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ON ω-US SPACES

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Abstract: In this paper, we introduce and study ω-US topological spaces. Also, ω-convergency, sequentially ω-compact sets, sequentially ω-continuous functions, sequentially nearly ω-continuous functions, sequentially ω-compact preserving functions and sequentially sub-ω-continuous functions have been introduced.

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1. Introduction

Quite recently, as generalization of closed sets, ω-closed sets were introduced and studied by Sundaram and Sheik John [5]. This notion was further studied by Maki et. al. [2, 3]. The aim of this paper is to introduce and study ω-US spaces; ω-convergency, sequentially ω-compact sets, sequentially ω-continuous functions, sequentially nearly ω-continuous functions, sequentially ω-compact preserving functions and sequentially sub-ω-continuous functions.

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2. Preliminaries

Throughout the present paper, spaces X and Y mean topological spaces. For a subset A of a topological space X, $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$ denote the closure of A and the interior of A, respectively.

We recall the following definitions, which are used in the sequel.

Definition 1. A subset A of a space X is called semiopen [1] if $A \subset \mathrm{Cl}(\mathrm{Int}(A))$. The complement of a semiopen set is called a semiclosed set.

Definition 2. A subset A of a space X is called ω-closed [5] if $\mathrm{Cl}(A) \subset U$ whenever $A \subset U$ and U is semiopen in X. The complement of an ω-closed set is called an ω-open set.

The intersection of all ω-closed sets containing A is called the ω-closure [4] of A and is denoted by $\omega\,\mathrm{Cl}(A)$. The family of all ω-open sets of (X, τ) is denoted by $\omega O(X)$.

Definition 3. A space X is said to be ω-T_1 [3] if for each pair of distinct points x and y of X, there exist ω-open sets and U and V such that $x \in U$, $y \notin U$ and $x \notin V$, $y \in V$.

Definition 4. A space X is said to be ω-T_2 [3] if for each pair of distinct points x and y of X, there exist ω-open sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

Definition 5. A space X is said to be ω-R_1 [2] if for $x, y \in X$ with $\omega\,\mathrm{Cl}(\{x\}) \neq \omega\,\mathrm{Cl}(\{y\})$, there exist disjoint ω-open sets U and V such that $\omega\,\mathrm{Cl}(\{x\}) \subset U$ and $\omega\,\mathrm{Cl}(\{y\}) \subset V$.

Theorem 6. *[2] Let X be a topological space. Then X is ω-T_2 if and only if it is ω-R_1 and ω-T_0.*

3. On ω-US-Spaces

Definition 7. A sequence (x_n) is said to be ω-convergence to a point x of X, denoted by $(x_n) \underset{\rightarrow}{\omega} x$, if (x_n) is eventually in every ω-open set containing x.

Definition 8. A space X is said to be ω-US if every ω-convergent sequence (x_n) in X ω-converges to a unique point.

Definition 9. A set S is said to be sequentially ω-closed if every sequence in S ω-converging in X ω-converges to a point in S.

Definition 10. A subset S of a space X is said to be sequentially ω-compact if every sequence in S has a subsequence which ω-converges to a point in S.

Theorem 11. *Every ω-T_2 space is ω-US.*

Proof. Let X be an ω-T_2 space and (x_n) be a sequence in X. Suppose that (x_n) ω-converges to two distinct points x and y. That is, (x_n) is eventually in every ω-open set containing x and also in every ω-open set containing y. This is a contradiction since X is ω-T_2. This shows that the space X is ω-US. □

Theorem 12. *Every ω-US space is ω-T_1.*

Proof. Let X be an ω-US space and x and y be two distinct points of X. Consider the sequence (x_n) where $x_n \neq x$ for every n. Clearly (x_n) ω-converges to x. Also, since $x \neq y$ and X is ω-US, (x_n) cannot ω-converges to y, that is, there exists an ω-open set V containing y but not x. Similarly, if we consider the sequence (y_n) where $y_n \neq y$ for all n, and proceeding as above we get an ω-open set U containing x but not y. This shows that the space X is ω-T_1. □

Theorem 13. *In an ω-US space, every sequentially ω-compact set is sequentially ω-closed.*

Proof. Let X be an ω-US space and Y be a sequentially ω-compact subset of X. Let (x_n) be a sequence in Y. Suppose that (x_n) ω-converges to a point in $X - Y$. Let (x_{nk}) be a subsequence of (x_n) which ω-converges to a point $y \in Y$ since Y is sequentially ω-compact. Also, a subsequence (x_{nk}) of (x_n) ω-converges to $x \in X - Y$. Since (x_{nk}) is a sequence in the ω-US space X, $x = y$. This shows that Y is sequentially ω-closed set. □

Lemma 14. *Let A and Y be subsets of a space X, if $A \in \omega O(X)$ and Y is open in X, then $A \cap Y \in \omega O(Y)$.*

Theorem 15. *Every open subset of an ω-US space in ω-US.*

Proof. Let X be an ω-US space and Y be an open subset of X. Let (x_n) be a sequence in Y. Suppose that (x_n) ω-converge to x and y in Y. We shall prove that (x_n) ω-converges to x and y in X. Let U be any ω-open subset of X containing x and U be any ω-open set of X containing y. Then by Lemma 14, $U \cap Y$ and $V \cap Y$ are ω-open sets in Y. Therefore, (x_n) is eventually in $U \cap Y$ and $V \cap Y$ and so in U and V. Since X is ω-US, this implies that $x = y$ and hence the subspace Y is ω-US. □

Theorem 16. *A space X is ω-T_2 if and only if it is both ω-R_1 and ω-US.*

Proof. Let X be an ω-T_2 space. Then X is ω-R_1 by Theorem 6 and ω-US by Theorem 11. Conversely, let X be both ω-R_1 and ω-US space. By Theorem 12, we obtain that every ω-US space is ω-T_1 and X is both ω-T_1 and ω-R_1, it follows from Theorem 6 that X is ω-T_2. □

Now, we introduce the notion of sequentially ω-continuous functions.

Definition 17. A function $f : X \to Y$ is said to be

1. sequentially ω-continuous at $x \in X$ if $f(x_n)$ ω-converges to $f(x)$ whenever (x_n) is a sequence ω-converging to x;

2. sequentially ω-continuous if f is sequentially ω-continuous at all $x \in X$.

Definition 18. A function $f : X \to Y$ is said to be sequentially nearly ω-continuous if for each point $x \in X$ and each sequence (x_n) in X ω-converging to x, there exists a subsequence (x_{nk}) of (x_n) such that $f(x_{nk}) \underset{\to}{\omega} f(x)$.

Theorem 19. *Let $f : X \to Y$ and $g : Y \to Z$ be two sequentially ω-continuous functions. If Y is ω-US, then the set $A = \{x : f(x) = g(x)\}$ is sequentially ω-closed.*

Proof. Let Y be an ω-US space and suppose that there exists a sequence (x_n) in A ω-converging to $x \in X$. Since f and g are sequentially ω-continuous functions, $f(x_n) \underset{\to}{\omega} f(x)$ and $g(x_n) \underset{\to}{\omega} g(x)$. Hence $f(x) = g(x)$ and $x \in A$. Therefore, we obtain A is sequentially ω-closed. □

Definition 20. A function $f : X \to Y$ is said to be sequentially sub-ω-continuous if for each point $x \in X$ and each sequence (x_n) in X ω-converging to x, there exists a subsequence (x_{nk}) of (x_n) and a point $y \in Y$ such that $f(x_{nk}) \underset{\to}{\omega} y$.

Definition 21. A function $f : X \to Y$ is said to be sequentially ω-compact preserving if the image $f(K)$ of every sequentially ω-compact set K of X is sequentially ω-compact in Y.

Theorem 22. *Every function $f : X \to Y$ is sequentially sub-ω-continuous if Y is sequentially ω-compact.*

Proof. Let (x_n) be a sequence in X ω-converging to a point x of X. Then $(f(x_n))$ is a sequence in Y and as Y is sequentially ω-compact, there exists a subsequence $(f(x_{nk}))$ of $(f(x_n))$ ω-converging to a point $y \in Y$. This shows that $f : X \to Y$ is sequentially sub-ω-continuous. □

Theorem 23. *Every sequentially nearly ω-continuous function is sequentially ω-compact preserving.*

Proof. Suppose that $f : X \to Y$ is a sequentially nearly ω-continuous function and let A be any sequentially ω-compact set of Y. Let (y_n) be any sequence in $f(A)$. Then for each positive integer n, there exists a point $x_n \in A$ such that $f(x_n) = y_n$. Since (x_n) is a sequence in the sequentially ω-compact set A, there exists a subsequence (x_{n_k}) of (x_n) ω-converging to a point $x \in A$. Since f is sequentially nearly ω-continuous, then there exists a subsequence (x_j) of (x_{nk}) such that $f(x_j) \underset{\to}{\omega} f(x)$. Therefore, there exists a subsequence (y_j) of (y_n) ω-converging to $f(x) \in f(A)$. This shows that $f(M)$ is sequentially ω-compact set in Y. □

Theorem 24. *Every sequentially ω-compact preserving function is sequentially sub-ω-continuous.*

Proof. Suppose $f : X \to Y$ is a sequentially ω-compact preserving function. Let x be any point of X and (x_n) any sequence in X ω-converging to x. We shall denoted the set $\{x_n\colon\ n = 1,2,....\}$ by A and $B = A \cup \{x\}$. Then B is sequentially ω-compact since $x_n \underset{\to}{\omega} x$. Since f is sequentially ω-compact set preserving, it follows that $f(B)$ is a sequentially ω compact set of Y. Since $(f(x_n))$ is a sequence in $f(B)$, there exists a subsequence $(f(x_{nk}))$ of $(f(x_n))$ ω-converging to a point $y \in f(B)$. This implies that f is sequentially sub-ω-continuous. □

Theorem 25. *A function $f : X \to Y$ is sequentially ω-compact preserving if and only if $f_{|M}$: $M \to f(M)$ is sequentially sub-ω-continuous for each sequentially ω-compact subset M of X.*

Proof. Suppose that $f : X \to Y$ is a sequentially ω-compact preserving function. Then $f(M)$ is sequentially ω-compact set M of X. Therefore, by Theorem 22, $f_{|M}$: $M \to f(M)$ is sequentially sub-ω-continuous function. Conversly, let M be any sequentially ω-compact set in Y. We shall show that $f(M)$ is sequentially ω-compact set in Y. Let (y_n) be any sequence in $f(M)$. Then for each positive integer n, there exists a point $x_n \in M$ such that $f(x_n) = y_n$. Since (x_n) is a sequence in a sequentially ω-compact set M, there exists a subsequence (x_{nk}) of (x_n) ω-converging to a point $x \in M$. Since $f_{|M}$: $M \to f(M)$ is sequentially sub-ω-continuous, there exists a subseqeunce (y_{nk}) of (y_n) ω-converging to a point $y \in f(M)$. This implies that $f(M)$ is sequentially ω-compact set in Y. Thus, $f : X \to Y$ is sequentially ω-compact preserving function. □

The following theorem gives a sufficient condition for a sequentially sub-ω-continuous function to be a sequentially ω-compact preserving.

Theorem 26. *If a function $f : X \to Y$ is sequentially sub-ω-continuous and $f(M)$ is sequentially ω-closed set in Y for each sequentially ω-compact set M of X, then f is sequentially ω-compact preserving function.*

Proof. We use the previous Theorem. It suffices to prove that $f_{|M} : M \to f(M)$ is sequentially sub-ω-continuous for each sequentialy ω-compact subset M of X. Let (x_n) be any sequence in M ω-converging to a point $x \in M$. Then since f is sequentially sub-ω-continuous, there exists a subsequence (x_{nk}) of (x_n) and a point $y \in Y$ such that $f(x_{nk})$ ω-converges to y. Since $f(x_{nk})$ is a sequence in the sequentially ω-closed set $f(M)$ of Y, we obtain $y \in f(M)$. This implies that $f_{|M}$: $M \to f(M)$ is sequentially sub-ω-continuous. □

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