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ON THE CHEBYSHEV-HALLEY FAMILY OF ITERATION FUNCTIONS AND THE n-TH ROOT COMPUTATION PROBLEM

F. Dubeau[1 §], C. Gnang[2]

[1,2]Mathematics Department
University of Sherbrooke
2500, University Boul., Sherbrooke (Qc), CANADA, J1K 2R1

Abstract: We revisit the way the Chebyshev-Halley family of iteration functions of order 3 is obtained by considering a linear combination of two Newton's iteration functions. We also make some remarks on the iteration functions when they are applied to the n-th root computation problem.

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1. Introduction

The Chebyshev-Halley family of iteration functions (IFs) to solve $f(x) = 0$ has been introduced by Werner [10]. It can also also be found in [1] and [5], as reported in [9]. Each member of this family is obtained as an improvement of the Newton's IF, depends on a real parameter β, and can be written as

$$G_\beta(x) = x - \frac{f(x)}{f^{(1)}(x)} \left[\frac{1 - (\beta - 1/2)L_f(x)}{1 - \beta L_f(x)}\right]$$

where $L_f(x) = \frac{f(x)f^{(2)}(x)}{\left[f^{(1)}(x)\right]^2}$. These IFs are of order 3 when we look for an $\alpha \in \mathbb{R}$ such that $f(\alpha) = 0$ and α is a simple root of $f(x)$.

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§Correspondence author

In this paper we obtain this family of IFs from a linear combination of two Newton's IFs. We also consider the best parameter β for the n-th root computation problem. For this problem the best parameter depends only on n. In particular, when we compare the Halley ($\beta = 1/2$) and Super-Halley ($\beta = 1$) IFs, we show that Super-Halley IF is the locally best one to compute the n-th root for $n = 2, 3, 4$, Halley's IF is the locally best one for $n \geq 6$, and they are equivalent for $n = 5$. Both are locally better than the Chebyshev ($\beta = 0$) IF. The term "locally better" here means that the asymptotic constant is the smallest one.

2. Newton's method

Let $f(x)$ be a regular enough function. We look for $\alpha \in \mathbb{R}$ such that $f(\alpha) = 0$ and $f^{(1)}(\alpha) \neq 0$, a simple zero of $f(x)$. The Newton's IF to find α is given by

$$N(x) = x - \frac{f(x)}{f^{(1)}(x)}.$$

We have

$$N^{(1)}(x) = \frac{f(x)f^{(2)}(x)}{\left[f^{(1)}(x)\right]^2} = L_f(x),$$

$$N^{(2)}(x) = L_f^{(1)}(x) = \frac{f^{(2)}(x)}{f^{(1)}(x)} + f(x)\left[\frac{f^{(1)}(x)f^{(3)}(x) - 2\left[f^{(2)}(x)\right]^2}{\left[f^{(1)}(x)\right]^3}\right],$$

and

$$\begin{aligned} N^{(3)}(x) &= L_f^{(2)}(x) \\ &= 2\frac{f^{(3)}(x)}{f^{(1)}(x)} - 3\left[\frac{f^{(2)}(x)}{f^{(1)}(x)}\right]^2 + f(x)\left[\frac{f^{(1)}(x)f^{(3)}(x) - 2\left[f^{(2)}(x)\right]^2}{\left[f^{(1)}(x)\right]^3}\right]^{(1)}. \end{aligned}$$

It follows that for $x = \alpha$

$$N(\alpha) = \alpha, \quad N^{(1)}(\alpha) = 0, \quad N^{(2)}(\alpha) = \frac{f^{(2)}(\alpha)}{f^{(1)}(\alpha)},$$

and

$$N^{(3)}(\alpha) = 2\frac{f^{(3)}(\alpha)}{f^{(1)}(\alpha)} - 3\left[\frac{f^{(2)}(\alpha)}{f^{(1)}(\alpha)}\right]^2.$$

This last expression is known as the Schwarzian derivative of $f(x)$ at $x=\alpha$ [6]. In general the Newton's IF is of order 2 (in fact when $f^{(1)}(\alpha)f^{(2)}(\alpha)\neq 0$), and its asymptotic constant is

$$K_2(\alpha;N)=\frac{N^{(2)}(\alpha)}{2!}=\frac{f^{(2)}(\alpha)}{2!f^{(1)}(\alpha)}.$$

Now let us replace $f(x)$ by $f(x)/\left[f^{(1)}(x)\right]^{\beta}$ with $\beta\in\mathbb{R}$ (we suppose $f^{(1)}(\alpha)>0$, if $f^{(1)}(\alpha)<0$ we use $-f(x)$ instead of $f(x)$). We get

$$\begin{aligned} N_\beta(x) &= x-\frac{f(x)/\left[f^{(1)}(x)\right]^{\beta}}{\left[f(x)/\left[f^{(1)}(x)\right]^{\beta}\right]^{(1)}} \\ &= x-\frac{f(x)}{f^{(1)}(x)}\left[\frac{1}{1-\beta L_f(x)}\right]. \end{aligned}$$

Since

$$\left[\frac{f(x)}{\left[f^{(1)}(x)\right]^{\beta}}\right]^{(1)}=\frac{1}{\left[f^{(1)}(x)\right]^{\beta-1}}-\beta f(x)\frac{f^{(2)}(x)}{\left[f^{(1)}(x)\right]^{\beta+1}},$$

$$\left[\frac{f(x)}{\left[f^{(1)}(x)\right]^{\beta}}\right]^{(2)}=(1-2\beta)\frac{f^{(2)}(x)}{\left[f^{(1)}(x)\right]^{\beta}}-\beta f(x)\frac{f^{(1)}(x)f^{(3)}(x)-(\beta+1)\left[f^{(2)}(x)\right]^2}{\left[f^{(1)}(x)\right]^{\beta+2}},$$

and

$$\begin{aligned} \left[\frac{f(x)}{\left[f^{(1)}(x)\right]^{\beta}}\right]^{(3)} &= \frac{(1-3\beta)f^{(1)}(x)f^{(3)}(x)+3\beta^2\left[f^{(2)}(x)\right]^2}{\left[f^{(1)}(x)\right]^{\beta+1}} \\ &\quad -\beta f(x)\left[\frac{f^{(1)}(x)f^{(3)}(x)-(\beta+1)\left[f^{(2)}(x)\right]^2}{\left[f^{(1)}(x)\right]^{\beta+2}}\right]^{(1)}, \end{aligned}$$

we obtain for $x=\alpha$

$$\left[\frac{f(\alpha)}{\left[f^{(1)}(\alpha)\right]^{\beta}}\right]^{(1)}=\frac{1}{\left[f^{(1)}(\alpha)\right]^{\beta-1}},\quad \left[\frac{f(\alpha)}{\left[f^{(1)}(\alpha)\right]^{\beta}}\right]^{(2)}=(1-2\beta)\frac{f^{(2)}(\alpha)}{\left[f^{(1)}(\alpha)\right]^{\beta}},$$

and

$$\left[\frac{f(\alpha)}{\left[f^{(1)}(\alpha)\right]^{\beta}}\right]^{(3)}=\frac{(1-3\beta)f^{(1)}(\alpha)f^{(3)}(\alpha)+3\beta^2\left[f^{(2)}(\alpha)\right]^2}{\left[f^{(1)}(\alpha)\right]^{\beta+1}}.$$

Moreover

$$N_\beta(\alpha)=\alpha,\quad N_\beta^{(1)}(\alpha)=0,\quad N_\beta^{(2)}(\alpha)=(1-2\beta)\frac{f^{(2)}(\alpha)}{f^{(1)}(\alpha)},$$

and

$$N_\beta^{(3)}(\alpha)=2(1-3\beta)\frac{f^{(3)}(\alpha)}{f^{(1)}(\alpha)}-3(1-4\beta+2\beta^2)\left[\frac{f^{(2)}(\alpha)}{f^{(1)}(\alpha)}\right]^2.$$

These IFs are of order 2 with an asymptotic constant

$$K_2\left(\alpha;N_\beta\right)=\frac{N_\beta^{(2)}(\alpha)}{2!}=\frac{1-2\beta}{2!}\frac{f^{(2)}(\alpha)}{f^{(1)}(\alpha)}.$$

Let us remark that for $\beta=1/2$ we obtain the Halley IF which is of order 3.

3. Chebyshev-Halley family of iteration functions

In this section we combine two IFs of order 2 to obtain a new IF of order 3. Let $\beta\neq 0$ and define

$$\begin{aligned}G_\beta(x) &= \frac{1}{2\beta}\left[N_\beta(x)-(1-2\beta)N_0(x)\right]\\ &= x-\frac{f(x)}{f^{(1)}(x)}\left[\frac{1-(\beta-1/2)L_f(x)}{1-\beta L_f(x)}\right]\end{aligned}$$

which corresponds to the Chebyshev-Halley IF family. Indeed, for $\beta=1$ we get the Super-Halley IF, for $\beta=1/2$ we get the Halley IF, and for $\beta=0$, which is a limit case, we get the Chebyshev IF. In this case, we can also define $G_0(x)$ by the limit

$$\begin{aligned}G_0(x) &= \lim_{\beta\to 0}G_\beta(x)\\ &= N_0(x)+\frac{1}{2}\lim_{\beta\to 0}\frac{N_\beta(x)-N_0(x)}{\beta}\\ &= N_0(x)+\frac{1}{2}\frac{\partial}{\partial\beta}N_\beta(x)|_{\beta=0}\\ &= N_0(x)-\frac{1}{2}\frac{f(x)}{f^{(1)}(x)}L_f(x).\end{aligned}$$

We verify that

$$G_\beta(\alpha) = \alpha, \quad G_\beta^{(1)}(\alpha) = 0, \quad G_\beta^{(2)}(\alpha) = 0,$$

and

$$G_\beta^{(3)}(\alpha) = -\frac{f^{(3)}(\alpha)}{f^{(1)}(\alpha)} + 3(1-\beta)\left[\frac{f^{(2)}(\alpha)}{f^{(1)}(\alpha)}\right]^2.$$

Then for any β we have an IF of order 3 with an asymptotic constant

$$K_3(\alpha; G_\beta) = \frac{G_\beta^{(3)}(\alpha)}{3!} = \frac{1}{3!}\left[-\frac{f^{(3)}(\alpha)}{f^{(1)}(\alpha)} + 3(1-\beta)\left[\frac{f^{(2)}(\alpha)}{f^{(1)}(\alpha)}\right]^2\right]. \tag{1}$$

For example, if $\beta = 1$, $G_1(x)$ is the Super-Halley IF which is of order 4 for a quadratic equation, because in this case $f^{(3)}(x) = 0$. More generally, if this expression can be set to 0 by a good choice of β, we obtain an IF of order 4.

Let us define

$$H_\beta(\xi) = \frac{1-(\beta-1/2)\xi}{1-\beta\xi}$$

we have

$$G_\beta(x) = x - \frac{f(x)}{f^{(1)}(x)} H_\beta(L_f(x)).$$

It is well known [8, 4] that any method $G(x)$ of order 3 can be written as

$$G(x) = x - \frac{f(x)}{f^{(1)}(x)}\left[H_\beta(L_f(x)) + b(x)f^2(x)\right] = G_\beta(x) + O(f^3(x))$$

where $b(x)$ is a bounded function on a neighborhood of α.

4. Other order 3 iteration functions

If we take two different values of β, say β_1 and β_2, and consider the linar combination

$$\begin{aligned} G_{\beta_1,\beta_2}(x) &= \frac{1}{2(\beta_2-\beta_1)}\left[(1-2\beta_1)N_{\beta_2}(x) - (1-2\beta_2)N_{\beta_1}(x)\right] \\ &= x - \frac{f(x)}{f^{(1)}(x)}\left[\frac{1-\left[(\beta_1+\beta_2)-1/2\right]L_f(x)}{(1-\beta_1 L_f(x))(1-\beta_2 L_f(x))}\right] \end{aligned}$$

$$= \quad x - \frac{f(x)}{f^{(1)}(x)} \left[\frac{1 - [(\beta_1 + \beta_2) - 1/2]\, L_f(x)}{1 - [\beta_1 + \beta_2]\, L_f(x) + \beta_1 \beta_2 L_f^2(x)} \right]$$

we obtain an IF of order 3 because $G^{(2)}_{\beta_1,\beta_2}(x) = 0$. Moreover

$$\begin{aligned} K_3(\alpha; G_{\beta_1,\beta_2}) &= \frac{G^{(3)}_{\beta_1,\beta_2}(\alpha)}{3!} \\ &= \frac{1}{3!} \left[-\frac{f^{(3)}(\alpha)}{f^{(1)}(\alpha)} + 3[1 - (\beta_1 + \beta_2) + 2\beta_1\beta_2] \left[\frac{f^{(2)}(\alpha)}{f^{(1)}(\alpha)} \right]^2 \right]. \end{aligned}$$

If this expression can be set to 0 by a good choice of β_1 and β_2, we obtain an IF of order 4, and possibly of order 5 (but $G^{(4)}_{\beta_1,\beta_2}(\alpha)$ is not easy to compute).

5. Example: n-th root computation

The problem of finding the n-th root $\alpha = r^{1/n}$ of a positive real number r is equivalent to solve the equation $f(x) = x^n - r = 0$. Since $f^{(1)}(x) = nx^{n-1}$, $f^{(2)}(x) = n(n-1)x^{n-2}$, and $f^{(3)}(x) = n(n-1)(n-2)x^{n-3}$, it follows that

$$K_3(\alpha; G_\beta) = \frac{(n-1)^2}{2\alpha^2} C(\beta)$$

where

$$C(\beta) = \frac{2n-1}{3(n-1)} - \beta,$$

and for

$$\beta^*(n) = \frac{2n-1}{3(n-1)},$$

which does not depend on α but only on n, we have $C(\beta^*(n)) = 0$ and $K_3(\alpha; G_{\beta^*(n)}) = 0$, and the method is of order 4.

In Table 1 we compare the values of $C(\beta)$ for the 3 popular IFs applied on this particular problem. We observe that for low values of n ($2 \leq n \leq 4$) the Super-Halley IF has the lowest asymptotic constant, lest than the Halley IF. For values of $n \geq 6$ the Halley IF has the lowest asymptotic constant and for $n = 5$ they are equivalent. The Chebyshev IF has always a greater asymptotic constant compared to the two other IFs.

Finally, for G_{β_1,β_2} we have

$$K_3(\alpha; G_{\beta_1,\beta_2}) = \frac{(n-1)^2}{2\alpha^2} \left[\frac{2n-1}{3(n-1)} - (\beta_1 + \beta_2) + 2\beta_1\beta_2 \right].$$

which can also be set to 0.

	$C(\beta) = \frac{2n-1}{3(n-1)} - \beta$			
	Chebyshev	Halley	Super-Halley	Optimal parameter
n	$\beta = 0$	$\beta = 1/2$	$\beta = 1$	$\beta^*(n) = \frac{2n-1}{3(n-1)}$
2	1	1.5/3	0*	3/3
3	5/6	2/6	−1/6*	5/6
4	7/9	2.5/9	−2/9*	7/9
5	9/12	3/12*	−3/12*	9/12
6	11/15	3.5/15*	−4/15	11/15
7	13/18	4/18*	−5/18	13/18
8	15/21	4.5/21*	−6/21	15/21
9	17/24	5/24*	−7/24	17/24
10	19/27	5.5/27*	−8/27	19/27
⋮	⋮	⋮	⋮	⋮
∞	2/3	1/6*	−1/3	2/3

Table 1: Comparaison of the parameter $C(\beta)$ of the asymptotic constants for Chebychev, Halley and Super-Halley IFs for n-th root computation (an * indicates the smallest asymptotic constant).

6. Concluding remarks

The Chebyshev IF and the Halley IF have a long history [3, 4]. The Super-Halley IF is relatively new (introduced in 1980 [10]) and it reappears in [1, 5]. Recently it has been rediscovered by using the Adomian decomposition method [2]. Later in [7], without recognizing the Super-Halley IF, it has been compared to 5 other IFs, among them was the Chebyshev IF. It has been claimed that in general the Chebyshev IF was better than the Super-Halley IF. It appears here that for the n-th root computation the Chebyshev IF is always worse than the Super-Halley and the Halley IFs. We can say that the behaviour of an IF depends on its asymptotic constant which depends on the function under study, so it is not clear to obtain a general classification of the (order 3, or more generally fixed order p) IFs.

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