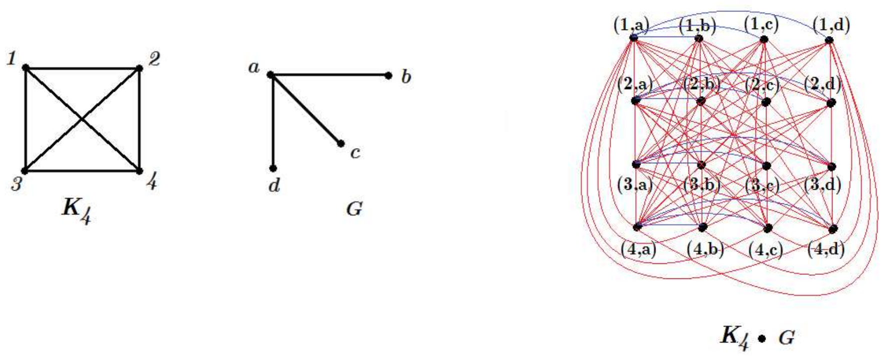

Figure 1: The graph of $K_4 \bullet G$

Next, we give the definitions about some graph parameters. A dominating set (or domset) of graph G is a subset D of the vertex set V of G such that each vertex of $V - D$ is adjacent to at least one vertex of D. The minimum cardinality of a dominating set of a graph G is called the domination number of G, denote by $\gamma(G)$.

A subset T of the edge set E of G is said to be an edge domination set of graph G, if each edge of G either is in T, or adjacent to an edge of T. The minimum cardinality of an edge domination set of G is called the edge dominating number of G, denoted by $\gamma'(G)$.

By definitions of domination number and edge domination number, clearly that $\gamma(K_n) = 1, \gamma'(K_n) = \lceil \frac{n}{3} \rceil = \gamma'(C_n) + \gamma'(C_k)$ where $n \equiv k(mod\lceil \frac{n}{3} \rceil), \lceil \frac{n}{3} \rceil - 2 \leq k \leq \lceil \frac{n}{3} \rceil$.

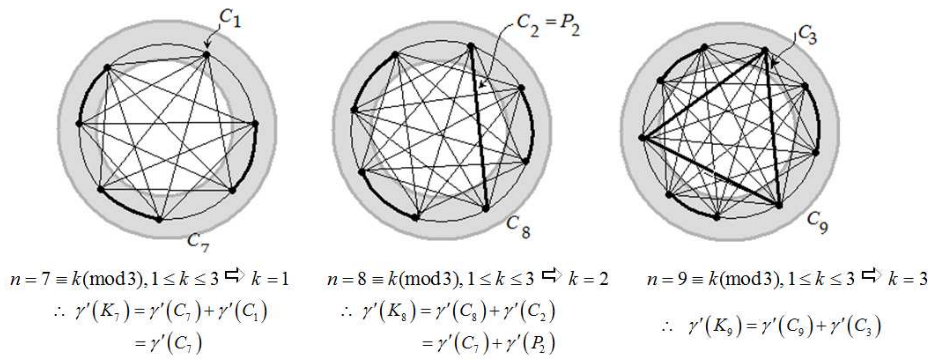

Figure 2: The edge domination number of K_7, K_8 and K_9

2. Domination Number of the Graph of $K_n \otimes G$

We begin this section by giving the theorem 4, that shows a character of minimum dominating set.

Theorem 4. *A dominating set D is a minimum dominating set if and only if for each vertex $u \in D$, one of following two conditions holds:*

(a) $N(u) \subseteq V - D$;

(b) there exists a vertex $v \in V - D$ for which $N(v) \cap D = \{u\}$.

Next, we giving the lemma 5 which show character of domination number for each H_{ij} and R_i.

Lemma 5. *Let $K_n \bullet G \cong [(\bigcup_{i=1}^{n-1} H_i)] \cup \bigcup_{i=1}^{n} R_i$; $H_i = \bigcup_{j=i+1}^{n} H_{ij}$, then $\gamma(H_{ij}) = |V(G)|$ and $\gamma(R_i) = \gamma(G)$.*

Proof. By proposition 3, we get $H_{ij} \cong K_{|V(G)|,|V(G)|}$, $R_i \cong G$.
Hence $\gamma(H_{ij}) = |V(G)|$ and $\gamma(R_i) = \gamma(G)$. □

Next, we establish theorem 6 for a domination number of $K_n \bullet G$

Theorem 6. *Let G be connected graph of order m, then $\gamma(K_n \bullet G) = \gamma(G)$.*

Proof. Let $V(K_n) = \{u_i, i = 1, 2, ..., n\}$, $V(G) = \{v_i, i = 1, 2, ..., m\}$, $S_i = \{(u_i, v_j) \in V(K_n \bullet G)/j = 1, 2, ..., m\}, i = 1, 2, ..., n$ and from $\gamma(K_n) = 1$. Assume that the minimum dominating set of G be $D_2 = \{v_1, v_2, ..., v_k\}$.

For H_1, by lemma 5 we have $\gamma(H_{ij}) = |V(G)|$; $j = 2, 3, ..., n$. We get H_{1j} and H_{1k}; $j, k = 2, 3, ..., n$; $j \neq k$ have $\gamma(G)$ common vertices in their dominating set. So the dominating set of H_1 is in $[(\bigcup_{i=1}^{n} W_i)]$

Similarly, for the dominating set of $H_2, H_3, ..., H_{n-1}$ have $\gamma(G)$ common vertices in their dominating set which is in $W_2, W_3, ..., W_{n-1}$, respectively.

But the dominating set of $H_2, H_3, ..., H_{n-1}$ are subset of the dominating set of H_1.

Considers for each R_i, we get their dominating set is $\{(u, v)/v \in D_2\}$.

We have the dominating set of $K_n \bullet G$ be $D = A \cap \{(u, v)/v \in D_2\}$.

Hence $\gamma(K_n \bullet G) \leq \gamma(G)$.

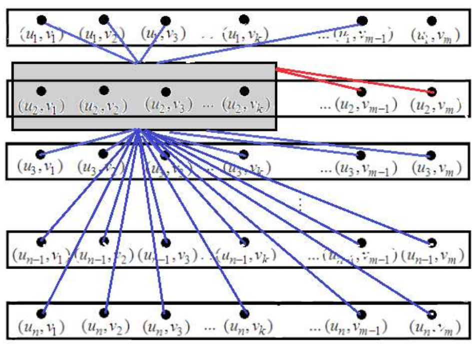

Figure 3: The region of dominating set D

Suppose that $\gamma(K_n \bullet G) < \gamma(G)$. It is impossible, because $N((u, v)) \subseteq V(K_n \bullet G) - D$ for all $(u, v) \in D$. By Theorem 4, we get D is a minimum dominating set of $K_n \bullet G$.

Hence $\gamma(K_n \bullet G) = \gamma(G)$. □

3. Edge Domination Number of the Graph of $K_n \otimes G$

We begin this section by giving the lemma 7 that shows character of edge dominating set for each H_i, H_n and R_i.

Lemma 7. *Let* $C_n \bullet G \cong [(\bigcup_{i=1}^{n-1} H_i) \cup H_n] \cup \bigcup_{i=1}^{n} R_i$; $H_i = \bigcup_{j=i+1}^{n} H_{ij}$, *then* $\gamma'(H_{ij}) = |V(G)|$ *and* $\gamma'(R_i) = \gamma'(G)$.

Proof. By proposition 3, we get $H_{ij} \cong K_{|V(G)|,|V(G)|}$, $R_i \cong G$.
Hence $\gamma'(H_{ij}) = |V(G)|$ and $\gamma'(R_i) = \gamma'(G)$. □

Next, we establish theorem 8 for a minimum edge covering number of $K_n \bullet G$.

Theorem 8. *Let G be connected graph of order m, then*

$$\gamma'(K_n \bullet G) = \begin{cases} \frac{mn}{2}, & n \text{ is even,} \\ \frac{mn}{2} + m, & n \text{ is odd.} \end{cases}$$

Proof. Let $V(K_n) = \{u_i, i = 1, 2, ..., n\}$, $V(G) = \{v_j, j = 1, 2, ..., m\}$, $S_i = \{(u_i, v_j) \in V(C_n \bullet G)/j = 1, 2, ..., m\}$, $i = 1, 2, ..., n$.

By Lemma 5, we have $K_n \bullet G \cong [(\bigcup_{i=1}^{n-1} H_i) \cup H_n] \cup \bigcup_{i=1}^{n} R_i$; $H_i = \bigcup_{j=i+1}^{n} H_{ij}$ which have a edge dominating set is

$$K = (\bigcup_{1,3,\ldots,2\lfloor \frac{n}{2} \rfloor - 1} H_k;\ H_k = \{(u_k, v_j)(u_{k+1}, v_j)/$$

for all $j = 1, 2, ..., m\}; k = 1, 3, ..., 2\lfloor \frac{n}{2} \rfloor - 1$ where n is even. We can see that every edges in R_i are adjacent by edges in K.

Similarly, in the case n is odd. We get a dominating set is $K \cup H_{n-1}$.

Hence,

$$\gamma'(K_n \bullet G) \leq \begin{cases} \frac{mn}{2}, & \text{n is even,} \\ \frac{mn}{2} + m, & \text{n is odd.} \end{cases}$$

If n is even, suppose that $\gamma'(K_n \bullet G) < \frac{mn}{2}$, then there exists edge such that is not in K, which is not adjacent with edge in the edge dominating set of H_i or R_i. That is not true. So $\gamma'(K_n \bullet G) = \frac{mn}{2}$ where n is even.

Similarly, in case n is odd, we get $\gamma'(K_n \bullet G) = \frac{mn}{2} + m$.

Hence

$$\gamma'(K_n \bullet G) = \begin{cases} \frac{mn}{2}, & \text{n is even,} \\ \frac{mn}{2} + m, & \text{n is odd.} \end{cases}$$

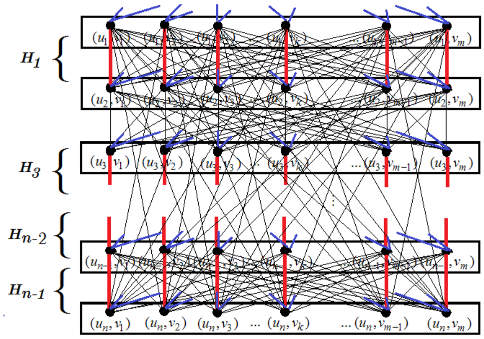

Figure 4: The edge domination set of $K_n \bullet G$ where n is odd

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