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REGULAR ELEMENTS OF THE COMPLETE SEMIGROUPS OF BINARY RELATIONS OF THE CLASS $\sum_7(X, 8)$

Barış Albayrak[1], I. Yasha Diasamidze[2], Neşet Aydin[3]

[1,3]Department of Mathematics
Faculty of Science and Art
Canakkale Onsekiz Mart University
Canakkale, TURKEY

[2]Shota Rustaveli State University
35, Ninoshvili St., Batumi 6010, GEORGIA

Abstract: In this paper let $Q = \{T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8\}$ be a subsemilattice of $X-$semilattice of unions D where $T_1 \subset T_2 \subset T_3 \subset T_5 \subset T_6 \subset T_8$, $T_1 \subset T_2 \subset T_3 \subset T_5 \subset T_7 \subset T_8$, $T_1 \subset T_2 \subset T_4 \subset T_5 \subset T_6 \subset T_8$, $T_1 \subset T_2 \subset T_4 \subset T_5 \subset T_7 \subset T_8$, $T_1 \neq \emptyset$, $T_4 \backslash T_3 \neq \emptyset$, $T_3 \backslash T_4 \neq \emptyset$, $T_6 \backslash T_7 \neq \emptyset$, $T_7 \backslash T_6 \neq \emptyset$, $T_3 \cup T_4 = T_5$, $T_6 \cup T_7 = T_8$, then we characterize the class each element of which is isomorphic to Q by means of the characteristic family of sets, the characteristic mapping and the generate set of Q. Moreover, we calculate the number of regular elements of $B_X(D)$ for a finite set X.

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1. Introduction

Let X be an arbitrary nonempty set. Recall that a binary relation on X is a subset of the cartesian product $X \times X$. The binary operation $\circ$ on B_X (the set

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of all binary relations on X) defined by for $\alpha,\ \beta \in B_X$

$$(x,z) \in \alpha \circ \beta \Leftrightarrow (x,y) \in \alpha \text{ and } (y,z) \in \beta, \text{ for some } y \in X$$

is associative. Therefore B_X is a semigroup with respect to the operation $\circ$. This semigroup is called the *semigroup of all binary relations* on the set X.

Let D be a nonempty set of subsets of X which is closed under the union i.e., $\cup D' \in D$ for any nonempty subset D' of D. In that case, D is called a *complete* $X-$ *semilattice of unions.* The union of all elements of D is denoted by the symbol $\breve{D}$. Clearly, $\breve{D}$ is the largest element of D.

Let X be an arbitrary nonempty set and m be an arbitrary cardinal number. $\Sigma\,(X,m)$ is the class of all complete $X-$ semilattices of unions of power m.

Let $\widetilde{D}$ and D' be some nonempty subsets of the complete $X-$ semilattices of unions. We say that a subset $\widetilde{D}$ generates a set D' if any element from D' is a set-theoretic union of the elements from $\widetilde{D}$.

Note that the semilattice D is partially ordered with respect to the set-theoretic inclusion. Let $\emptyset \neq D' \subseteq D$ and

$$N(D,D') = \left\{ Z \in D \mid Z \subseteq Z' \text{ for any } Z' \in D' \right\}.$$

It is clear that $N(D,D')$ is the set of all lower bounds of D'. If $N(D,D') \neq \emptyset$ then $\Lambda(D,D') = \cup N(D,D')$ belongs to D and it is *the greatest lower bound* of D'.

Further, let $x,y \in X$, $Y \subseteq X$, $\alpha \in B_X$, $T \in D$, $\emptyset \neq D' \subseteq D$ and $t \in \breve{D}$. Then we have the following notations,

$$\begin{aligned}
y\alpha &= \{x \in X \mid (y,x) \in \alpha\} \quad , \ Y\alpha = \bigcup_{y \in Y} y\alpha, \\
V(D,\alpha) &= \{Y\alpha \mid Y \in D\} \quad , \ D_t = \{Z' \in D \mid t \in Z'\}\ , \\
D'_T &= \{Z' \in D' \mid T \subseteq Z'\} \quad , \ \ddot{D}'_T = \{Z' \in D' \mid Z' \subseteq T\}\,.
\end{aligned}$$

Let f be an arbitrary mapping from X into D. Then one can construct a binary relation α_f on X by $\alpha_f = \bigcup_{x \in X} (\{x\} \times f(x))$.The set of all such binary relations is denoted by $B_X(D)$. It is easy to prove that $B_X(D)$ is a semigroup with respect to the operation $\circ$. In this case $B_X(D)$, is called a *complete semigroup of binary relations* defined by an $X-$semilattice of unions D. This structure was comprehensively investigated in Diasemidze [6].

If $\alpha \circ \beta \circ \alpha = \alpha$ for some $\beta \in B_X(D)$ then a binary relation α is called a *regular element* of $B_X(D)$.

Let $\alpha \in B_X$, $Y_T^\alpha = \{y \in X \mid y\alpha = T\}$ and

$$V[\alpha] = \begin{cases} V(X^*, \alpha), \text{ if } \emptyset \notin D, \\ V(X^*, \alpha), \text{ if } \emptyset \in V(X^*, \alpha), \\ V(X^*, \alpha) \cup \{\emptyset\}, \text{ if } \emptyset \notin V(X^*, \alpha) \text{ and } \emptyset \in D. \end{cases}$$

Then a representation of a binary relation α of the form $\alpha = \bigcup_{T \in V[\alpha]} (Y_T^\alpha \times T)$ is called quasinormal. Note that, if $\alpha = \bigcup_{T \in V[\alpha]} (Y_T^\alpha \times T)$ is a quasinormal representation of the binary relation α, then $X = \bigcup_{T \in V(X^*, \alpha)} Y_T^\alpha$ and $Y_T^\alpha \cap Y_{T'}^\alpha \neq \emptyset$ for $T, T' \in V(X^*, \alpha)$ which $T \neq T'$. In [7] they show that, if β is regular element of $B_X(D)$, then $V[\beta] = V(D, \beta)$ and a complete $X-$semilattice of unions D is an $XI-$ semilattice of unions if $\Lambda(D, D_t) \in D$ for any $t \in \breve{D}$ and $Z = \bigcup_{t \in Z} \Lambda(D, D_t)$ for any nonempty element Z of D.

Let D' be an arbitrary nonempty subset of the complete $X-$semilattice of unions D. A nonempty element $T \in D'$ is a *nonlimiting element* of D' if $T \backslash l(D', T) = T \backslash \cup (D' \backslash D'_T) \neq \emptyset$. A nonempty element $T \in D'$ is *limiting element* of D' if $T \backslash l(D', T) = \emptyset$.

The family $C(D)$ of pairwise disjoint subsets of the set $\breve{D} = \cup D$ is the *characteristic family* of sets of D if the following hold

a) $\cap D \in C(D)$

b) $\cup C(D) = \breve{D}$

c) There exists a subset $C_Z(D)$ of the set $C(D)$ such that $Z = \cup C_Z(D)$ for all $Z \in D$.

A mapping $\theta : D \to C(D)$ is called *characteristic mapping* if $Z = (\cap D) \cup \bigcup_{Z' \in \hat{D}} \theta(Z')$ for all $Z \in D$.

The existence and the uniqueness of characteristic family and characteristic mapping is given in Diasemidze [8]. Moreover, it is shown that every $Z \in D$ can be written as $Z = \theta(\breve{Q}) \cup \bigcup_{T \in \hat{Q}(Z)} \theta(T)$, where $\hat{Q}(Z) = Q \backslash \{T \in Q \mid Z \subseteq T\}$.

A one-to-one mapping φ between two complete $X-$ semilattices of unions D' and D'' is called a *complete isomorphism* if $\varphi(\cup D_1) = \bigcup_{T' \in D_1} \varphi(T')$ for each nonempty subset D_1 of the semilattice D'. Also, let $\alpha \in B_X(D)$. A complete isomorphism φ between $XI-$semilattice of unions Q and D is called a *complete $\alpha-$ isomorphism* if $Q = V(D,\alpha)$ and $\varphi(\emptyset) = \emptyset$ for $\emptyset \in V(D,\alpha)$ and $\varphi(T)\alpha = T$ for any $T \in V(D,\alpha)$.

Let Q and D' are respectively some XI and $X-$ subsemilattices of the complete $X-$ semilattice of unions D. Then

$$R_\varphi\left(Q, D'\right) = \{\alpha \in B_X(D) \mid \alpha \text{ regular element, } \varphi \text{ complete } \alpha-\text{isomorphism}\}$$

where $\varphi : Q \to D'$ complete isomorphism and $V(D,\alpha) = Q$. Besides, let us denote

$$R(Q,D') = \bigcup_{\varphi \in \Phi(Q,D')} R_\varphi(Q,D') \text{ and } R(D') = \bigcup_{Q' \in \Omega(Q)} R(Q',D').$$

where

$$\Phi\left(Q, D'\right) = \left\{\varphi \mid \varphi : Q \to D' \text{ is a complete } \alpha-\text{isomorhism } \exists\alpha \in B_X(D)\right\},$$

$$\Omega\left(Q\right) = \left\{Q' \mid Q' \text{ is } XI-\text{subsemilattices of } D \text{ which is complete isomorphic to } Q\right\}.$$

E. Schröder described the theory of binary relations in detail in the 1890s ([1]). The basic concepts and the properties of the theory were introduced in "Principia mathemetica" Whitehead and Russell([2]). The theory of binary relations has been improved by Riguet ([3] – [4]). Many researcher studied this theory using partial transformations as Vagner did ([5]). Regular elements of semigroup play an importent role in semigroup theory. Therefore Diasamidze generate systmatic rules for understanding structure of a semigroup of binary relations and characterization of regular elements of these semigroup in ([6] – [9]). In general he studied semigroups but, in particular, he investigates complete semigroups of the binary relations.

In this paper, we take in particular, $Q = \{T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8\}$ subsemilattice of $X-$semilattice of unions D where the elements T_i' s, $i = 1, 2, \ldots, 8$ are satisfying the following properties, $T_1 \subset T_2 \subset T_3 \subset T_5 \subset T_6 \subset T_8$, $T_1 \subset T_2 \subset T_3 \subset T_5 \subset T_7 \subset T_8$, $T_1 \subset T_2 \subset T_4 \subset T_5 \subset T_6 \subset T_8$, $T_1 \subset T_2 \subset T_4 \subset T_5 \subset T_7 \subset T_8$, $T_1 \neq \emptyset$, $T_4 \backslash T_3 \neq \emptyset$, $T_3 \backslash T_4 \neq \emptyset$, $T_6 \backslash T_7 \neq \emptyset$, $T_7 \backslash T_6 \neq \emptyset$, $T_3 \cup T_4 = T_5$, $T_6 \cup T_7 = T_8$. We will investigate the properties of regular elemant $\alpha \in B_X(D)$ satisfying $V(D,\alpha) = Q$. Moreover, we will calculate the number of regular elements of $B_X(D)$ for a finite set X.

As general, we study the properties and calculate the number of regular elements of $B_X(D)$ satisfying $V(D,\alpha) = Q'$ where Q' is a semilattice isomorph to Q. So, we characterize the class for each element of which is isomorphic to Q by means of the characteristic family of sets, the characteristic mapping and the generate set of D.

2. Preliminaries

Theorem 2.1. *[9, Theorem 10] Let α and σ be binary relations of the semigroup $B_X(D)$ such that $\alpha \circ \sigma \circ \alpha = \alpha$. If $D(\alpha)$ is some generating set of the semilattice $V(D,\alpha)\setminus\{\emptyset\}$ and $\alpha = \bigcup_{T\in V(D,\alpha)} (Y_T^\alpha \times T)$ is a quasinormal representation of the relation α, then $V(D,\alpha)$ is a complete $XI-$ semilattice of unions. Moreover, there exists a complete $\alpha-$isomorphism φ between the semilattice $V(D,\alpha)$ and $D' = \{T\sigma \mid T \in V(D,\alpha)\}$, that satisfies the following conditions:*

a) $\varphi(T) = T\sigma$ and $\varphi(T)\alpha = T$ for all $T \in V(D,\alpha)$

b) $\bigcup_{T'\in \ddot{D}(\alpha)_T} Y_{T'}^\alpha \supseteq \varphi(T)$ for any $T \in D(\alpha)$,

c) $Y_T^\alpha \cap \varphi(T) \neq \emptyset$ for all nonlimiting element T of the set $\ddot{D}(\alpha)_T$,

d) If T is a limiting element of the set $\ddot{D}(\alpha)_T$, then the equality $\cup B(T) = T$ is always holds for the set $B(T) = \left\{Z \in \ddot{D}(\alpha)_T \mid Y_Z^\alpha \cap \varphi(T) \neq \emptyset\right\}$.

On the other hand, if $\alpha \in B_X(D)$ such that $V(D,\alpha)$ is a complete $XI-$ semilattice of unions. If for a complete $\alpha-$isomorphism φ from $V(D,\alpha)$ to a subsemilattice D' of D satisfies the conditions $b) - d)$ of the theorem, then α is a regular element of $B_X(D)$.

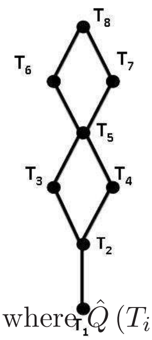
Theorem 2.2. *[7, Theorem 1.18.2] Let $D_j = \{T_1, \ldots, T_j\}$, X be finite set and $\emptyset \neq Y \subseteq X$. If f is a mapping of the set X, on the D_j, for which $f(y) = T_j$ for some $y \in Y$, then the numbers of those mappings f of the sets X on the set D_j can be calculated by the formula $s = j^{|X\setminus Y|} \cdot \left(j^{|Y|} - (j-1)^{|Y|}\right)$.*

Theorem 2.3. *[7, Theorem 6.3.5] Let X is a finite set. If φ is a fixed element of the set $\Phi(D,D')$ and $|\Omega(D)| = m_0$ and q is a number of all automorphisms of the semilattice D then $|R(D')| = m_0 \cdot q \cdot |R_\varphi(D,D')|$.*

3. Results

Let X be a finite set, D be a complete $X-$semilattice of unions and $Q = \{T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8\}$ be $X-$subsemilattice of unions of D satisfies the following conditions

$$\begin{array}{ll} T_1 \subset T_2 \subset T_3 \subset T_5 \subset T_6 \subset T_8, & T_1 \subset T_2 \subset T_3 \subset T_5 \subset T_7 \subset T_8, \\ T_1 \subset T_2 \subset T_4 \subset T_5 \subset T_6 \subset T_8, & T_1 \subset T_2 \subset T_4 \subset T_5 \subset T_7 \subset T_8, \\ T_4 \backslash T_3 \neq \emptyset,\ T_3 \backslash T_4 \neq \emptyset, & T_6 \backslash T_7 \neq \emptyset,\ T_7 \backslash T_6 \neq \emptyset, \\ T_3 \cup T_4 = T_5, T_6 \cup T_7 = T_8 & T_1 \neq \emptyset. \end{array}$$



The diagram of the Q is shown in Figure 3.1. Let $C(Q) = \{P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8\}$ is characteristic family of sets of Q and $\theta : Q \to C(Q)$, $\theta(T_i) = P_i$ $(i = 1, 2, \ldots, 8)$ is characteristic mapping.

Then, by using properties of characteristic family and characteristic mapping for each element $T_i \in Q$ we can write

$$T_i = \theta(\breve{Q}) \cup \bigcup_{T \in \hat{Q}(T_i)} \theta(T), (i = 1, 2, \ldots, 8)$$

where $\hat{Q}(T_i) = Q \backslash \{Z \in Q \mid T_i \subseteq Z\}, \breve{Q} = \cup Q = T_8$ and $\theta(\breve{Q}) = \theta(T_8) = P_8$.

Hence,

$$\begin{aligned} T_8 &= P_8 \cup \bigcup_{T \in \hat{Q}(T_8)} \theta(T) = P_8 \cup P_7 \cup P_6 \cup P_5 \cup P_4 \cup P_3 \cup P_2 \cup P_1, \\ T_7 &= P_8 \cup \bigcup_{T \in \hat{Q}(T_7)} \theta(T) = P_8 \cup P_6 \cup P_5 \cup P_4 \cup P_3 \cup P_2 \cup P_1, \\ T_6 &= P_8 \cup \bigcup_{T \in \hat{Q}(T_6)} \theta(T) = P_8 \cup P_7 \cup P_5 \cup P_4 \cup P_3 \cup P_2 \cup P_1, \\ T_5 &= P_8 \cup \bigcup_{T \in \hat{Q}(T_5)} \theta(T) = P_8 \cup P_4 \cup P_3 \cup P_2 \cup P_1, \\ T_4 &= P_8 \cup \bigcup_{T \in \hat{Q}(T_4)} \theta(T) = P_8 \cup P_3 \cup P_2 \cup P_1, \\ T_3 &= P_8 \cup \bigcup_{T \in \hat{Q}(T_3)} \theta(T) = P_8 \cup P_4 \cup P_2 \cup P_1, \\ T_2 &= P_8 \cup \bigcup_{T \in \hat{Q}(T_2)} \theta(T) = P_8 \cup P_1, \\ T_1 &= P_8 \cup \bigcup_{T \in \hat{Q}(T_1)} \theta(T) = P_8 \cup \emptyset = P_8 \end{aligned} \tag{3.1}$$

are obtained.

Lemma 3.1. *Q is $XI-$ semilattice of unions.*

Proof. Let us show that the conditions of definition of $XI-$ semilattice of unions hold. First, let determine the greatest lower bounds of the each semilattice Q_t in Q for $t \in T_8$. Since $T_8 = P_8 \cup P_7 \cup P_6 \cup P_5 \cup P_4 \cup P_3 \cup P_2 \cup P_1$ and P_i $(i = 1, 2, \ldots, 8)$ are pairwise disjoint sets, by Equation (3.1) and the definition of Q_t, we get

$$Q_t = \begin{cases} Q & , t \in P_8 \\ \{T_8, T_6\} & , t \in P_7 \\ \{T_8, T_7\} & , t \in P_6 \\ \{T_8, T_7, T_6\} & , t \in P_5 \\ \{T_8, T_7, T_6, T_5, T_3\} & , t \in P_4 \\ \{T_8, T_7, T_6, T_5, T_4\} & , t \in P_3 \\ \{T_8, T_7, T_6, T_5, T_4, T_3\} & , t \in P_2 \\ \{T_8, T_7, T_6, T_5, T_4, T_3, T_2\} & , t \in P_1 \end{cases}. \tag{3.2}$$

By using Equation (3.2) and the definition of $N(Q, Q_t)$, we get

$$N(Q, Q_t) = \begin{cases} \{T_1\} & , t \in P_8 \\ \{T_1, T_2, T_3, T_4, T_5, T_6\} & , t \in P_7 \\ \{T_1, T_2, T_3, T_4, T_5, T_7\} & , t \in P_6 \\ \{T_1, T_2, T_3, T_4, T_5\} & , t \in P_5 \\ \{T_1, T_2, T_3\} & , t \in P_4 \\ \{T_1, T_2, T_4\} & , t \in P_3 \\ \{T_1, T_2\} & , t \in P_2 \\ \{T_1, T_2\} & , t \in P_1 \end{cases}. \tag{3.3}$$

From the Equation (3.3) the greatest lower bounds for each semilattice Q_t

$$\cup N(Q, Q_t) = \Lambda(Q, Q_t) = \begin{cases} T_1 & , t \in P_8 \\ T_6 & , t \in P_7 \\ T_7 & , t \in P_6 \\ T_5 & , t \in P_5 \\ T_3 & , t \in P_4 \\ T_4 & , t \in P_3 \\ T_2 & , t \in P_2 \\ T_2 & , t \in P_1 \end{cases} \tag{3.4}$$

are obtained. So, we get $\Lambda(D, D_t) \in D$ for any $t \in T_8$. Now Using the Equation (3.4), we have

$$
\begin{aligned}
&t \in T_1 = P_8 \Rightarrow T_1 = \Lambda(Q, Q_t),\\
&t \in T_2 = P_8 \cup P_1 \Rightarrow t \in P_8 \text{ or } t \in P_1 \Rightarrow \Lambda(Q, Q_t) \in \{T_1, T_2\}\\
&\qquad \Rightarrow T_2 = T_1 \cup T_2 = \bigcup_{t \in T_2} \Lambda(Q, Q_t),\\
&t \in T_3 = P_8 \cup P_4 \cup P_2 \cup P_1 \Rightarrow \Rightarrow \Lambda(Q, Q_t) \in \{T_1, T_2, T_3\}\\
&\qquad \Rightarrow T_3 = T_1 \cup T_2 \cup T_3 = \bigcup_{t \in T_3} \Lambda(Q, Q_t),\\
&t \in T_4 = P_8 \cup P_3 \cup P_2 \cup P_1 \Rightarrow \Rightarrow \Lambda(Q, Q_t) \in \{T_1, T_2, T_4\}\\
&\qquad \Rightarrow T_4 = T_1 \cup T_2 \cup T_4 = \bigcup_{t \in T_4} \Lambda(Q, Q_t),\\
&t \in T_5 = P_8 \cup P_4 \cup P_3 \cup P_2 \cup P_1 \Rightarrow \Lambda(Q, Q_t) = \{T_1, T_2, T_3, T_4\}\\
&\qquad \Rightarrow T_5 = T_1 \cup T_2 \cup T_3 \cup T_4 = \bigcup_{t \in T_5} \Lambda(Q, Q_t),\\
&t \in T_6 = P_8 \cup P_7 \cup P_5 \cup \ldots \cup P_1 \Rightarrow \Lambda(Q, Q_t) = \{T_1, T_2, T_3, T_4, T_5, T_6\}\\
&\qquad \Rightarrow T_6 = T_1 \cup \ldots \cup T_6 = \bigcup_{t \in T_6} \Lambda(Q, Q_t),\\
&t \in T_7 = P_8 \cup P_6 \cup P_5 \cup \ldots \cup P_1 \Rightarrow \Lambda(Q, Q_t) = \{T_1, T_2, T_3, T_4, T_5, T_7\}\\
&\qquad \Rightarrow T_7 = T_1 \cup \ldots \cup T_5 \cup T_7 = \bigcup_{t \in T_7} \Lambda(Q, Q_t),\\
&t \in T_8 = T_7 \cup T_6 \Rightarrow \Lambda(Q, Q_t) = \{T_1, T_2, T_3, T_4, T_5, T_6, T_7\}\\
&\qquad \Rightarrow T_8 = T_6 \cup T_7 = \bigcup_{t \in T_8} \Lambda(Q, Q_t).
\end{aligned}
$$

Then Q is a $XI-$ semilattice of unions. □

Lemma 3.2. *Following equalities are true for Q where P_i's are pairwise disjoint sets and union of these sets equals Q.*

$$
\begin{aligned}
&P_1 = T_2 \backslash T_1, && P_2 = (T_4 \cap T_3) \backslash T_2, && P_3 = T_4 \backslash T_3, && P_4 = T_3 \backslash T_4,\\
&P_5 = (T_7 \cap T_6) \backslash T_5, && P_6 = T_7 \backslash T_6, && P_7 = T_6 \backslash T_7, && P_8 = T_1.
\end{aligned}
$$

Proof. Considering the (3.1), it is easy to see that equalities are true. □

Lemma 3.3. *Let $G = \{T_1, T_2, T_3, T_4, T_5, T_6, T_7\}$ be a generating set of Q. Then the elements $T_1, T_2, T_3, T_4, T_6, T_7$ are nonlimiting elements of the set $\ddot{G}_{T_1}$, $\ddot{G}_{T_2}$, $\ddot{G}_{T_3}$, $\ddot{G}_{T_4}$, $\ddot{G}_{T_6}$, $\ddot{G}_{T_7}$ respectively and T_5 is limiting eleman of the set $\ddot{G}_{T_5}$.*

Proof. Definition of $\ddot{D}'_T$, following equations

$$\begin{aligned}
\ddot{G}_{T_1} &= \{T_1\}\,,\\
\ddot{G}_{T_2} &= \{T_1, T_2\}\,,\\
\ddot{G}_{T_3} &= \{T_1, T_2, T_3\}\,,\\
\ddot{G}_{T_4} &= \{T_1, T_2, T_4\}\,,\\
\ddot{G}_{T_5} &= \{T_1, T_2, T_3, T_4, T_5\}\,,\\
\ddot{G}_{T_6} &= \{T_1, T_2, T_3, T_4, T_5, T_6\}\,,\\
\ddot{G}_{T_7} &= \{T_1, T_2, T_3, T_4, T_5, T_7\}\,.
\end{aligned} \tag{3.5}$$

are obtained. Now we get the sets $l(\ddot{G}_{T_i}, T_i)$, $i \in \{1, 2, \dots, 7\}$,

$$\begin{array}{lcll}
l(\ddot{G}_{T_1}, T_1) &=& \cup(\ddot{G}_{T_1} \setminus \{T_1\}) &= \emptyset,\\
l(\ddot{G}_{T_2}, T_2) &=& \cup(\ddot{G}_{T_2} \setminus \{T_2\}) &= T_1,\\
l(\ddot{G}_{T_3}, T_3) &=& \cup(\ddot{G}_{T_3} \setminus \{T_3\}) &= T_2,\\
l(\ddot{G}_{T_4}, T_4) &=& \cup(\ddot{G}_{T_4} \setminus \{T_4\}) &= T_2,\\
l(\ddot{G}_{T_5}, T_5) &=& \cup(\ddot{G}_{T_5} \setminus \{T_5\}) &= T_5,\\
l(\ddot{G}_{T_6}, T_6) &=& \cup(\ddot{G}_{T_6} \setminus \{T_6\}) &= T_5,\\
l(\ddot{G}_{T_7}, T_7) &=& \cup(\ddot{G}_{T_7} \setminus \{T_7\}) &= T_5.
\end{array}$$

Then we find nonlimiting and limiting elements of $\ddot{G}_{T_i}$, $i \in \{1, 2, \dots, 7\}$.

$$\begin{array}{llll}
T_1 \setminus l(\ddot{G}_{T_1}, T_1) & = T_1 \setminus \emptyset & = T_1 \neq \emptyset, & T_1 \text{ nonlimiting element}\\
T_2 \setminus l(\ddot{G}_{T_2}, T_2) & = T_2 \setminus T_1 & \neq \emptyset & T_2 \text{ nonlimiting element}\\
T_3 \setminus l(\ddot{G}_{T_3}, T_3) & = T_3 \setminus T_2 & \neq \emptyset & T_3 \text{ nonlimiting element}\\
T_4 \setminus l(\ddot{G}_{T_4}, T_4) & = T_4 \setminus T_2 & \neq \emptyset & T_4 \text{ nonlimiting element}\\
T_5 \setminus l(\ddot{G}_{T_5}, T_5) & = T_5 \setminus T_5 & = \emptyset & T_5 \text{ limiting element}\\
T_6 \setminus l(\ddot{G}_{T_6}, T_6) & = T_6 \setminus T_5 & \neq \emptyset & T_6 \text{ nonlimiting element}\\
T_7 \setminus l(\ddot{G}_{T_7}, T_7) & = T_7 \setminus T_5 & \neq \emptyset & T_7 \text{ nonlimiting element}
\end{array}$$

Therefore, the elements $T_1, T_2, T_3, T_4, T_6, T_7$ are nonlimiting elements of the sets $\ddot{G}_{T_1}$, $\ddot{G}_{T_2}$, $\ddot{G}_{T_3}$, $\ddot{G}_{T_4}$, $\ddot{G}_{T_6}$, $\ddot{G}_{T_7}$, respectively and T_5 is limiting eleman of the set $\ddot{G}_{T_5}$. □

Now, we determine properties of a reguler element α of $B_X(Q)$ where $V(D, \alpha) = Q$ and $\alpha = \bigcup_{i=1}^{8} (Y_i^\alpha \times T_i\ )$.

Theorem 3.4. *Let $\alpha \in B_X(Q)$ be a quasinormal representation of the form $\alpha = \bigcup_{i=1}^{8} (Y_i^\alpha \times T_i)$ such that $V(D, \alpha) = Q$. $\alpha \in B_X(D)$ is a regular iff for* *some complete α-isomorphism $\varphi : Q \to D' \subseteq D$, the following conditions are satisfied:*

$$\begin{array}{l} Y_1^\alpha \supseteq \varphi(T_1), \\ Y_1^\alpha \cup Y_2^\alpha \supseteq \varphi(T_2), \\ Y_1^\alpha \cup Y_2^\alpha \cup Y_3^\alpha \supseteq \varphi(T_3), \\ Y_1^\alpha \cup Y_2^\alpha \cup Y_4^\alpha \supseteq \varphi(T_4), \\ Y_1^\alpha \cup Y_2^\alpha \cup Y_3^\alpha \cup Y_4^\alpha \cup Y_5^\alpha \cup Y_6^\alpha \supseteq \varphi(T_6), \\ Y_1^\alpha \cup Y_2^\alpha \cup Y_3^\alpha \cup Y_4^\alpha \cup Y_5^\alpha \cup Y_7^\alpha \supseteq \varphi(T_7), \\ Y_2^\alpha \cap \varphi(T_2) \neq \emptyset,\ Y_3^\alpha \cap \varphi(T_3) \neq \emptyset, \\ Y_4^\alpha \cap \varphi(T_4) \neq \emptyset,\ Y_6^\alpha \cap \varphi(T_6) \neq \emptyset, \\ Y_7^\alpha \cap \varphi(T_7) \neq \emptyset. \end{array} \tag{3.6}$$

Proof. Let $G = \{T_1, T_2, T_3, T_4, T_5, T_6, T_7\}$ be a generating set of Q.

$\Rightarrow$: Since $\alpha \in B_X(D)$ is regular and $V(D, \alpha) = Q$ $XI-$semilattice of unions, by Theorem 2.1, there exits a complete isomorphism $\varphi : Q \to D'$. By Theorem 2.1 (a), satisfying $\varphi(T)\alpha = T$ for all $T \in V(D, \alpha)$. So, φ is complete α-isomorphism. Applying the Theorem 2.1 (b) we have

$$\begin{array}{l} Y_1^\alpha \supseteq \varphi(T_1) \\ Y_1^\alpha \cup Y_2^\alpha \supseteq \varphi(T_2) \\ Y_1^\alpha \cup Y_2^\alpha \cup Y_3^\alpha \supseteq \varphi(T_3) \\ Y_1^\alpha \cup Y_2^\alpha \cup Y_4^\alpha \supseteq \varphi(T_4) \\ Y_1^\alpha \cup Y_2^\alpha \cup Y_3^\alpha \cup Y_4^\alpha \cup Y_5^\alpha \supseteq \varphi(T_5) \\ Y_1^\alpha \cup Y_2^\alpha \cup Y_3^\alpha \cup Y_4^\alpha \cup Y_5^\alpha \cup Y_6^\alpha \supseteq \varphi(T_6), \\ Y_1^\alpha \cup Y_2^\alpha \cup Y_3^\alpha \cup Y_4^\alpha \cup Y_5^\alpha \cup Y_7^\alpha \supseteq \varphi(T_7) \end{array} \tag{3.7}$$

Moreover, considering that the elements $T_1, T_2, T_3, T_4, T_6, T_7$ are nonlimiting and using the Theorem 2.1 (c), following properties

$$\begin{array}{l} Y_1^\alpha \cap \varphi(T_1) \neq \emptyset,\ Y_2^\alpha \cap \varphi(T_2) \neq \emptyset, \\ Y_3^\alpha \cap \varphi(T_3) \neq \emptyset,\ Y_4^\alpha \cap \varphi(T_4) \neq \emptyset, \\ Y_6^\alpha \cap \varphi(T_6) \neq \emptyset,\ Y_7^\alpha \cap \varphi(T_7) \neq \emptyset. \end{array} \tag{3.8}$$

are obtained. From $Y_1^\alpha \supseteq \varphi(T_1)$, $Y_1^\alpha \cap \varphi(T_1) \neq \emptyset$ always ensured. Also by using $Y_1^\alpha \cup Y_2^\alpha \cup Y_3^\alpha \supseteq \varphi(T_3)$ and $Y_1^\alpha \cup Y_2^\alpha \cup Y_4^\alpha \supseteq \varphi(T_4)$, we get

$$\begin{aligned} Y_1^\alpha \cup Y_2^\alpha \cup Y_3^\alpha \cup Y_4^\alpha \cup Y_5^\alpha &= (Y_1^\alpha \cup Y_2^\alpha \cup Y_3^\alpha) \cup (Y_1^\alpha \cup Y_2^\alpha \cup Y_4^\alpha) \\ &\supseteq \varphi(T_3) \cup \varphi(T_4) \cup Y_5^\alpha \\ &= \varphi(T_5) \cup Y_5^\alpha \\ &\supseteq \varphi(T_5) \end{aligned}$$

Thus there is no need the condition $Y_1^\alpha \cup Y_2^\alpha \cup Y_3^\alpha \cup Y_4^\alpha \cup Y_5^\alpha \supseteq \varphi(T_5)$. Therefore there exist an $\alpha-$isomorphism φ which holds given conditions.

$\Leftarrow$: Since $V(D,\alpha) = Q$, $V(D,\alpha)$ is $XI-$semilattice of unions. Let $\varphi : Q \to D' \subseteq D$ be complete $\alpha-$isomorphism which holds given conditions. So, considering Equation (3.6), satisfying Theorem 2.1 $(a)-(c)$. Remembering that T_5 is a limiting element of the set $\ddot{G}_{T_5}$, we constitute the set $B(T_5) = \left\{Z \in \ddot{G}_{T_5} \mid Y_Z^\alpha \cap \varphi(T_5) \neq \emptyset\right\}$. If $Y_4^\alpha \cap \varphi(T_5) = \emptyset$ we have

$$\begin{aligned} Y_1^\alpha \cup Y_2^\alpha \cup Y_3^\alpha \cup Y_4^\alpha &= (Y_1^\alpha \cup Y_2^\alpha \cup Y_3^\alpha) \cup (Y_1^\alpha \cup Y_2^\alpha \cup Y_4^\alpha) \\ &\supseteq \varphi(T_3) \cup \varphi(T_4) = \varphi(T_5) \end{aligned}$$

So we get $Y_1^\alpha \cup Y_2^\alpha \cup Y_3^\alpha \supseteq \varphi(T_5) \supseteq \varphi(T_4)$ which is a contradiction with $Y_4^\alpha \cap \varphi(T_4) \neq \emptyset$. Therefore $T_4 \in B(T_5)$. If $Y_3^\alpha \cap \varphi(T_5) = \emptyset$ we have

$$\begin{aligned} Y_1^\alpha \cup Y_2^\alpha \cup Y_3^\alpha \cup Y_4^\alpha &= (Y_1^\alpha \cup Y_2^\alpha \cup Y_3^\alpha) \cup (Y_1^\alpha \cup Y_2^\alpha \cup Y_4^\alpha) \\ &\supseteq \varphi(T_3) \cup \varphi(T_4) = \varphi(T_5) \end{aligned}$$

So we get $Y_1^\alpha \cup Y_2^\alpha \cup Y_4^\alpha \supseteq \varphi(T_5) \supseteq \varphi(T_3)$ which is a contradiction with $Y_3^\alpha \cap \varphi(T_3) \neq \emptyset$. Therefore $T_3 \in B(T_5)$. We have $\cup B(T_5) = T_3 \cup T_4 = T_5$. By Theorem 2.1, we conclude that α is the regular element of the $B_X(D)$. □

Now we calculate the number of regular elements α, satisfying the hyphothesis of Theorem 3.4. Let $\alpha \in B_X(D)$ be a regular element which is quasinormal representation of the form $\alpha = \bigcup_{i=1}^{8}(Y_i^\alpha \times T_i)$ and $V(D,\alpha) = Q$. Then there exist a complete $\alpha-$ isomorphism $\varphi : Q \to D' = \{\varphi(T_1), \varphi(T_2), \ldots, \varphi(T_8)\}$ satisfying the hyphothesis of Theorem 3.4. So, $\alpha \in R_\varphi(Q, D')$. We will denote $\varphi(T_i) = \overline{T}_i$, $i = 1, 2, \ldots 8$. Diagram of the $D' = \left\{\overline{T}_1, \overline{T}_2, \overline{T}_3, \overline{T}_4, \overline{T}_5, \overline{T}_6, \overline{T}_7, \overline{T}_8\right\}$ is shown in Figure 3.2. Then the Equation (3.6) reduced to below equation.

$$\begin{aligned} & Y_1^\alpha \supseteq \overline{T}_1 \\ & Y_1^\alpha \cup Y_2^\alpha \supseteq \overline{T}_2 \\ & Y_1^\alpha \cup Y_2^\alpha \cup Y_3^\alpha \supseteq \overline{T}_3 \\ & Y_1^\alpha \cup Y_2^\alpha \cup Y_4^\alpha \supseteq \overline{T}_4 \\ & Y_1^\alpha \cup Y_2^\alpha \cup Y_3^\alpha \cup Y_4^\alpha \cup Y_5^\alpha \cup Y_6^\alpha \supseteq \overline{T}_6, \\ & Y_1^\alpha \cup Y_2^\alpha \cup Y_3^\alpha \cup Y_4^\alpha \cup Y_5^\alpha \cup Y_7^\alpha \supseteq \overline{T}_7 \\ & Y_2^\alpha \cap \varphi(T_2) \neq \emptyset, Y_3^\alpha \cap \varphi(T_3) \neq \emptyset, \\ & Y_4^\alpha \cap \varphi(T_4) \neq \emptyset, Y_6^\alpha \cap \varphi(T_6) \neq \emptyset, \\ & Y_7^\alpha \cap \varphi(T_7) \neq \emptyset. \end{aligned} \tag{3.9}$$

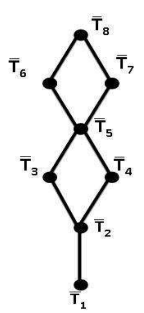


Figure 3.2

On the other hand, the image of the sets in Lemma 3.2 under the $\alpha-$ isomorphism φ

$$\overline{T}_1, (\overline{T}_3 \cap \overline{T}_4) \setminus \overline{T}_1, \overline{T}_4 \setminus \overline{T}_3, \overline{T}_3 \setminus \overline{T}_4, (\overline{T}_7 \cap \overline{T}_6) \setminus \overline{T}_5, \overline{T}_7 \setminus \overline{T}_6, \overline{T}_6 \setminus \overline{T}_7, X \setminus \overline{T}_8$$

are also pairwise disjoint sets and union of these sets equals X.

Lemma 3.5. *For every $\alpha \in R_{\varphi}(Q, D')$, there exists an ordered system of disjoint mappings which is defined $\{\overline{T}_1, (\overline{T}_3 \cap \overline{T}_4)\backslash\overline{T}_1, \overline{T}_4\backslash\overline{T}_3, \overline{T}_3\backslash\overline{T}_4, (\overline{T}_7 \cap \overline{T}_6)\backslash\overline{T}_5, \overline{T}_7\backslash\overline{T}_6, \overline{T}_6\backslash\overline{T}_7, X\backslash\overline{T}_8\}$. Also, ordered systems are different which correspond to different binary relations.*

Proof. Let $f_\alpha : X \to D$ be a mapping satisfying the condition $f_\alpha(t) = t\alpha$ for all $t \in X$. We consider the restrictions of the mapping f_α as $f_{1\alpha}$, $f_{2\alpha}$, $f_{3\alpha}$, $f_{4\alpha}$, $f_{5\alpha}$, $f_{6\alpha}$, $f_{7\alpha}$, $f_{8\alpha}$ on the sets $\overline{T}_1, (\overline{T}_3 \cap \overline{T}_4)\backslash\overline{T}_1, \overline{T}_4\backslash\overline{T}_3, \overline{T}_3\backslash\overline{T}_4, (\overline{T}_7 \cap \overline{T}_6)\backslash\overline{T}_5, \overline{T}_7\backslash\overline{T}_6, \overline{T}_6\backslash\overline{T}_7, X\backslash\overline{T}_8$, respectively.

Now, considering the definition of the sets Y_i^α, $i = 1, 2, \ldots, 8$, together with the Equation (3.9) we have

$$t \in \overline{T}_1 \Rightarrow t \in Y_1^\alpha \Rightarrow t\alpha = T_1 \Rightarrow f_{1\alpha}(t) = T_1, \ \forall t \in \overline{T}_1.$$

$$\begin{aligned} t \in (\overline{T}_3 \cap \overline{T}_4)\backslash\overline{T}_1 &\Rightarrow t \in (\overline{T}_3 \cap \overline{T}_4) \subseteq Y_1^\alpha \cup Y_2^\alpha \\ &\Rightarrow t\alpha \in \{T_1, T_2\} \\ &\Rightarrow f_{2\alpha}(t) \in \{T_1, T_2\}, \ \forall t \in (\overline{T}_3 \cap \overline{T}_4)\backslash\overline{T}_1. \end{aligned}$$

Since $Y_2^\alpha \cap \overline{T}_2 \neq \emptyset$, there is an element $t_2 \in Y_2^\alpha \cap \overline{T}_2$. Then $t_2\alpha = T_2$ and $t_2 \in \overline{T}_2$. If $t_2 \in \overline{T}_1$ then $t_2 \in \overline{T}_1 \subseteq Y_1^\alpha$. Therefore, $t_2\alpha = T_1$ which is in contradiction with the equality $t_2\alpha = T_2$. So $f_{2\alpha}(t_2) = T_2$ for some $t_2 \in \overline{T}_2\backslash\overline{T}_1$.

$$\begin{aligned} t \in \overline{T}_4\backslash\overline{T}_3 &\Rightarrow t \in \overline{T}_4\backslash\overline{T}_3 \subseteq \overline{T}_4 \subseteq Y_1^\alpha \cup Y_2^\alpha \cup Y_4^\alpha \\ &\Rightarrow t\alpha \in \{T_1, T_2, T_4\} \\ &\Rightarrow \ f_{3\alpha}(t) \in \{T_1, T_2, T_4\}, \ \forall t \in \overline{T}_4\backslash\overline{T}_3. \end{aligned}$$

$Y_4^\alpha \cap \overline{T}_4 \neq \emptyset$ so there is an element $t_4 \in Y_4^\alpha \cap \overline{T}_4$. Then $t_4\alpha = T_4$ and $t_4 \in \overline{T}_4$. If $t_4 \in \overline{T}_3$ then $t_4 \in \overline{T}_3 \subseteq Y_1^\alpha \cup Y_2^\alpha \cup Y_3^\alpha$. Thus $t_4\alpha \in \{T_1, T_2, T_3\}$ which is in contradiction with the equality $t_4\alpha = T_4$. So there is an element $t_4 \in \overline{T}_4\backslash\overline{T}_3$ with $f_{3\alpha}(t_4) = T_4$.

$$\begin{aligned} t \in \overline{T}_3\backslash\overline{T}_4 &\Rightarrow t \in \overline{T}_3\backslash\overline{T}_4 \subseteq \overline{T}_3 \subseteq Y_1^\alpha \cup Y_2^\alpha \cup Y_3^\alpha \\ &\Rightarrow t\alpha \in \{T_1, T_2, T_3\} \\ &\Rightarrow \ f_{4\alpha}(t) \in \{T_1, T_2, T_3\}, \ \forall t \in \overline{T}_3\backslash\overline{T}_4. \end{aligned}$$

Since $Y_3^\alpha \cap \overline{T}_3 \neq \emptyset$, there is an element t_3 with $t_3\alpha = T_3$ and $t_3 \in \overline{T}_3$. If $t_3 \in \overline{T}_4$ then $t_3 \in \overline{T}_4 \subseteq Y_1^\alpha \cup Y_2^\alpha \cup Y_4^\alpha$. Therefore, $t_3\alpha \in \{T_1, T_2, T_4\}$ which contradicts to the equality $t_3\alpha = T_3$. So there is an element $t_3 \in \overline{T}_3\backslash\overline{T}_4$ with $f_{4\alpha}(t_3) = T_3$.

$$\begin{aligned} t \in (\overline{T}_7 \cap \overline{T}_6)\backslash\overline{T}_5 &\Rightarrow t \in (\overline{T}_7 \cap \overline{T}_6)\backslash\overline{T}_5 \subseteq \overline{T}_7 \cap \overline{T}_6 \subseteq Y_1^\alpha \cup Y_2^\alpha \cup Y_3^\alpha \cup Y_4^\alpha \cup Y_5^\alpha \\ &\Rightarrow t\alpha \in \{T_1, T_2, T_3, T_4, T_5\} \\ &\Rightarrow f_{5\alpha}(t) \in \{T_1, T_2, T_3, T_4, T_5\}, \ \forall t \in (\overline{T}_7 \cap \overline{T}_6)\backslash\overline{T}_5. \end{aligned}$$

$$\begin{aligned} t\in \overline{T}_7\backslash\overline{T}_6 &\Rightarrow t\in \overline{T}_7\backslash\overline{T}_6\subseteq \overline{T}_7\subseteq Y_1^\alpha\cup Y_2^\alpha\cup Y_3^\alpha\cup Y_4^\alpha\cup Y_5^\alpha\cup Y_7^\alpha \\ &\Rightarrow t\alpha\in\{T_1,T_2,T_3,T_4,T_5,T_7\} \\ &\Rightarrow f_{6\alpha}(t)\in\{T_1,T_2,T_3,T_4,T_5,T_7\}\,,\ \forall t\in \overline{T}_7\backslash\overline{T}_6. \end{aligned}$$

Also, there is an element $t_7\in Y_7^\alpha\cap\overline{T}_7$ since $Y_7^\alpha\cap\overline{T}_7\neq\emptyset$. Then $t_7\alpha=T_7$ and $t_7\in\overline{T}_7$. If $t_7\in\overline{T}_6$ then $t_7\in\overline{T}_6\subseteq Y_1^\alpha\cup Y_2^\alpha\cup Y_3^\alpha\cup Y_4^\alpha\cup Y_5^\alpha\cup Y_6^\alpha$. So $t_7\alpha\in\{T_1,T_2,T_3,T_4,T_5,T_6\}$. However this contradicts to $t_7\alpha=T_7$. So $f_{6\alpha}(t_7)=T_7$ for some $t_7\in\overline{T}_7\backslash\overline{T}_6$.

$$\begin{aligned} t\in \overline{T}_6\backslash\overline{T}_7 &\Rightarrow t\in \overline{T}_6\backslash\overline{T}_7\subseteq \overline{T}_6\subseteq Y_1^\alpha\cup Y_2^\alpha\cup Y_3^\alpha\cup Y_4^\alpha\cup Y_5^\alpha\cup Y_6^\alpha \\ &\Rightarrow t\alpha\in\{T_1,T_2,T_3,T_4,T_5,T_6\} \\ &\Rightarrow f_{7\alpha}(t)\in\{T_1,T_2,T_3,T_4,T_5,T_6\}\,,\ \forall t\in \overline{T}_6\backslash\overline{T}_7. \end{aligned}$$

Smilarly there is an element t_6 with $t_6\alpha=T_6$ and $t_6\in\overline{T}_6$ since $Y_6^\alpha\cap\overline{T}_6\neq\emptyset$. If $t_6\in\overline{T}_7$ then $t_6\in\overline{T}_7\subseteq Y_1^\alpha\cup Y_2^\alpha\cup Y_3^\alpha\cup Y_4^\alpha\cup Y_5^\alpha\cup Y_7^\alpha$. Therefore, $t_6\alpha\in\{T_1,T_2,T_3,T_4,T_5,T_7\}$ which is in contradiction with the equality $t_6\alpha=T_6$. So $f_{7\alpha}(t_6)=T_6$ for some $t_6\in\overline{T}_6\backslash\overline{T}_7$.

$$t\in X\backslash\overline{T}_8\Rightarrow t\in X\backslash\overline{T}_8\subseteq X=\bigcup_{i=1}^{8}Y_i^\alpha\Rightarrow t\alpha\in Q\Rightarrow f_{8\alpha}(t)\in Q,\ \forall t\in X\backslash\overline{T}_8.$$

Therefore, for every binary relation $\alpha\in R_\varphi(Q,D')$ there exists an ordered system $(f_{1\alpha},\ f_{2\alpha},\ f_{3\alpha},\ f_{4\alpha},\ f_{5\alpha},\ f_{6\alpha},\ f_{7\alpha}, f_{8\alpha})$.

On the other hand, suppose that for $\alpha,\beta\in R_\varphi(Q,D')$ which $\alpha\neq\beta$, be obtained $f_\alpha=(f_{1\alpha},\ f_{2\alpha},\ f_{3\alpha},\ f_{4\alpha},\ f_{5\alpha},\ f_{6\alpha},\ f_{7\alpha},\ f_{8\alpha})$ and $f_\beta=(f_{1\beta},\ f_{2\beta},\ f_{3\beta},\ f_{4\beta},$ $f_{5\beta},\ f_{6\beta},\ f_{7\beta},\ f_{8\beta})$. If $f_\alpha=f_\beta$, we get

$$f_\alpha=f_\beta\Rightarrow f_\alpha(t)=f_\beta(t),\ \forall t\in X\Rightarrow t\alpha=t\beta,\ \forall t\in X\Rightarrow\alpha=\beta$$

which contradicts to $\alpha\neq\beta$. Therefore different binary relations's ordered systems are different. □

Lemma 3.6. *Let $f=(f_1,\ f_2,\ f_3,\ f_4,\ f_5,\ f_6,\ f_7, f_8)$ be ordered system from*

X in the semilattice D such that

f_1:$\overline{T}_1 \to \{T_1\}$,$f_1(t) = T_1$,
f_2: $(\overline{T}_3 \cap \overline{T}_4)\backslash\overline{T}_1 \to \{T_1, T_2\}$, $f_2(t) \in \{T_1, T_2\}$ *and* $f_2(t_2) = T_2 \ \exists\ t_2 \in \overline{T}_2\backslash\overline{T}_1$,
f_3:$\overline{T}_4\backslash\overline{T}_3 \to \{T_1, T_2, T_4\}$, $f_3(t) \in \{T_1, T_2, T_4\}$ *and* $f_3(t_4) = T_4 \ \exists\ t_4 \in \overline{T}_4\backslash\overline{T}_3$,
f_4:$\overline{T}_3\backslash\overline{T}_4 \to \{T_1, T_2, T_3\}$, $f_4(t) \in \{T_1, T_2, T_3\}$ *and* $f_4(t_3) = T_3 \ \exists\ t_3 \in \overline{T}_3\backslash\overline{T}_4$,
f_5: $(\overline{T}_7 \cap \overline{T}_6)\backslash\overline{T}_5 \to \{T_1, T_2, T_3, T_4, T_5\}$, $f_5(t) \in \{T_1, T_2, T_3, T_4, T_5\}$,
f_6: $\overline{T}_7\backslash\overline{T}_6 \to \{T_1, T_2, T_3, T_4, T_5, T_7\}$, $f_6(t) \in \{T_1, T_2, T_3, T_4, T_5, T_7\}$
and $f_6(t_7) = T_7 \ \exists\ t_7 \in \overline{T}_7\backslash\overline{T}_6$,
f_7: $\overline{T}_6\backslash\overline{T}_7 \to \{T_1, T_2, T_3, T_4, T_5, T_6\}$, $f_7(t) \in \{T_1, T_2, T_3, T_4, T_5, T_6\}$,
and $f_7(t_6) = T_6 \ \exists\ t_6 \in \overline{T}_6\backslash\overline{T}_7$,
f_8: $X\backslash\overline{T}_8 \to Q$, $f_{8\alpha}(t) \in Q$.

Then $\beta = \bigcup\limits_{x\in X} (\{x\}\times f(x)) \in B_X(D)$ *is regular and* φ *is complete* $\beta-$*isomorphism* θ. *So* $\beta \in R_\varphi(Q, D')$.

Proof. First we see that $V(D, \beta) = Q$. Considering $V(D, \beta) = \{Y\beta \mid Y \in D\}$, the properties of f mapping, $\overline{T}_i\beta = \bigcup\limits_{x\in\overline{T}_i} x\beta$ and $D' \subseteq D$, we get

$T_1 \in Q \Rightarrow \overline{T}_1\beta{=}T_1 \Rightarrow T_1 \in V(D, \beta)$,
$T_2 \in Q \Rightarrow \overline{T}_2\beta{=}\ T_1 \cup T_2 = T_2 \Rightarrow T_2 \in V(D, \beta)$,
$T_3 \in Q \Rightarrow \overline{T}_3\beta{=}\ T_1 \cup T_2 \cup T_3 = T_3 \Rightarrow T_3 \in V(D, \beta)$,
$T_4 \in Q \Rightarrow \overline{T}_4\beta{=}\ T_1 \cup T_2 \cup T_3 \cup T_4 \cup T_5 = T_4 \Rightarrow T_4 \in V(D, \beta)$,
$T_5 \in Q \Rightarrow \overline{T}_5\beta{=}\left(\overline{T}_3 \cup \overline{T}_4\right)\beta = \overline{T}_3\beta \cup \overline{T}_4\beta = T_3 \cup T_4 = T_5 \Rightarrow T_5 \in V(D, \beta)$,
$T_6 \in Q \Rightarrow \overline{T}_6\beta{=}T_1 \cup T_2 \cup T_3 \cup T_4 \cup T_5 \cup T_6 = T_6 \Rightarrow T_6 \in V(D, \beta)$,
$T_7 \in Q \Rightarrow \overline{T}_7\beta{=}T_1 \cup T_2 \cup T_3 \cup T_4 \cup T_5 \cup T_7 = T_7 \Rightarrow T_7 \in V(D, \beta)$,
$T_8 \in Q \Rightarrow \overline{T}_8\beta{=}\left(\overline{T}_6 \cup \overline{T}_7\right)\beta = \overline{T}_6\beta \cup \overline{T}_7\beta = T_6 \cup T_7 = T_8 \Rightarrow T_8 \in V(D, \beta)$.

Then $Q \subseteq V(D, \beta)$. Also,

$$\begin{aligned} Z \in V(D,\beta) &\Rightarrow Z = Y\beta,\ \exists Y \in D \\ &\Rightarrow Z = Y\beta = \bigcup_{y\in Y} y\beta = \bigcup_{y\in Y} f(y) \in Q \end{aligned}$$

since $f(y) \in Q$ and Q is closed set-theoretic union. Therefore, $V(D, \beta) \subseteq Q$. Hence $V(D, \beta) = Q$.

Also, $\beta = \bigcup\limits_{T\in V(X^*,\beta)} \left(Y_T^\beta \times T\right)$ is quasinormal representation of β since $\emptyset \notin Q$. From the definition of β, $f(x) = x\beta$ for all $x \in X$. It is easily seen that

$V(X^*,\beta)=V(D,\beta)=Q$. We get $\beta=\bigcup\limits_{i=1}^{8}\left(Y_i^\beta\times T_i\right)$.

On the other hand

$$\begin{aligned}
&t\in\overline{T}_1\Rightarrow t\beta=f(t)=T_1\Rightarrow t\in Y_1^\beta\Rightarrow\overline{T}_1\subseteq Y_1^\beta,\\
&t\in\overline{T}_2=\overline{T}_1\cup\left((\overline{T}_3\cap\overline{T}_4)\setminus\overline{T}_1\right)\Rightarrow t\beta=f(t)\in\{T_1,T_2\}\ \Rightarrow t\in Y_1^\beta\cup Y_2^\beta\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\Rightarrow\overline{T}_2\subseteq Y_1^\beta\cup Y_2^\beta\\
&t\in\overline{T}_3=\overline{T}_1\cup\left((\overline{T}_3\cap\overline{T}_4)\setminus\overline{T}_1\right)\cup\left(\overline{T}_3\setminus\overline{T}_4\right)\ \Rightarrow t\beta=f(t)\in\{T_1,T_2,T_3\}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\Rightarrow t\in Y_1^\beta\cup Y_2^\beta\cup Y_3^\beta\\
&\qquad\qquad\qquad\qquad\qquad\qquad\Rightarrow\overline{T}_4\subseteq Y_1^\beta\cup Y_2^\beta\cup Y_3^\beta,\\
&t\in\overline{T}_4=\overline{T}_1\cup\left((\overline{T}_3\cap\overline{T}_4)\setminus\overline{T}_1\right)\cup\left(\overline{T_4}\setminus\overline{T}_3\right)\ \Rightarrow t\beta=f(t)\in\{T_1,T_2,T_4\}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\Rightarrow t\in Y_1^\beta\cup Y_2^\beta\cup Y_4^\beta\\
&\qquad\qquad\qquad\qquad\qquad\qquad\Rightarrow\overline{T}_4\subseteq Y_1^\beta\cup Y_2^\beta\cup Y_4^\beta,\\
&t\in\overline{T}_6\ =\left(\overline{T}_6\setminus\overline{T}_7\right)\cup\left((\overline{T}_7\cap\overline{T}_6)\setminus\overline{T}_5\right)\cup\overline{T}_3\cup\overline{T}_4\\
&\qquad\Rightarrow t\beta=f(t)\in\{T_1,T_2,T_3,T_4,T_5,T_6\}\\
&\qquad\Rightarrow t\in Y_1^\beta\cup Y_2^\beta\cup Y_3^\beta\cup Y_4^\beta\cup Y_5^\beta\cup Y_6^\beta\\
&\qquad\Rightarrow\overline{T}_6\subseteq Y_1^\beta\cup Y_2^\beta\cup Y_3^\beta\cup Y_4^\beta\cup Y_5^\beta\cup Y_6^\beta,\\
&t\in\overline{T}_7\ =\left(\overline{T}_7\setminus\overline{T}_6\right)\cup\left((\overline{T}_7\cap\overline{T}_6)\setminus\overline{T}_5\right)\cup\overline{T}_3\cup\overline{T}_4\\
&\qquad\Rightarrow t\beta=f(t)\in\{T_1,T_2,T_3,T_4,T_5,T_7\}\\
&\qquad\Rightarrow t\in Y_1^\beta\cup Y_2^\beta\cup Y_3^\beta\cup Y_4^\beta\cup Y_5^\beta\cup Y_7^\beta\\
&\qquad\Rightarrow\overline{T}_6\subseteq Y_1^\beta\cup Y_2^\beta\cup Y_3^\beta\cup Y_4^\beta\cup Y_5^\beta\cup Y_7^\beta,
\end{aligned}$$

Also, by using $f_2(t_2)=T_2,\ \exists\ t_2\in\overline{T}_2\setminus\overline{T}_1$, we obtain $Y_2^\beta\cap\overline{T}_2\neq\emptyset$. Similarly, from properties of $f_3,\ f_4,\ f_6,\ f_7$, be seen $Y_3^\beta\cap\overline{T}_3\neq\emptyset,\ Y_4^\beta\cap\overline{T}_4\neq\emptyset,\ Y_6^\beta\cap\overline{T}_6\neq\emptyset$ and $Y_7^\beta\cap\overline{T}_7\neq\emptyset$. Therefore the mapping $\varphi:Q\to D'=\left\{\overline{T}_1,\overline{T}_2,\ldots,\overline{T}_8\right\}$ to be defined $\varphi(T_i)=\overline{T}_i$ satisfy the conditions in (3.9) for β. Hence φ is complete $\beta-$isomorhism because of $\varphi(T)\beta=\overline{T}\beta=T$, for all $T\in V(D,\beta)$. By Theorem 3.4, $\beta\in R_\varphi(Q,D')$. □

Therefore, there is one to one correspondence between the elements of $R_\varphi(Q,D')$ and the set of ordered systems of disjoint mappings.

Theorem 3.7. *Let X be a finite set and Q be $XI-$ semilattice. If*

$$D'=\left\{\overline{T}_1,\overline{T}_2,\overline{T}_3,\overline{T}_4,\overline{T}_5,\overline{T}_6,\overline{T}_7,\overline{T}_8\right\}$$

is $\alpha-$ isomorphic to Q and $\Omega(Q)=m_0$, then

$$\left|R(D')\right|=m_0\cdot 4\cdot\left(2^{\left|(\overline{T}_3\cap\overline{T}_4)\setminus\overline{T}_2\right|}(2^{\left|\overline{T}_2\setminus\overline{T}_1\right|}-1)\right)\cdot\left(3^{\left|\overline{T}_4\setminus\overline{T}_3\right|}-2^{\left|\overline{T}_4\setminus\overline{T}_3\right|}\right)$$

$$\cdot\left(3^{|\overline{T}_3\setminus\overline{T}_4|}-2^{|\overline{T}_3\setminus\overline{T}_4|}\right)\cdot 5^{|(\overline{T}_7\cap\overline{T}_6)\setminus\overline{T}_5|}\cdot\left(6^{|\overline{T}_7\setminus\overline{T}_6|}-5^{|\overline{T}_7\setminus\overline{T}_6|}\right)$$
$$\cdot\left(6^{|\overline{T}_6\setminus\overline{T}_7|}-5^{|\overline{T}_6\setminus\overline{T}_7|}\right)\cdot 8^{|X\setminus\overline{T}_8|}$$

Proof. Lemma 3.5 and Lemma 3.6 show us that the number of the ordered system of disjoint mappings $(f_{1\alpha},\ f_{2\alpha},\ f_{3\alpha},\ f_{4\alpha},\ f_{5\alpha},\ f_{6\alpha},\ f_{7\alpha},\ f_{8\alpha})$ is equal to $|R_\varphi(Q,D')|$, which $\alpha\in B_X(D)$ regular element, $V(D,\alpha)=Q$ and $\varphi:Q\to D'$ is a complete $\alpha-$isomorphism.

From the Theorem 2.2, the number of the mappings $f_{1\alpha}$, $f_{2\alpha}$, $f_{3\alpha}$, $f_{4\alpha}$, $f_{5\alpha}$, $f_{6\alpha}$, $f_{7\alpha}$ and $f_{8\alpha}$ are respectively as

$$1,\left(2^{|(\overline{T}_3\cap\overline{T}_4)\setminus\overline{T}_2|}(2^{|\overline{T}_2\setminus\overline{T}_1|}-1)\right),\left(3^{|\overline{T}_4\setminus\overline{T}_3|}-2^{|\overline{T}_4\setminus\overline{T}_3|}\right),\left(3^{|\overline{T}_3\setminus\overline{T}_4|}-2^{|\overline{T}_3\setminus\overline{T}_4|}\right),$$
$$5^{|(\overline{T}_7\cap\overline{T}_6)\setminus\overline{T}_5|},\left(6^{|\overline{T}_7\setminus\overline{T}_6|}-5^{|\overline{T}_7\setminus\overline{T}_6|}\right),\left(6^{|\overline{T}_6\setminus\overline{T}_7|}-5^{|\overline{T}_6\setminus\overline{T}_7|}\right),8^{|X\setminus\overline{T}_8|}.$$

Now, we determine the number of regular elements

$$\left|R_\varphi(Q,D')\right|=\left(2^{|(\overline{T}_3\cap\overline{T}_4)\setminus\overline{T}_2|}(2^{|\overline{T}_2\setminus\overline{T}_1|}-1)\right)\cdot\left(3^{|\overline{T}_4\setminus\overline{T}_3|}-2^{|\overline{T}_4\setminus\overline{T}_3|}\right)$$
$$\cdot\left(3^{|\overline{T}_3\setminus\overline{T}_4|}-2^{|\overline{T}_3\setminus\overline{T}_4|}\right)\cdot 5^{|(\overline{T}_7\cap\overline{T}_6)\setminus\overline{T}_5|}\cdot\left(6^{|\overline{T}_7\setminus\overline{T}_6|}-5^{|\overline{T}_7\setminus\overline{T}_6|}\right)$$
$$\cdot\left(6^{|\overline{T}_6\setminus\overline{T}_7|}-5^{|\overline{T}_6\setminus\overline{T}_7|}\right)\cdot 8^{|X\setminus\overline{T}_8|}$$

The number of all automorphisms of the semilattice Q is $q=4$. These are

$$I_Q=\begin{pmatrix}T_1&T_2&T_3&T_4&T_5&T_6&T_7&T_8\\T_1&T_2&T_3&T_4&T_5&T_6&T_7&T_8\end{pmatrix}\quad\varphi=\begin{pmatrix}T_1&T_2&T_3&T_4&T_5&T_6&T_7&T_8\\T_1&T_2&T_3&T_4&T_5&T_7&T_6&T_8\end{pmatrix}$$
$$\theta=\begin{pmatrix}T_1&T_2&T_3&T_4&T_5&T_6&T_7&T_8\\T_1&T_2&T_4&T_3&T_5&T_7&T_6&T_8\end{pmatrix}\quad\tau=\begin{pmatrix}T_1&T_2&T_3&T_4&T_5&T_6&T_7&T_8\\T_1&T_2&T_4&T_3&T_5&T_6&T_7&T_8\end{pmatrix}.$$

Therefore by using Theorem 2.3,

$$\left|R(D')\right|=m_0\cdot 4\cdot\left(2^{|(\overline{T}_3\cap\overline{T}_4)\setminus\overline{T}_2|}(2^{|\overline{T}_2\setminus\overline{T}_1|}-1)\right)\cdot\left(3^{|\overline{T}_4\setminus\overline{T}_3|}-2^{|\overline{T}_4\setminus\overline{T}_3|}\right)$$
$$\cdot\left(3^{|\overline{T}_3\setminus\overline{T}_4|}-2^{|\overline{T}_3\setminus\overline{T}_4|}\right)\cdot 5^{|(\overline{T}_7\cap\overline{T}_6)\setminus\overline{T}_5|}\cdot\left(6^{|\overline{T}_7\setminus\overline{T}_6|}-5^{|\overline{T}_7\setminus\overline{T}_6|}\right)$$
$$\cdot\left(6^{|\overline{T}_6\setminus\overline{T}_7|}-5^{|\overline{T}_6\setminus\overline{T}_7|}\right)\cdot 8^{|X\setminus\overline{T}_8|}$$

is obtained. □

Example 1. *Let* $X=\{1,2,3,4,5,6\}$ *and*

$$D=\ \{T_1=\{1\},\ T_2=\{1,2\},\ T_3=\{1,2,3\},\ T_4=\{1,2,4\},\ T_5=\{1,2,3,4\},\ T_6=\{1,2,3,4,5\},\ T_7=\{1,2,3,4,6\},\ T_8=\{1,2,3,4,5,6\}\}.$$

D is an $X-$semilattice of unions since D is closed the union of sets. Moreover D satisfies the conditions in (3.1). Then, D is an $XI-$semilattice. Let $D = Q$. Therefore $|\Omega(Q)| = 1$. Besides, the number of all automorphisms of Q is $q = 4$. By using Theorem 3.7

$$\begin{aligned}
|R(Q)| &= 1 \cdot 4 \cdot \left(2^{\left|(\overline{T}_3 \cap \overline{T}_4) \setminus \overline{T}_2\right|}(2^{\left|\overline{T}_2 \setminus \overline{T}_1\right|} - 1)\right) \cdot \left(3^{\left|\overline{T}_4 \setminus \overline{T}_3\right|} - 2^{\left|\overline{T}_4 \setminus \overline{T}_3\right|}\right) \\
&\cdot \left(3^{\left|\overline{T}_3 \setminus \overline{T}_4\right|} - 2^{\left|\overline{T}_3 \setminus \overline{T}_4\right|}\right) \cdot 5^{\left|(\overline{T}_7 \cap \overline{T}_6) \setminus \overline{T}_5\right|} \cdot \left(6^{\left|\overline{T}_7 \setminus \overline{T}_6\right|} - 5^{\left|\overline{T}_7 \setminus \overline{T}_6\right|}\right) \\
&\cdot \left(6^{\left|\overline{T}_6 \setminus \overline{T}_7\right|} - 5^{\left|\overline{T}_6 \setminus \overline{T}_7\right|}\right) \cdot 8^{\left|X \setminus \overline{T}_8\right|} \\
&= 4
\end{aligned}$$

is obtained.

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