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T-ROUGH IDEALS IN TERNARY SEMIGROUPS

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Abstract: The purpose of this paper is to introduce and discuss the concept of T-rough ternary subsemigroups, T-rough ideals, T-rough bi-ideals and T-rough interior ideals in ternary semigroups.

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1. Introduction

The notion of a rough set was proposed by Pawlak [1] in 1982. Since then the subject has been investigated in many papers. There are many mathematicians who applied the rough set theory to algebraic structures, for instance, Biswas and Nanda [2], Xiao and Zhang [3], Petchkhaew and Chinram [4], Kuroki [5], Shabir and Rehman [6], Yaqoob et al. [7-16], Davvaz [17, 18, 19], Kazancı et al. [20] and Leoreanu [21].

The majority studies on rough sets for algebraic structures such as semigroups, groups, rings and modules have been concentrated on a congruence relation. However, the congruence relation restricts the application of the generalized rough set model for algebraic sets. To solve this problem, Davvaz [22]

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introduced the concept of set-valued homomorphism in groups. Xiao [23], studied the properties of T-roughness in semigroups. Hosseini et al. [24, 25], applied T-rough set theory to semigroups and modules.

In this paper, the concept of set-valued homomorphism of ternary semigroups is defined. We constructed the generalized rough approximation operators in ternary semigroups by means of set-valued mapping and the properties of them are studied.

2. Preliminaries and Basic Definitions

In this section, we give some basic notions and results about generalized rough sets and ternary semigroups.

Definition 2.1. [23] Let U and W be two non-empty universes. Let T be a set-valued mapping given by $T : U \to P(W)$. Then the triple (U, W, T) is referred to as a generalized approximation space or generalized rough set. Any set-valued function from U to $P(W)$ defines a binary relation from U to W by setting $R_T = \{(x, y)|y \in T(x)\}$. Obviously, if R is an arbitrary relation from U to W, then it can be defined as a set-valued mapping $T_R : U \to P(W)$ by $T_R(x) = \{y \in W|(x, y) \in R\}$ where $x \in U$. For any set $A \subseteq W$, the lower and upper approximations $\underline{Apr}_T(A)$ and $\overline{Apr}_T(A)$ are defined by

$$\begin{aligned} \underline{Apr}_T(A) &= \{x \in U|T(x) \subseteq A\} \\ \overline{Apr}_T(A) &= \{x \in U|T(x) \cap A \neq \emptyset\}. \end{aligned}$$

The pair $(\underline{Apr}_T(A), \overline{Apr}_T(A))$ is referred to as a generalized rough set, and $\underline{Apr}_T$ and $\overline{Apr}_T$ are referred to as lower and upper generalized approximation operators, respectively.

If a subset $A \subseteq W$ satisfies that $\underline{Apr}_T(A) = \overline{Apr}_T(A)$, then A is called a definable set of (U, W, T). We denote all the definable sets of (U, W, T) by Def(T).

Theorem 2.2. *[23] Let (U, W, T) be a generalized approximation space, its lower and upper approximation operators satisfy the following properties:*

For all $A, B \in P(W)$,

$(L1)$ $\underline{Apr}_T(A) = (\overline{Apr}_T(A^c))^c$,	$(U1)$ $\overline{Apr}_T(A) = (\underline{Apr}_T(A^c))^c$,
$(L2)$ $\underline{Apr}_T(W) = U$,	$(U2)$ $\overline{Apr}_T(\emptyset) = \emptyset$,
$(L3)$ $\underline{Apr}_T(A \cap B) = \underline{Apr}_T(A) \cap \underline{Apr}_T(B)$,	$(U3)$ $\overline{Apr}_T(A \cup B) = \overline{Apr}_T(A) \cup \overline{Apr}_T(B)$,
$(L4)$ $A \subseteq B \Rightarrow \underline{Apr}_T(A) \subseteq \underline{Apr}_T(B)$,	$(U4)$ $A \subseteq B \Rightarrow \overline{Apr}_T(A) \subseteq \overline{Apr}_T(B)$,
$(L5)$ $\underline{Apr}_T(A \cup B) \supseteq \underline{Apr}_T(A) \cup \underline{Apr}_T(B)$,	$(U5)$ $\overline{Apr}_T(A \cap B) \subseteq \overline{Apr}_T(A) \cap \overline{Apr}_T(B)$.

where A^c *is the complement of the set* A.

Definition 2.3. A ternary semigroup is an algebraic structure (S, f) such that S is a non-empty set and $f : S^3 \to S$ is a ternary operation satisfying the following associative law:

$$f(f(a, b, c), d, e) = f(a, f(b, c, d), e) = f(a, b, f(c, d, e)).$$

For simplicity we write $f(a, b, c)$ as abc.

Definition 2.4. A non-empty subset T of a ternary semigroup S is said to be a ternary subsemigroup of S if $TTT = T^3 \subseteq T$, that is $abc \in T$ for all $a, b, c \in T$.

Definition 2.5. By a left (right, lateral (middle)) ideal of a ternary semigroup S we mean a non-empty subset A of S such that $SSA \subseteq A$ ($ASS \subseteq A$, $SAS \subseteq A$). By a two sided ideal, we mean a subset of S which is both a left and a right ideal of S. If a non-empty subset of S is a left, right and lateral ideal of S, then it is called an ideal of S.

Definition 2.6. An ideal A of a semigroup S is a prime ideal of S such that $xyz \in A$ for some $x, y, z \in S$ implies $x \in A$ or $y \in A$ or $z \in A$.

Definition 2.7. A non-empty subset A of a ternary semigroup S is called a bi-ideal of S if $AAA \subseteq A$ and $ASASA \subseteq A$.

Definition 2.8. A bi-ideal A of a ternary semigroup S is said to be a prime bi-ideal of S, if for $x, y, z \in S$, $xaybz \in A$ implies $x \in A$ or $y \in A$ or $z \in A$ for all $a, b \in S$.

Definition 2.9. A non-empty subset A of S is called an interior ideal of S if $SSASS \subseteq A$.

Definition 2.10. A non-empty subset A of S is called a quasi-ideal of S if $ASS \cap SAS \cap SSA \subseteq A$ and $ASS \cap SSASS \cap SSA \subseteq A$.

Definition 2.11. Let ρ be a congruence on ternary semigroup S, that is, ρ is an equivalence relation on S such that

$$(a,b)\in\rho \text{ implies } (xya,xyb)\in\rho,\ (xay,xby)\in\rho \text{ and } (axy,bxy)\in\rho$$

for all $a,b,x,y\in S$.

We denote by $[a]_\rho$ the ρ-congruence class containing the element $a\in S$. It is obvious that $[a]_\rho[b]_\rho[c]_\rho\subseteq[abc]_\rho$ for all $a,b,c\in S$. A congruence ρ on S is called complete if $[a]_\rho[b]_\rho[c]_\rho=[abc]_\rho$ for all $a,b,c\in S$.

3. T-rough Ideals in Ternary Semigroups

In what follows, let S denote a ternary semigroup unless otherwise specified. In this section, we will discuss some results on T-roughness in ternary semigroups.

Definition 3.1. Let S_1 and S_2 be two ternary semigroups and $B\subseteq S_2$. Let $T:S_1\to P(S_2)$ be a set-valued mapping where $P(S_2)$ denotes the set of all non-empty subsets of S_2. The generalized lower and upper approximations of B are defined by

$$\begin{aligned}\underline{Apr}_T(B) &= \{x\in S_1 : T(x)\subseteq B\}\\ \overline{Apr}_T(B) &= \{x\in S_1 : T(x)\cap B\neq\emptyset\}.\end{aligned}$$

Definition 3.2. A set-valued homomorphism T from a ternary semigroup S_1 to a ternary semigroup S_2 is a mapping from S_1 to $P(S_2)$ that preserves the group operation, that is, $T(a)T(b)T(c)\subseteq T(abc)$ for all $a,b,c\in S_1$. T is called a strong set-valued homomorphism, if $T(a)T(b)T(c)=T(abc)$ for all $a,b,c\in S_1$.

Example 3.3. Let $S_1=\{x,y,z\}$ and $S_2=\{a,b,c,d,e\}$ be two ternary semigroups with the following multiplication tables:

	x	y	z
x	x	x	x
y	x	y	y
z	x	z	z

	a	b	c	d	e
a	d	b	b	d	b
b	b	b	b	b	b
c	b	b	c	b	c
d	d	b	b	d	b
e	b	b	c	b	c

Assume $T(x)=\{a,d\}$, $T(y)=\{b,c\}$ and $T(z)=\{e\}$. Here T is a set-valued homomorphism from S_1 to S_2. But T is not a strong set-valued homomorphism from S_1 to S_2, because

$$\{d\}=T(x)T(x)T(x)\neq T(xxx)=\{a,d\}.$$

Theorem 3.4. *Let θ be a (complete) congruence relation on a ternary semigroup S. Define $T_\theta : S \to P(S)$ by $T_\theta(x) = [x]_\theta$. Then T_θ is a (strong) set-valued homomorphism.*

Theorem 3.5. *Let S_1 and S_2 be two ternary semigroups and $T : S_1 \to P(S_2)$ be a set-valued homomorphism. If A, B, C are three non-empty subsets of S_2, then*

$$\overline{Apr}_T(A)\overline{Apr}_T(B)\overline{Apr}_T(C) \subseteq \overline{Apr}_T(ABC).$$

Proof. Assume that $t \in \overline{Apr}_T(A)\overline{Apr}_T(B)\overline{Apr}_T(C)$, there exist $x \in \overline{Apr}_T(A)$, $y \in \overline{Apr}_T(B)$ and $z \in \overline{Apr}_T(C)$ such that $t = xyz$. So there exist $a \in T(x) \cap A$, $b \in T(y) \cap B$ and $c \in T(z) \cap C$, then $abc \in ABC$. Since T is a set–valued homomorphism, we have

$$abc \in T(x)T(y)T(z) \subseteq T(xyz) = T(t).$$

So $abc \in T(t) \cap ABC$, which implies $t \in T(ABC)$. □

Theorem 3.6. *Let S_1 and S_2 be two ternary semigroups and $T : S_1 \to P(S_2)$ be a strong set-valued homomorphism. If A, B, C are three non-empty subsets of S_2, then*

$$\underline{Apr}_T(A)\underline{Apr}_T(B)\underline{Apr}_T(C) \subseteq \underline{Apr}_T(ABC).$$

Proof. Assume that $t \in \underline{Apr}_T(A)\underline{Apr}_T(B)\underline{Apr}_T(C)$, there exist $x \in \underline{Apr}_T(A)$, $y \in \underline{Apr}_T(B)$ and $z \in \underline{Apr}_T(C)$ such that $t = xyz$. Since T is a strong set–valued homomorphism, we have

$$T(t) = T(xyz) = T(x)T(y)T(z) \subseteq ABC.$$

Thus $t \in T(ABC)$. □

Let T_1, T_2 and T_3 be three set-valued mappings from S_1 to $P(S_2)$, we define

$$(T_1 \cap T_2 \cap T_3)(x) = T_1(x) \cap T_2(x) \cap T_3(x).$$

Theorem 3.7. *Let T_1, T_2 and T_3 be three set-valued mappings from S_1 to $\wp^*(S_2)$. If A is a non-empty subset of S_2, then*

$$\begin{aligned} &(1)\ \overline{Apr}_{T_1\cap T_2\cap T_3}(A) \subseteq \overline{Apr}_{T_1}(A) \cap \overline{Apr}_{T_2}(A) \cap \overline{Apr}_{T_3}(A) \\ &(2)\ \underline{Apr}_{T_1\cap T_2\cap T_3}(A) \supseteq \underline{Apr}_{T_1}(A) \cap \underline{Apr}_{T_2}(A) \cap \underline{Apr}_{T_3}(A). \end{aligned}$$

Proof. (1) Assume that $x \in \overline{Apr}_{T_1 \cap T_2 \cap T_3}(A)$, there exists $y \in (T_1 \cap T_2 \cap T_3)(x) \cap A$. So $y \in T_1(x) \cap A$, $y \in T_2(x) \cap A$, and $y \in T_3(x) \cap A$, which imply that $x \in \overline{Apr}_{T_1}(A) \cap \overline{Apr}_{T_2}(A) \cap \overline{Apr}_{T_3}(A)$.

(2) Assume that $x \in \underline{Apr}_{T_1}(A) \cap \underline{Apr}_{T_2}(A) \cap \underline{Apr}_{T_3}(A)$, we have $T_1(x) \subseteq A$, $T_2(x) \subseteq A$ and $T_3(x) \subseteq A$. So $T_1(x) \cap T_2(x) \cap T_3(x) \subseteq A$. Thus $(T_1 \cap T_2 \cap T_3)(x) \subseteq A$, which implies that $x \in \underline{Apr}_{T_1 \cap T_2 \cap T_3}(A)$. □

Theorem 3.8. *Let S_1 and S_2 be two ternary semigroups and $T : S_1 \to P(S_2)$ be a set-valued homomorphism. If A, B and C are a right ideal, lateral ideal and a left ideal of S_2, respectively. Then*

$$\begin{array}{lrcl} (1) & \overline{Apr}_T(ABC) & \subseteq & \overline{Apr}_T(A) \cap \overline{Apr}_T(B) \cap \overline{Apr}_T(C) \\ (2) & \underline{Apr}_T(ABC) & \subseteq & \underline{Apr}_T(A) \cap \underline{Apr}_T(B) \cap \underline{Apr}_T(C). \end{array}$$

Proof. (1) Since A is a right ideal of S_2, so $ABC \subseteq AS_2S_2 \subseteq A$. Since B is a lateral ideal of S_2, so $ABC \subseteq S_2BS_2 \subseteq B$, also C is a left ideal of S_2, so $ABC \subseteq S_2S_2C \subseteq C$, thus $ABC \subseteq A \cap B \cap C$. Then by Theorem 2.2(U4) and Theorem 2.2(U5), we have

$$\begin{aligned} \overline{Apr}_T(ABC) &\subseteq \overline{Apr}_T(A \cap B \cap C) \\ &\subseteq \overline{Apr}_T(A) \cap \overline{Apr}_T(B) \cap \overline{Apr}_T(C). \end{aligned}$$

(2) The proof is similar to (1). □

Theorem 3.9. *Let S_1 and S_2 be two ternary semigroups and $T : S_1 \to P(S_2)$ be a set-valued homomorphism.*

(1) If A is a ternary subsemigroup of S_2. Then $\overline{Apr}_T(A)$ is, if it is non-empty, a ternary subsemigroup of S_1.

(2) If A is a left [right, lateral] ideal of S_2 and $T(x) \neq \emptyset$ for all $x \in S$. Then $\overline{Apr}_T(A)$ is, if it is non-empty, a left [right, lateral] ideal of S_1.

Proof. (1) Since A is a ternary subsemigroup of S_2, we have $AAA \subseteq A$. By Theorem 2.2 and Theorem 3.5, we have

$$\overline{Apr}_T(A)\overline{Apr}_T(A)\overline{Apr}_T(A) \subseteq \overline{Apr}_T(AAA) \subseteq \overline{Apr}_T(A).$$

So $\overline{Apr}_T(A)$ is a ternary subsemigroup of S_1.

(2) Since A is a left ideal of S_2, we have $S_2S_2A \subseteq A$. So $\overline{Apr}_T(S_2S_2A) \subseteq \overline{Apr}_T(A)$. Since $T(x) \neq \emptyset$ for all $x \in S$, we have $\overline{Apr}_T(S_2) = S_1$. So

$$S_1S_1\overline{Apr}_T(A) \quad = \quad \overline{Apr}_T(S_2)\overline{Apr}_T(S_2)\overline{Apr}_T(A)$$

$$\subseteq \overline{Apr}_T(S_2S_2A) \subseteq \overline{Apr}_T(A).$$

This means that $\overline{Apr}_T(A)$ is, if it is non-empty, a left ideal of S. The other cases can be seen in a similar way. □

Theorem 3.10. *Let S_1 and S_2 be two ternary semigroups and $T : S_1 \rightarrow P(S_2)$ be a strong set-valued homomorphism.*

(1) If A is a ternary subsemigroup of S_2. Then $\underline{Apr}_T(A)$ is, if it is non-empty, a ternary subsemigroup of S_1.

(2) If A is a left [right, lateral] ideal of S_2. Then $\underline{Apr}_T(A)$ is, if it is non-empty a left [right, lateral] ideal of S_1.

Proof. (1) Since A be a ternary subsemigroup of S_2, we have $AAA \subseteq A$. By Theorem 2.2 and Theorem 3.6, we have

$$\underline{Apr}_T(A)\underline{Apr}_T(A)\underline{Apr}_T(A) \subseteq \underline{Apr}_T(AAA) \subseteq \underline{Apr}_T(A).$$

So $\underline{Apr}_T(A)$ is a ternary subsemigroup of S_1.

(2) Since A is a left ideal of S_2, we have $S_2S_2A \subseteq A$. So $\underline{Apr}_T(S_2S_2A) \subseteq \underline{Apr}_T(A)$, we have $\underline{Apr}_T(S_2) = S_1$. So

$$\begin{aligned} S_1S_1\underline{Apr}_T(A) &= \underline{Apr}_T(S_2)\underline{Apr}_T(S_2)\underline{Apr}_T(A) \\ &\subseteq \underline{Apr}_T(S_2S_2A) \subseteq \underline{Apr}_T(A). \end{aligned}$$

This means that $\underline{Apr}_T(A)$ is, if it is non-empty a left ideal of S_1.
The other cases can be seen in a similar way. □

Theorem 3.11. *Let S_1 and S_2 be two ternary semigroups and $T : S_1 \rightarrow P(S_2)$ be a strong set-valued homomorphism. If A is a prime ideal of S_2, then*

(1) $\overline{Apr}_T(A)$ is, if it is non-empty, a prime ideal of S_1,

(2) $\underline{Apr}_T(A)$ is, if it is non-empty, a prime ideal of S_1.

Proof. By Theorem 3.9(2) and Theorem 3.10(2), we know $\overline{Apr}_T(A)$ and $\underline{Apr}_T(A)$ are ideals of S_1.

(1) Assume that $xyz \in \overline{Apr}_T(A)$, then $T(xyz) \cap A \neq \emptyset$. Since T is a strong set-valued homomorphism, we have $T(xyz) = T(x)T(y)T(z)$. So there exist $u \in T(x)$, $v \in T(y)$ and $w \in T(z)$ such that $uvw \in A$. Since A is a prime ideal of S_2, we have $u \in A$ or $v \in A$ or $w \in A$. So $x \in \overline{Apr}_T(A)$ or $y \in \overline{Apr}_T(A)$ or $z \in \overline{Apr}_T(A)$. Therefore, $\overline{Apr}_T(A)$ is, if it is non-empty, a prime ideal of S_1.

(2) We suppose that $\underline{Apr}_T(A)$ is not a prime ideal of S_1, then there exist $x, y, z \in S_1$ such that $xyz \in \underline{Apr}_T(A)$ but $x \notin \underline{Apr}_T(A)$, $y \notin \underline{Apr}_T(A)$ and

$z \notin \underline{Apr}_T(A)$. So there exist $a \in T(x)$, $b \in T(y)$ and $c \in T(z)$ but $a, b, c \notin A$. Thus

$$abc \in T(x)T(y)T(z) = T(xyz) \subseteq A.$$

Since A is a prime ideal of S_2, we have $a \in A$ or $b \in A$ or $c \in A$. It contradicts the supposition. This means that $\underline{Apr}_T(A)$ is, if it is non-empty, a prime ideal of S_1. □

Theorem 3.12. *Let S_1 and S_2 be two ternary semigroups and $T : S_1 \to P(S_2)$ be a set-valued homomorphism. If A is a bi-ideal of S_2 and $T(x) \neq \emptyset$ for all $x \in S$, then $\overline{Apr}_T(A)$ is, if it is non-empty, a bi-ideal of S_1.*

Proof. Since A is a bi-ideal of S_2, we have $AS_2AS_2A \subseteq A$. Since $T(x) \neq \emptyset$ for all $x \in S$, we have $\overline{Apr}_T(S_2) = S_1$. By Theorem 3.5, we have

$$\begin{aligned}&\overline{Apr}_T(A)S_1\overline{Apr}_T(A)S_1\overline{Apr}_T(A)\\ &\quad = \overline{Apr}_T(A)\overline{Apr}_T(S_2)\overline{Apr}_T(A)\overline{Apr}_T(S_2)\overline{Apr}_T(A)\\ &\quad \subseteq \overline{Apr}_T(AS_2AS_2A) \subseteq \overline{Apr}_T(A).\end{aligned}$$

By this and Theorem 3.9(1), $\overline{Apr}_T(A)$ is, if it is non-empty, a bi-ideal of S_1. □

Theorem 3.13. *Let S_1 and S_2 be two ternary semigroups and $T : S_1 \to P(S_2)$ be a strong set-valued homomorphism. If A is a bi-ideal of S_2, then $\underline{Apr}_T(A)$ is, if it is non-empty, a bi-ideal of S_1.*

Proof. Since A is a bi-ideal of S_2, we have $AS_2AS_2A \subseteq A$. So $T(AS_2AS_2A) \subseteq T(A)$, we have $\overline{Apr}_T(S_2) = S_1$. By Theorem 3.6, we have

$$\begin{aligned}&\underline{Apr}_T(A)S_1\underline{Apr}_T(A)S_1\underline{Apr}_T(A)\\ &\quad = \underline{Apr}_T(A)\underline{Apr}_T(S_2)\underline{Apr}_T(A)\underline{Apr}_T(S_2)\underline{Apr}_T(A)\\ &\quad \subseteq \underline{Apr}_T(AS_2AS_2A) \subseteq \underline{Apr}_T(A).\end{aligned}$$

By this and Theorem 3.10(1), $\underline{Apr}_T(A)$ is, if it is non-empty a bi-ideal of S_1. □

Theorem 3.14. *Let S_1 and S_2 be two ternary semigroups and $T : S_1 \to P(S_2)$ be a strong set-valued homomorphism. If A is a prime bi-ideal of S_2, then*

(1) $\overline{Apr}_T(A)$ is, if it is non-empty, a prime bi-ideal of S_1,
(2) $\underline{Apr}_T(A)$ is, if it is non-empty, a prime bi-ideal of S_1.

Proof. By Theorem 3.12 and 3.13, we know $\overline{Apr}_T(A)$ and $\underline{Apr}_T(A)$ are bi-ideals of S_1.

(1) Assume that $xaybz \in \overline{Apr}_T(A)$, then $T(xaybz) \cap A \neq \emptyset$. Since T is a strong set-valued homomorphism, we have

$$T(xaybz) = T(x)T(a)T(y)T(b)T(z).$$

So there exist $s \in T(x)$, $t \in T(a)$, $u \in T(y)$, $v \in T(b)$ and $w \in T(z)$ such that $stuvw \in A$. Since A is a prime bi-ideal of S_2, we have $s \in A$ or $u \in A$ or $w \in A$. So $x \in \overline{Apr}_T(A)$ or $y \in \overline{Apr}_T(A)$ or $z \in \overline{Apr}_T(A)$. Therefore, $\overline{Apr}_T(A)$ is, if it is non-empty, a prime bi-ideal of S_1.

(2) We suppose that $\underline{Apr}_T(A)$ is not a prime bi-ideal of S_1, then there exist $x, a, y, b, z \in S_1$ such that $xaybz \in \underline{Apr}_T(A)$ but $x \notin \underline{Apr}_T(A)$, $y \notin \underline{Apr}_T(A)$ and $z \notin \underline{Apr}_T(A)$. So there exist $s \in T(x)$, $t \in T(a)$, $u \in T(y)$, $v \in T(b)$ and $w \in T(z)$ but $s, t, u, v, w \notin A$. Thus

$$stuvw \in T(x)T(a)T(y)T(b)T(z) = T(xaybz) \subseteq A.$$

Since A is a prime bi-ideal of S_2, we have $s \in A$ or $u \in A$ or $w \in A$. It contradicts the supposition. This means that $\underline{Apr}_T(A)$ is, if it is non-empty, a prime bi-ideal of S_1. □

Theorem 3.15. *Let S_1 and S_2 be two ternary semigroups and $T : S_1 \to P(S_2)$ be a set-valued homomorphism. If A is an interior ideal of S_2 and $T(x) \neq \emptyset$ for all $x \in S$, then $\overline{Apr}_T(A)$ is, if it is non-empty, an interior ideal of S_1.*

Proof. The proof is similar to the proof of Theorem 3.12. □

Theorem 3.16. *Let S_1 and S_2 be two ternary semigroups and $T : S_1 \to P(S_2)$ be a strong set-valued homomorphism. If A is an interior ideal of S_2, then $\underline{Apr}_T(A)$ is, if it is non-empty, an interior ideal of S_1.*

Proof. The proof is similar to the proof of Theorem 3.13. □

Theorem 3.17. *Let S_1 and S_2 be two ternary semigroups and $T : S_1 \to P(S_2)$ be a strong set-valued homomorphism. If A is a quasi-ideal of S_2, then $\underline{Apr}_T(A)$ is, if it is non-empty, a quasi-ideal of S_1.*

Proof. Let A be a quasi-ideal of S_2, we have $AS_2S_2 \cap S_2AS_2 \cap S_2S_2A \subseteq A$ and $AS_2S_2 \cap S_2S_2AS_2S_2 \cap S_2S_2A \subseteq A$. So $T(AS_2S_2 \cap S_2AS_2 \cap S_2S_2A) \subseteq$

$\underline{T}(A)$ and $T(AS_2S_2 \cap S_2S_2AS_2S_2 \cap S_2S_2A) \subseteq T(A)$. By Theorem 2.2, we have $\overline{Apr}_T(S_2) = S_1$. Now by Theorem 2.2 and Theorem 3.6, we get

$$\begin{aligned}
& \underline{Apr}_T(A)S_1S_1 \cap S_1\underline{Apr}_T(A)S_1 \cap S_1S_1\underline{Apr}_T(A) \\
= \; & \underline{Apr}_T(A)\underline{Apr}_T(S_2)\underline{Apr}_T(S_2) \cap \underline{Apr}_T(S_2)\underline{Apr}_T(A)\underline{Apr}_T(S_2) \\
& \cap\underline{Apr}_T(S_2)\underline{Apr}_T(S_2)\underline{Apr}_T(A) \\
\subseteq \; & \underline{Apr}_T(AS_2S_2) \cap \underline{Apr}_T(S_2AS_2) \cap \underline{Apr}_T(S_2S_2A) \\
= \; & \underline{Apr}_T(AS_2S_2 \cap S_2AS_2 \cap S_2S_2A) \\
\subseteq \; & \underline{Apr}_T(A).
\end{aligned}$$

Similarly we can prove that

$$\underline{Apr}_T(A)S_1S_1 \cap S_1S_1\underline{Apr}_T(A)S_1S_1 \cap S_1S_1\underline{Apr}_T(A) \subseteq \overline{Apr}_T(A).$$

Thus we obtain that $\underline{Apr}_T(A)$ is, if it is non-empty, a quasi-ideal of S_1. □

4. T-Rough Ideals in the Quotient Ternary Semigroups

Suppose $S_1/T = \{T(x) : x \in S_1\}$. It is clear that S_1/T is a ternary semigroup.

Definition 4.1. Let S_1 and S_2 be two ternary semigroups and $B \subseteq S_2$. Let $T : S_1 \to \wp^*(S_2)$ be a set-valued mapping where $\wp^*(S_2)$ denotes the set of all non-empty subsets of S_2. The generalized lower and upper approximations of A in the quotient ternary semigroups are defined by

$$\begin{aligned}
\underline{\underline{Apr}}_T(A) &= \{T(x) \in S_1/T : T(x) \subseteq A\} \\
\overline{\overline{Apr}}_T(A) &= \{T(x) \in S_1/T : T(x) \cap A \neq \emptyset\}.
\end{aligned}$$

Theorem 4.2. *Let S_1 and S_2 be two ternary semigroups and $T : S_1 \to P(S_2)$ be a set-valued homomorphism. If A is a ternary subsemigroup of S_2, then*

(1) $\overline{\overline{Apr}}_\rho(A)$ is a ternary subsemigroup of S_1/T.

(2) if T is strong, then $\underline{\underline{Apr}}_\rho(A)$ is, if it is non-empty, a ternary subsemigroup of S_1/T.

Proof. (1) Let $T(a), T(b), T(c) \in \overline{\overline{Apr}}_\rho(A)$. Then $T(a) \cap A \neq \emptyset$, $T(b) \cap A \neq \emptyset$ and $T(c) \cap A \neq \emptyset$. So there exist $x \in T(a) \cap A$, $y \in T(b) \cap A$ and $z \in T(c) \cap A$.

Since A is a ternary subsemigroup of S_2, by Theorem 3.9(1), $\overline{Apr}_T(A)$ is a ternary subsemigroup of S_1. We have $xyz \in A$ and

$$xyz \in T(a)T(b)T(c) \subseteq T(abc).$$

Thus $T(abc) \cap A \neq \emptyset$, which implies that $T(abc) \subseteq \overline{\overline{Apr}}_T(A)$. Hence $\overline{\overline{Apr}}_T(A)$ is a ternary subsemigroup of S_1/T.

(2) Let $T(a), T(b), T(c) \in \underline{\underline{Apr}}_\rho(A)$. Then $T(a) \subseteq A$, $T(b) \subseteq A$ and $T(c) \subseteq A$. Since A is a subsemigroup of S_2, by Theorem 3.10(1) $\underline{Apr}_T(A)$ is a ternary subsemigroup of S_1, we have

$$T(a)T(b)T(c) = T(abc) \subseteq AAA \subseteq A.$$

This means that $\underline{\underline{Apr}}_\rho(A)$ is a ternary subsemigroup of S_1/T. □

Theorem 4.3. *Let S_1 and S_2 be two ternary semigroups and $T : S_1 \to P(S_2)$ be a set-valued homomorphism. If A is a left (right, lateral) ideal of S_2, then*

(1) $\overline{\overline{Apr}}_T(A)$ is a left (right, lateral) ideal of S_1/T.

(2) if T is strong, then $\underline{\underline{Apr}}_\rho(A)$ is, if it is non-empty, a left (right, lateral) ideal of S_1/T.

Proof. (1) Let A be a left ideal of S_2. Let $T(x) \in \overline{\overline{Apr}}_\rho(A)$ and $T(y), T(z) \in S_1/T$. Then $T(x) \cap A \neq \emptyset$, hence $x \in \overline{Apr}_T(A)$. Since A is a left ideal of S_2, by Theorem 3.9(2), $\overline{Apr}_T(A)$ is a left ideal of S_1. So, we have

$$yzx \in \overline{Apr}_\rho(A).$$

Now, for every $m = yzx$, we have $T(m) \cap A \neq \emptyset$. On the other hand, from $m = yzx$, we obtain $T(m) \subseteq T(y)T(z)T(x)$. Therefore $T(y)T(z)T(x) \subseteq \overline{\overline{Apr}}_\rho(A)$. This means that $\overline{\overline{Apr}}_\rho(A)$ is a left ideal of S_1/T.

(2) Let A be a left ideal of S_2. Let $T(x) \in \overline{\overline{Apr}}_\rho(A)$ and $T(y), T(z) \in S_1/T$. Then, $T(x) \subseteq A$, which implies $x \in \underline{Apr}_T(A)$. Since A is a left ideal of S_2, by Theorem 3.10(2), $\underline{Apr}_T(A)$ is a left ideal of S_1/T. Thus, we have

$$yzx \in \underline{Apr}_\rho(A).$$

Now, for every $m = yzx$, we have $m \in \underline{Apr}_\rho(A)$, which implies that $T(m) \subseteq A$. Hence, $T(m) \in \underline{\underline{Apr}}_\rho(A)$. On the other hand, from $m = yzx$, we have $T(m) \subseteq$

$T(z)T(y)T(x)$. Therefore $T(z)T(y)T(x) \subseteq \underline{\underline{Apr}}_{\rho}(A)$. This means that $\underline{\underline{Apr}}_{\rho}(A)$ is, if it is non-empty, a left ideal of S_1/T. The other cases can be seen in a similar way. $\square$

Theorem 4.4. *Let S_1 and S_2 be two ternary semigroups and $T : S_1 \to P(S_2)$ be a set-valued homomorphism. If A is a bi-ideal of S_2, then*

(1) $\overline{\overline{Apr}}_T(A)$ is a bi-ideal of S_1/T.

(2) if T is strong, then $\underline{\underline{Apr}}_{\rho}(A)$ is, if it is non-empty, a bi-ideal of S_1/T.

Proof. The proof is similar to the proof of the Theorem 4.3. $\square$

Theorem 4.5. *Let S_1 and S_2 be two ternary semigroups and $T : S_1 \to P(S_2)$ be a set-valued homomorphism. If A is an interior ideal of S_2, then*

(1) $\overline{\overline{Apr}}_T(A)$ is an interior ideal of S_1/T.

(2) if T is strong, then $\underline{\underline{Apr}}_{\rho}(A)$ is, if it is non-empty, an interior ideal of S_1/T.

Proof. The proof is similar to the proof of the Theorem 4.3. $\square$

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