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EXACT SOLUTIONS FOR NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS AND ITS APPLICATIONS

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Abstract: In this article, we use the improved general mapping deformation method based on the generalized Jacobi elliptic functions expansion method with computerized symbolic computation to construct some new exact solutions for some nonlinear partial differential equations via the cubic nonlinear Klein - Gordon equation and the modified Kawahara equation. As a result, new generalized Jacobi elliptic function-like solutions are obtained by using this method. This method is more powerful and will be used in further works to establish more entirely new solutions for other kinds of nonlinear partial differential equations arising in mathematical physics when the balance is positive integer and the balance number is not positive integers.

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1. Introduction

The nonlinear partial differential equations play an important role to study many problems in physics and Geometry. The effort in finding exact solutions to nonlinear equations is important for the understanding of most nonlinear physical phenomena. For instance, the nonlinear wave phenomena observed in fluid dynamics, plasma and optical fibers are often modeled by the bell shaped sech solutions and the kink shaped tanh solutions. Many effective methods have been presented, such as inverse scattering transform method [1], Bäcklund transformation [2], Darboux transformation [3], Hirota bilinear method [4], variable separation approach [5], various tanh methods [6–9], homogeneous balance method [10] , similarity reductions method [11,12], $(\acute{G}/G)$ -expansion method [13,14] , the reduction mKdV equation method [15], the tri-function method [16], the projective Riccati equation method [17], the Weierstrass elliptic function method [18], the Sine- Cosine method [19,20], the Jacobi elliptic function expansion [21,22], the complex hyperbolic function method [23], the truncated Painleve expansion [24], the F-expansion method [25], the rank analysis method [26], the ansatz method [27,28], the exp-function expansion method [29], the sub- ODE. method [30], and so on. Recently, Hong et.al. [31] put a good new method to obtained the exact solutions for the general variable coefficients KdV equation by using the improve general mapping deformation method based on the generalized Jacobi elliptic functions expansion method. In this paper , we use the improve general mapping deformation method based on the generalized Jacobi elliptic functions expansion method to contract a several new families of the exact solutions for some nonlinear partial differential equations such as the cubic nonlinear Klein- Gordon equation and the modified Kawahara equation which are very important in the mathematical physics and have been paid attention by many researchers when the balance number not positive integer.

2. Description of the Improve General Mapping Deformation Method

Suppose that a non-linear partial differential equation is given by

$$F(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, ...) = 0, \tag{2.1}$$

where $u = u(x,t)$ is an unknown function, F is a polynomial in $u = u(x,t)$ and its partial derivatives, in which the highest order derivatives and non-linear terms are involved. In the following we give the main steps of a deformation method:

Step 1. The traveling wave variable

$$u(x,t) = u(\xi)\ , \qquad \xi = x - Vt, \tag{2.2}$$

where V is a constant, permits us reducing Eq. (2.1) to an ODE for $u = u(\xi)$ in the form

$$P(u, u', u'', ...) = 0. \tag{2.3}$$

where P is a polynomial of $u = u(\xi)$ and its total derivatives.

Step 2. Firstly, we assume that the solution of Eq. (2.1) has the following form:

$$u(\xi) = \sum_{i=0}^{n} A_i \phi^i(\xi) + \sum_{i=0}^{n} B_i \phi^{-i}(\xi), \tag{2.4}$$

where $\xi = x - Vt$ and$A_i, B_i (i = 0, 1, 2, \ldots, n), V$ are arbitrary constants to be determined later , while $\phi(\xi)$ satisfy the following nonlinear first order differential equation:

$$[\phi'(\xi)]^2 = \sum_{i=0}^{4} a_i \ \ \phi^i(\xi) \tag{2.5}$$

where a_i is a constant to be determined later.

Step 3. The positive integer "n" can be determined by considering the homogeneous balance between the highest order partial derivative and nonlinear terms appearing in Eq.(2.1). If "n" is not positive integer. In order to apply this method when "n" is not positive integer (fraction or negative integer), we make the following transformation:

1) When $n = \frac{p}{q}, q \neq 0$ is a fraction in lowest term, we take the transformation $u(\xi) = Y^{p/q}(\xi)$.

2) When n is negative integer, we take the transformation $u(\xi) = Y^{-n}(\xi)$, and return to determine the value of n again from the new equation. Therefore, we can get the value of n in Eq.(2.3) is positive integer.

Setp4. Substituting Eq.(2.4) into Eq.(2.3) with the conditions (2.5), we obtain polynomial in $\phi^j(\xi)[\phi'(\xi)]^s, (s = 0, 1; j = \cdots, -2, -1, 0, 1, 2, \ldots)$. Setting each coefficient of this polynomial to be zero, yields a set of algebraic equations for $A_i, B_i (i = 0, 1, 2, \ldots, n)$ and V.

Step 5. In order to obtain some new rational Jacobi elliptic solutions of Eq.(2.5), we assume that Eq. (2.5) have the following solutions:

$$\phi(\xi) = c_0 + c_1 e(\xi) + c_2 f(\xi) + c_3 g(\xi) + c_4 h(\xi) \tag{2.6}$$

where $c_i (i = 0, \dots, 4)$ are arbitrary constants to be determined later, the functions $e = e(\xi), f = f(\xi), g = g(\xi), h = h(\xi)$ are expressed as follows[32,33]:

$$\begin{aligned} e &= \frac{1}{p + q\ sn(\xi, m) + r\ cn(\xi, m) + l\ dn(\xi, m)}, \\ f &= \frac{sn(\xi, m)}{p + q\ sn(\xi, m) + r\ cn(\xi, m) + l\ dn(\xi, m)}, \\ g &= \frac{cn(\xi, m)}{p + q\ sn(\xi, m) + r\ cn(\xi, m) + l\ dn(\xi, m)}, \\ h &= \frac{dn(\xi, m)}{p + q\ sn(\xi, m) + r\ cn(\xi, m) + l\ dn(\xi, m)}, \end{aligned} \tag{2.7}$$

where p, q, r, l are arbitrary constants and $sn(\xi, m), cn(\xi, m), dn(\xi, m)$ are the Jacobi elliptic functions with the modulus m.

Step 6. Substituting Eqs.(2.6) and (2.7) into Eq.(2.5). Cleaning the denominator and collecting all terms with the same degree of $sn(\xi, m), cn(\xi, m)$, $dn(\xi, m)$ together, the left hand side of Eq. (2.5) is converted into a polynomial in $sn(\xi, m), cn(\xi, m), dn(\xi, m)$. Setting each coefficients $sn(\xi, m), cn(\xi, m)$, $dn(\xi, m))$ of these polynomials to be zero, we derive a set of algebraic equations for $c_i (i = 0, \dots, 4)$,p, q, r and l .

Step 7. With the help of symbolic software package as Maple or Mathematica, we solve the system of algebraic equation which obtained in step 4 with the system obtained in step 6 to calculate for $A_i, B_i (i = 0, 1, 2, \dots, n)$,V and $c_i (i = 0, \dots, 4), p, q, r, l$.

Step 8. Substituting the results in Step 7 into Eqs.(2.4),(2.6) and (2.7), we will construct many new exact solutions for the nonlinear partial differential equation (2.1).

3. Application of the Improved General Mapping Deformation Method

In this section, we apply the improved general mapping deformation method construct the Jacobi elliptic traveling wave solutions for some nonlinear partial differential equations, namely the nonlinear Klein- Gordon differential equations, the modified Kawahara differential equation which are very important in the mathematical physics and have been paid attention by many researchers.

3.1. Example 1. The Improved General Mapping Deformation Method to the Cubic Nonlinear Klein-Gordon Equation

In this subsection, we study the cubic nonlinear Klein Gordon equation [34] in the following form:

$$u_{tt} - k^2 u_{xx} + \alpha u - \beta u^3 + \gamma u^5 = 0, \tag{3.1}$$

where k, α, β, γ are arbitrary constant. These equations play an important role in many scientific applications, such as the solid state physics, the nonlinear optics, the quantum field theory and so on (see [19,34,35]). Wazwaz [19,35] investigated the nonlinear Klein - Gordon equations and found many types of exact traveling wave solutions including compact solutions, soliton solution, solitary patterns solutions and periodic solutions using the tanh- function method . We use the proposed method to solve the cubic nonlinear Klein- Gordon equation. The traveling wave variable (2.2) permits us converting equation (3.1) into the following ODE:

$$(V^2 - k^2)u'' + \alpha u - \beta u^3 + \gamma u^5 = 0. \tag{3.2}$$

Considering the homogeneous balance between the highest order derivative u'' and the nonlinear term u^5 in (3.2), we get $m = 1/2$. According to step 3, we use the transformation

$$u = [\psi(\xi)]^{1/2} \tag{3.3}$$

where $\psi(\xi)$ is a new function of ξ. Substituting (3.3) into (3.2), we get the new ODE:

$$-\frac{1}{4}(V^2 - k^2)[\psi']^2 + \frac{1}{2}(V^2 - k^2)\psi\psi'' + \alpha\psi - \beta\psi^3 + \gamma\psi^5 = 0. \tag{3.4}$$

Determining the balance number m of the new Eq. (3.4), we get $m = 1$. Consequently, we have the formal solution of Eq.(3.4) in the form:

$$u(\xi) = A_0 + A_1\phi(\xi) + \frac{B_1}{\phi(\xi)}, \tag{3.5}$$

where

$$[\phi'(\xi)]^2 = \sum_{i=0}^{4} a_i \ \ \phi^i(\xi) \tag{3.6}$$

Substituting Eq.(3.5) along with the conditions(3.6) into Eq.(3.4) and collecting all terms with the same power of $\phi^j(\xi)[\phi'(\xi)]^s, (s = 0, 1; j = \cdots, -2, -1, 0, 1, 2, \ldots)$. Setting each coefficients of this polynomial to be zero, we get a system of the algebraic equations for $A_0, A_1, B_1, a_0, a_1, a_2, a_3, a_4$ and V. Also we substitute Eqs.(2.6) and (2.7) into Eq.(3.6). Cleaning the denominator and collecting all terms with the same degree of $sn(\xi, m)$, $cn(\xi, m)$, $dn(\xi, m)$ together, the left hand side of Eq. (3.6) is converted into a polynomial in $sn(\xi, m)$, $cn(\xi, m)$, $dn(\xi, m)$. Setting each coefficients $sn(\xi, m)$, $cn(\xi, m)$, $dn(\xi, m)$ of these polynomials to be zero, we derive a system algebraic equations for $c_i (i = 0, \ldots, 4), p, q, r$ and l . With the help of Maple, we solve the system of the algebraic equations for $A_0, A_1, B_1, a_0, a_1, a_2, a_3, a_4$ and V with the system algebraic equations for $c_i (i = 0, \ldots, 4), p, q, r$ and l . We get the following results :

Case 1:

$$\begin{aligned}
A_0 &= \frac{8\alpha(m^2-1)}{\beta(5m^2-4)}, \quad A_1 = \frac{-4\alpha r(1-m^2)^{3/2}}{c_2\beta(5m^2-4)}, \quad B_1 = \frac{4c_2\alpha\sqrt{1-m^2}}{\beta r(5m^2-4)}, \\
\gamma &= \frac{3\beta^2(-4+5m^2)}{64\alpha(m^2-1)}, \quad a_0 = \frac{c_2^2}{4r^2}, \quad a_2 = \frac{1}{2}(1+m^2), \\
a_4 &= \frac{r^2(m^2-1)^2}{4c_2^2}, \quad V = \sqrt{\frac{4\alpha+4k^2-5k^2m^2}{4-5m^2}}, \quad l = -r \ , \\
a_1 &= a_3 = p = q = c_0 = c_1 = c_3 = c_4 = 0,
\end{aligned} \tag{3.7}$$

where α, β, c_2 and r are arbitrary constants. In this case the rational Jacobi elliptic solution has the following form:

$$\psi = \frac{8\alpha(m^2-1)}{\beta(5m^2-4)} - \frac{4\alpha(1-m^2)^{3/2} sn(\xi,m)}{\beta(5m^2-4)[cn(\xi,m)-dn(\xi,m)]} + \frac{4\alpha\sqrt{1-m^2}[cn(\xi,m)-dn(\xi,m)]}{\beta(5m^2-4)sn(\xi,m)}. \tag{3.8}$$

Consequently the exact solution of the cubic nonlinear Klein Gordon equation (3.1) takes the form

$$u(x,t) = \left[\frac{8\alpha(m^2-1)}{\beta(5m^2-4)} - \frac{4\alpha(1-m^2)^{3/2} sn(\xi,m)}{\beta(5m^2-4)[cn(\xi,m)-dn(\xi,m)]} + \frac{4\alpha\sqrt{(1-m^2}[cn(\xi,m)-dn(\xi,m)]}{\beta(5m^2-4)sn(\xi,m)}\right]^{1/2} \tag{3.9}$$

where

$$\xi = x - t\sqrt{\frac{4\alpha + 4k^2 - 5k^2m^2}{4 - 5m^2}}. \tag{3.10}$$

Case 2:

$$\begin{aligned}
A_0 &= \frac{2(m^2-1)(V^2-k^2)}{\beta}, A_1 = -\frac{p(m^2-1)(V^2-k^2)}{\beta c_3},\\
B_1 &= \frac{c_3(m^2-1)(V^2-k^2)}{p\beta},\\
\gamma &= \frac{3\beta^2}{16(m^2-1)(V^2-k^2)}, \quad a_0 = \frac{-(m^2-1)c_3^2}{4p^2},\\
a_2 &= \frac{1}{2}(1+m^2),\ q = -p,\\
a_4 &= -\frac{p^2(m^2-1)}{4c_3^2}, \alpha = \frac{1}{4}(4m^2-5)(V^2-k^2),\\
a_1 &= a_3 = l = r = c_0 = c_1 = c_2 = c_4 = 0,
\end{aligned} \tag{3.11}$$

where V, β, c_3 and p are arbitrary constants. In this case the rational Jacobi elliptic solution has the following form:

$$\psi = \frac{2(m^2-1)(V^2-k^2)}{\beta} - \frac{(m^2-1)(V^2-k^2)cn(\xi,m)}{\beta[1-sn(\xi,m)]} + \frac{(m^2-1)(V^2-k^2)[1-sn(\xi,m)]}{\beta cn(\xi,m)}. \tag{3.12}$$

Consequently the exact solution of the cubic nonlinear Klein Gordon equation (3.1) takes the form

$$u(x,t) = \left[\frac{2(m^2-1)(V^2-k^2)}{\beta} - \frac{(m^2-1)(V^2-k^2)cn(\xi,m)}{\beta[1-sn(\xi,m)]} + \frac{(m^2-1)(V^2-k^2)[1-sn(\xi,m)]}{\beta cn(\xi,m)}\right]^{1/2} \tag{3.13}$$

where

$$\xi = x - Vt$$

Case 3:

$$A_0 = \frac{-2(V^2-k^2)}{\beta}, \quad A_1 = \frac{q(V^2-k^2)}{\beta\sqrt{m^2c_1^2 - m^2c_3^2 + c_3^2}},$$

$$
\begin{aligned}
B_1 &= \frac{(V^2-k^2)}{\beta q}\sqrt{m^2c_1^2-m^2c_3^2+c_3^2},\\
\alpha &= \frac{1}{4}(m^2-5)(V^2-k^2), \quad \gamma=\frac{-3\beta^2}{16(V^2-k^2)},\\
a_0 &= \frac{m^2c_1^2-m^2c_3^2+c_3^2}{4q^2}, a_2=\frac{1}{2}-m^2,\\
a_4 &= \frac{q^2}{4(m^2c_1^2-m^2c_3^2+c_3^2)},\ l=q\sqrt{\frac{(c_3^2-c_1^2)}{m^2c_1^2-m^2c_3^2+c_3^2}},\\
a_1 &= a_3=p=r=c_0=c_2=c_4=0, \qquad (3.14)
\end{aligned}
$$

where V, β, c_1, c_3 and q are arbitrary constants. In this case the rational Jacobi elliptic solution has the following form:

$$
\psi=\frac{-2(V^2-k^2)}{\beta}+\frac{(V^2-k^2)(c_1+c_3cn(\xi,m))}{\beta\sqrt{m^2c_1^2-m^2c_3^2+c_3^2}[sn(\xi,m)+\sqrt{\frac{(c_3^2-c_1^2)}{m^2c_1^2-m^2c_3^2+c_3^2}}dn(\xi,m)]}
$$

$$
+\frac{(V^2-k^2)\sqrt{m^2c_1^2-m^2c_3^2+c_3^2}}{\beta(c_1+c_3cn(\xi,m))}[sn(\xi,m)+\sqrt{\frac{(c_3^2-c_1^2)}{m^2c_1^2-m^2c_3^2+c_3^2}}dn(\xi,m)]
$$

Consequently the exact solution of the cubic nonlinear Klein Gordon equation (3.1) takes the form

$$
\begin{aligned}
u(x,t)=\Bigg\{&\frac{-2(V^2-k^2)}{\beta}\\
&+\frac{(V^2-k^2)(c_1+c_3cn(\xi,m))}{\beta\sqrt{m^2c_1^2-m^2c_3^2+c_3^2}[sn(\xi,m)+\sqrt{\frac{(c_3^2-c_1^2)}{m^2c_1^2-m^2c_3^2+c_3^2}}dn(\xi,m)]}\\
&+\frac{(V^2-k^2)\sqrt{m^2c_1^2-m^2c_3^2+c_3^2}}{\beta(c_1+c_3cn(\xi,m))}[sn(\xi,m)\\
&+\sqrt{\frac{(c_3^2-c_1^2)}{m^2c_1^2-m^2c_3^2+c_3^2}}dn(\xi,m)]\Bigg\}^{1/2}, \qquad (3.15)
\end{aligned}
$$

where $\xi = x - Vt$.

3.2. Example 2. The Modified Kawahara Equation

We start with the modified Kawahara equation [36,37] in the form:

$$
u_t+u_x+u^2u_x+\beta\ u_{xxx}+\alpha\ u_{xxxxx}=0, \qquad (3.16)
$$

where α and β are arbitrary constants. This equation has been derived by Kawahara [36] as a model for water waves in the long- wave regime for moderate values of surface tension. The Kawahara equation (3.17) gives an appropriate description of several phenomena observed in the dynamics of the water- wave problem.

The traveling wave variable (2.2) permits us converting equation (3.17) into the following ODE:

$$3(1-V)u + u^3 + 3\beta u'' + 3\alpha u^{(4)} + 3C = 0, \tag{3.17}$$

where C is the integration constant. Considering the homogeneous balance between the highest order derivative $u^{(4)}$ and the nonlinear term u^3 in (3.18), we get $M = 2$. Consequently, we have the formal solution of Eq.(3.18) in the form:

$$u(\xi) = A_0 + A_1\phi(\xi) + A_2\ \phi^2(\xi) + \frac{B_1}{\phi(\xi)} + \frac{B_2}{\phi^2(\xi)}, \tag{3.18}$$

where

$$[\phi'(\xi)]^2 = \sum_{i=0}^{4} a_i\ \ \phi^i(\xi) \tag{3.19}$$

Substituting Eq.(3.19) along with the conditions(3.20) into Eq.(3.18) and collecting all terms with the same power of $\phi^j(\xi)[\phi'(\xi)]^s, (s = 0, 1; j = \cdots, -2, -1, 0, 1, 2, \ldots)$. Setting each coefficients of this polynomial to be zero, we get a system of the algebraic equations for $A_0, A_1, A_2, B_1, B_2, a_0, a_1, a_2, a_3, a_4$ and V. Also we substitute Eqs.(2.6) and (2.7) into Eq.(3.20). Cleaning the denominator and collecting all terms with the same degree of $sn(\xi, m)$, $cn(\xi, m)$, $dn(\xi, m)$ together, the left hand side of Eq. (3.20) is converted into a polynomial in $sn(\xi, m)$, $cn(\xi, m)$, $dn(\xi, m)$. Setting each coefficients $sn(\xi, m)$, $cn(\xi, m)$, $dn(\xi, m)$ of these polynomials to be zero, we derive a system algebraic equations for $c_i(i = 0, \ldots, 4), p, q, r$ and l . With the help of Maple, we solve the system of the algebraic equations for $A_0, A_1, A_2, B_1, B_2, a_0, a_1, a_2, a_3, a_4$ and V with the system algebraic equations for $c_i(i = 0, \ldots, 4), p, q, r$ and l . We get the following results :

Case 1:

$$\begin{aligned} A_0 &= \frac{\pm[10\alpha(m^2+1)+\beta]}{\sqrt{-10\alpha}},\ A_2 = \pm\frac{3(m^2-1)p^2}{c_3^2}\sqrt{\frac{-5\alpha}{2}}, \\ V &= \frac{-1}{10\alpha}[15m^4\alpha^2 + 210m^2\alpha^2 + 15\alpha^2 + \beta^2 - 10\alpha], \end{aligned}$$

$$
\begin{aligned}
C &= \frac{\pm 1}{75\alpha^2}\sqrt{\frac{-5\alpha}{2}}[1650m^2\alpha^3(m^2+1) - 15\alpha^2\beta(m^4+14m^2+1) \\
&\quad - 50\alpha^3(m^6+1) + \beta^3] \qquad (3.20)\\
a_0 &= \frac{-c_3^2(m^2-1)}{4p^2},\ a_2 = \frac{1}{2}(1+m^2),\ a_4 = \frac{-p^2(m^2-1)}{4c_3^2},\ q = \pm p \qquad (3.21)\\
A_1 &= B_1 = B_2 = a_1 = a_3 = L = r = c_0 = c_1 = c_2 = c_4 = 0
\end{aligned}
$$

where p, β and c_3 are arbitrary constants and $\alpha < 0$. In this case the rational Jacobi elliptic solution has the following form:

$$
u = \pm\frac{[10\alpha(m^2+1)+\beta]}{\sqrt{-10\alpha}} \pm 3\sqrt{\frac{-5\alpha}{2}}\frac{(m^2-1)cn^2(\xi,m)}{[1\pm sn(\xi,m)]^2}, \qquad (3.22)
$$

where

$$
\xi = x + \frac{t}{10\alpha}[15m^4\alpha^2 + 210m^2\alpha^2 + 15\alpha^2 + \beta^2 - 10\alpha]. \qquad (3.23)
$$

Case 2:

$$
\begin{aligned}
A_0 &= \frac{\pm[10\alpha(m^2+1)+\beta]}{\sqrt{-10\alpha}},\ A_2 = \frac{\pm 3(m^2-1)p^2}{c_3^2}\sqrt{\frac{-5\alpha}{2}},\\
B_2 &= \frac{\pm 3(m^2-1)c_3^2}{p^2}\sqrt{\frac{-5\alpha}{2}},\\
V &= \frac{-1}{10\alpha}[240\alpha^2(m^4-m^2+1)+\beta^2-10\alpha], a_0 = \frac{-c_3^2(m^2-1)}{4p^2},\\
a_2 &= \frac{1}{2}(1+m^2),\\
a_4 &= \frac{-p^2(m^2-1)}{4c_3^2},\ q = \pm p,\\
C &= \frac{\pm 1}{75\alpha^2}\sqrt{\frac{-5\alpha}{2}}[4800m^2\alpha^3(m^2+1) - 3200\alpha^3(m^6+1)\\
&\quad - 240\alpha^2\beta(m^4-m^2+1)+\beta^3],\\
A_1 &= B_1 = a_1 = a_3 = l = r = c_0 = c_1 = c_2 = c_4 = 0. \qquad (3.24)
\end{aligned}
$$

where p, β, c_3 are arbitrary constants and $\alpha < 0$. In this case the rational Jacobi elliptic solution has the following form:

$$
u = \frac{\pm[10\alpha(m^2+1)+\beta]}{\sqrt{-10\alpha}} \pm \frac{3(m^2-1)cn^2(\xi,m)}{[1\pm sn(\xi,m)]^2}\sqrt{\frac{-5\alpha}{2}}
$$

$$\pm 3(m^2-1)\sqrt{\frac{-5\alpha}{2}}\frac{[1\pm sn(\xi,m)]^2}{cn^2(\xi,m)}, \quad (3.25)$$

where

$$\xi = x + \frac{t}{10\alpha}[240\alpha^2(m^4-m^2+1)+\beta^2-10\alpha]. \quad (3.26)$$

Case 3:

$$\begin{aligned}
A_0 &= \frac{\pm[-20\alpha m^2+10\alpha+\beta]}{\sqrt{-10\alpha}}, A_2 = \pm\frac{3q^2\sqrt{-5\alpha}}{\sqrt{2}[c_3^2(m^2-1)-m^2c_1^2]},\\
V &= \frac{-1}{10\alpha}[240\alpha^2m^2(m^2-1)+15\alpha^2+\beta^2-10\alpha],\ l=\pm q\sqrt{\frac{c_3^2-c_1^2}{c_1^2m^2-(m^2-1)c_3^2}}\\
C &= \frac{\pm 1}{75\alpha^2}\sqrt{\frac{-2}{5\alpha}}[\beta^3-50\alpha^3-15\alpha^2\beta-1500m^2\alpha^3+4800m^4\alpha^3\\
&\quad -3200\alpha^3m^6-240\alpha^2\beta m^4+240\alpha^2m^2\beta],\\
a_0 &= \frac{c_1^2m^2-(m^2-1)c_3^2}{4q^2}, a_2=\frac{1}{2}-m^2, a_4=\frac{q^2}{4[c_1^2m^2-(m^2-1)c_3^2]}, \quad (3.27)\\
A_1 &= B_1=B_2=a_1=a_3=p=r=c_0=c_1=c_2=c_4=0,
\end{aligned}$$

where p, β, c_1, c_3 are arbitrary constants and $\alpha < 0$. In this case the rational Jacobi elliptic solution has the following form:

$$\begin{aligned}
u = &\frac{\pm[-20\alpha m^2+10\alpha+\beta]}{\sqrt{-10\alpha}}\\
&\pm\frac{3\sqrt{-5\alpha}\{c_1+c_3cn(\xi,m)\}^2}{\sqrt{2}[c_3^2(m^2-1)-m^2c_1^2]\{sn(\xi,m)+\sqrt{\frac{c_3^2-c_1^2}{c_1^2m^2-(m^2-1)c_3^2}}dn(\xi,m)\}^2}, \quad (3.28)
\end{aligned}$$

where

$$\xi = x+\frac{t}{10\alpha}[240\alpha^2m^2(m^2-1)+15\alpha^2+\beta^2-10\alpha].$$

Case 4:

$$\begin{aligned}
A_0 &= \pm\frac{[-20\alpha m^4+10\alpha+\beta]}{\sqrt{-10\alpha}},\ A_2=\pm\frac{3q^2\sqrt{-5\alpha}}{\sqrt{2}[c_3^2(m^2-1)-m^2c_1^2]},\\
l &= \pm q\sqrt{\frac{c_3^2-c_1^2}{c_1^2m^2-(m^2-1)c_3^2}},
\end{aligned}$$

$$\begin{aligned}
V &= \frac{-1}{10\alpha}[240\alpha^2(m^4-m^2+1)+\beta^2-10\alpha],\\
B_2 &= \pm\frac{3\sqrt{-5\alpha}}{\sqrt{2}q^2}[c_3^2(m^2-1)-m^2c_1^2],\\
C &= \frac{\pm1}{75\alpha^2}\sqrt{\frac{-5\alpha}{2}}[4800\alpha^3m^2(m^2+1)-3200\alpha^3(m^6+1)\\
&\quad -240\alpha^2\beta(m^4-m^2+1)],\\
a_0 &= \frac{c_1^2m^2-(m^2-1)c_3^2}{4q^2},\ a_2=\frac{1}{2}-m^2,\\
a_4 &= \frac{q^2}{4[c_1^2m^2-(m^2-1)c_3^2]},\\
A_1 &= B_1=a_1=a_3=p=r=c_0=c_2=c_4=0,
\end{aligned} \tag{3.29}$$

where p,β,c_1,c_3 are arbitrary constants and $\alpha<0$. In this case the rational Jacobi elliptic solution has the following form:

$$\begin{aligned}
u=&\pm\frac{[-20\alpha m^4+10\alpha+\beta]}{\sqrt{-10\alpha}}\\
&\pm\frac{3\sqrt{-5\alpha}[c_1+c_3cn(\xi,m)]^2}{\sqrt{2}[c_3^2(m^2-1)-m^2c_1^2][sn(\xi,m)\pm\sqrt{\frac{c_3^2-c_1^2}{c_1^2m^2-(m^2-1)c_3^2}}dn(\xi,m)]^2}\pm\\
&\frac{3\sqrt{-5\alpha}[c_3^2(m^2-1)-m^2c_1^2]}{\sqrt{2}[c_1+c_3cn(\xi,m)]^2}[sn(\xi,m)\pm\sqrt{\frac{c_3^2-c_1^2}{c_1^2m^2-(m^2-1)c_3^2}}dn(\xi,m)]^2
\end{aligned} \tag{3.30}$$

where

$$\xi=x+\frac{t}{10\alpha}[240\alpha^2(m^4-m^2+1)+\beta^2-10\alpha]$$

4. Conclusions

In this paper the improved general mapping deformation method based on the generalized Jacobi elliptic functions expansion method with computerized symbolic computation used to construct the exact solutions for some nonlinear partial differential equations in the mathematical physics when the balance number are positive integer or not positive integer. This method allowed us for calculating many new exact solutions for nonlinear partial differential equations in mathematical physics. The Jacobi elliptic solutions which obtained by this method is the generalization to the hyperbolic function solutions and trigonometric function solutions when the modulus $m \to 1$ and $m \to 0$ respectively. This method is reliable , concise and gives more exact solutions compared to the other methods.

References

[1] Ablowitz MJ, Clarkson PA (1991). Soliton Nonlinear Evolution Equations and Inverse Scattering. Cambridge University Press.

[2] Gu CH (1990). Soliton Theory and Its Application, Zhejiang Science and Technology Press.

[3] Matveev VB, Salle MA (1991). Darboux Transformation and Soliton. Springer Berlin.

[4] Hirota R(2004). The Direct Method in Soliton Theory. Cambridge University Press.

[5] Lou SY, Lu JZ (1996). Special solutions from variable separation approach: Davey-Stewartson equation. J. Phys. A. Math. 29:4209-4215.

[6] Parkes EJ, Duffy BR (1997). Travelling solitary wave solutions to a compound KdV-Burgers equation. Phys. Lett. A. 229:217-220.

[7] Fan E (2000). Extended tanh-function method and its applications to nonlinear equations. Phys. Lett. A. 277:212-218.

[8] Yan ZY(2001). New explicit travelling wave solutions for two new integrable coupled nonlinear evolution equations. Phys. Lett. A. 292:100-106.

[9] Chen Y, Zheng Y (2003). Generalized extended tanh-function method to construct new explicit exact solutions for the approximate equations for long water waves. Int. J. Mod. Phys. C. 14(4):601-611.

[10] Wang ML (1996). Application of a homogeneous balance method to exact solutions of nonlinear equations in mathematical physics. Phys. Lett. A. 216 : 67-75.

[11] Bluman GW, Kumei S (1989). Symmetries and Differential Equations. Springer-Verlag. New York.

[12] Olver PJ (1986). Applications of Lie Groups to Differential Equations. Springer-Verlag. New York.

[13] Zayed EME, Gepreel KA (2009). The ($\acute{G}$/G)- expansion method for finding traveling wave solutions of nonlinear PDEs in mathematical physics. J. Math. Phys. 50:013502-013514.

[14] Gepreel KA(2011). Exact Solutions for Nonlinear PDEs with the Variable Coefficients in Mathematical Physics. J. Information and Computing Science. 6(1):3-14.

[15] Yan ZY (2003). A reduction mKdV method with symbolic computation to constract new doubly- periodic solutions for nonlinear wave equations. Int. J. Mod. Phys. C. 14:661-672.

[16] Yan ZY (2008). The new tri-function method to multiple exact solutions of nonlinear wave equations, Phys. Scr. 78(3):035001.

[17] Lu DC, Hong BJ (2008). New exact solutions for the (2+1)-dimensional Generalized Broer-Kaup system. Appl. Math. Comput. 199:572-580.

[18] Porubov AV (1996). Periodical solution to the nonlinear dissipative equation for surface waves in a convecting liquid. Phys. Lett. A. 221:391-394.

[19] Wazwaz M (2005). The tanh and sine- cosine method for compact and noncompact solutions of nonlinear Klein Gordon equation. Appl. Math. Comput. 167 :1179-1195.

[20] Yan ZY, Zhang HQ (2001). New explicit solitary wave solutions and periodic wave solutions for Whitham–Broer–Kaup equation in shallow water. Phys. Lett. A. 285 :355-362.

[21] Lu DA(2005). Jacobi elliptic functions solutions for two variant Boussinesq equations. Chaos, Solitons and Fractals. 24:1373-1385.

[22] Yan ZY (2003). Abundant families of Jacobi elliptic functions of the (2+1) dimensional integrable Davey- Stawartson-type equation via a new method. Chaos, Solitons and Fractals. 18:299-309.

[23] Bai CL, Zhao H. Generalized method to construct the solitonic solutions to (3+1)- dimensional nonlinear equation. Phys. Lett. A. 354:428-436.

[24] Cariello F, Tabor M (1991), Similarity reductions from extended Painleve expansions for nonintegrable evolution equations. Phys. D. 53:59-70.

[25] Wang M, Li X (2005). Extended F-expansion and periodic wave solutions for the generalized Zakharov equations. Phys. Lett. A. 343:48-54.

[26] Feng X(2000). Exploratory approach to explicit solution of nonlinear evolution equations. Int. J. Theor. Phys. 39(1): 207-222.

[27] Hu JL (2001). Explicit solutions to three nonlinear physical models. Phys. Lett. A. 287:81-89.

[28] Hu JL (2001). A new method for finding exact traveling wave solutions to nonlinear partial differential equations. Phys. Lett. A. 286:175-179.

[29] He JH, Wu XH (2006). Exp-function method for nonlinear wave equations. Chaos, Solitons and Fractals. 30:700-708.

[30] Li XZ, Wang ML (2007). A sub-ODE method for finding exact solutions of a generalized KdV- mKdV equation with higher order nonlinear terms. Phys. Lett. A. 361:115-118.

[31] Hong BJ, Lu D (2012). New Jacobi elliptic function-like solutions for the general KdV equation with variable coefficients. Math and Computer Modelling 55:1594-1600.

[32] Hong BJ (2009). New Jacobi elliptic functions solutions for the variable-coefficient mKdV equation. Appl. Math. Comput. 215(8):2908-2913.

[33] Hong BJ (2010). New Jacobi elliptic functions solutions for the generalized coupled Hirota-Satsuma KdV system. Appl. Math. Comput. 217(2):472-479.

[34] Zayed E.M, Gepreel K A.The modified ($G'/$G)- expansion method and its applications to construct exact solutions for nonlinear PDEs,WSEAS TRANSACTIONS on MATHEMATICS, 10 (2011) 270-278

[35] Wazwaz A M. Compacton soliton and periodic solution for some forms of nonlinear Klein Gordon equations,Chaos, Solitons and Fractals, 28 (2006), pp.1005–1013.

[36] Wazwaz A M. New solitary solution to the modified Kawahara equation. Phys Letters A, 2007, 360: 588-592.