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ON WEAKLY ω-CONTINUOUS FUNCTIONS

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Abstract: In this paper, we introduce a new class of functions called weakly ω-continuous functions and investigate some of their fundamental properties.

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1. Introduction

Recent progress in the study of characterizations and generalizations of continuity, compactness, connectedness, separation axioms etc. has been done by means of several generalized closed sets. The notion of generalized closed sets has been studied extensively in recent years by many topologists [see [7], [4]] because generalized closed sets are only natural generalization of closed sets. More importantly, they also suggest several new properties of topological spaces. As generalization of closed sets, ω-closed sets were introduced and studied by Sundaram and Sheik John [7]. In this paper, we introduce a new class of functions called weakly ω-continuous functions and investigate some of their fundamental properties.

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2. Preliminaries

Throughout this paper, spaces always means topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a space (X, τ), $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$ denote the closure of A and the interior of A in X, respectively. A subset A of X is said to be semiopen [2] if $A \subset \mathrm{Cl}(\mathrm{Int}(A))$. The complement of a semiopen set is called a semiclosed set [1].

Definition 1. [6] Let (X, τ) be a topological space. A subset A of (X, τ) is said to be *ω-closed* in (X, τ) if $Cl(A) \subset U$ whenever $A \subset U$ and U is semiopen in X. A subset B of (X, τ) is said to be *ω-open* if $X \setminus B$ is ω-closed.

The family of all ω-open (resp. ω-closed) sets of (X, τ) is denoted by $\omega(X, \tau)$ (resp. $\omega C(X, \tau)$) and sometimes they are denoted by $\omega(\tau)$ (resp. $\omega C(\tau)$). We set $\omega(X, x) = \{V \in \omega(\tau) | x \in V\}$ for $x \in X$. Note that the family of ω-open subsets of (X, τ) forms a topology.

Definition 2. The union (resp. intersection) of all ω-open (resp. ω-closed) sets each contained in (resp. containing) a set A in a space X is called the ω-interior (resp. ω-closure) of A and is denoted by $\omega\,\mathrm{Int}(A)$ (resp. $\omega\,\mathrm{Cl}(A)$) [7].

Lemma 3. *[6] Let A be a subset of a topological space (X, τ). Then $x \in \omega\,\mathrm{Cl}(A)$ if and only if $U \cap A \neq \emptyset$ for every $U \in \omega O(X, x)$.*

Definition 4. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be ω-continuous [7] (resp. ω-irresolute [6]) if $f^{-1}(V) \in \omega(\tau)$ for every open set V of Y (resp. $V \in \omega(\sigma)$).

Definition 5. A topological space (X, τ) is said to be ω-regular [6] if for each closed set F and each $x \notin F$, there exist disjoint ω-open sets U and V such that $x \in U$ and $F \subset V$.

Lemma 6. *For a topological space (X, τ), the following are equivalent:*

1. *X is ω-regular;*
2. *for each open set U and each $x \in U$, there exists $V \in \omega(\tau)$ such that $x \in V \subset \omega\,\mathrm{Cl}(V) \subset U$.*

3. Weakly ω-Continuous Functions

Definition 7. A function $f : (X, \tau) \to (Y, \sigma)$ is called weakly ω-continuous if for each $x \in X$ and each open set V containing $f(x)$ there exists $U \in \omega O(X, x)$

such that $f(U) \subseteq \omega\,\mathrm{Cl}(V)$.

It is clear that every ω-continuous function is weakly ω-continuous but not converse.

Example 8. *Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{b\}, X\}$ and $\sigma = \{\emptyset, \{a\}, X\}$. Then the identity function $f : (X, \tau) \to (X, \sigma)$ is weakly ω-continuous but not ω-continuous.*

Theorem 9. *Let (X, τ) be an ω-regular space. Then $f : (X, \tau) \to (Y, \sigma)$ is an ω-continuous if and only if it is weakly ω-continuous.*

Proof. The proof follows from Lemma 6. □

Theorem 10. *For a function $f : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

1. *f is weakly ω-continuous;*
2. *$f^{-1}(V) \subset \omega\,\mathrm{Int}(f^{-1}(\omega\,\mathrm{Cl}(V)))$ for every open set V of Y;*
3. *$\omega\,\mathrm{Cl}(f^{-1}(\omega\,\mathrm{Int}(F))) \subset f^{-1}(F)$ for every closed set F of Y;*
4. *$\omega\,\mathrm{Cl}(f^{-1}(\omega\,\mathrm{Int}(\mathrm{Cl}(B)))) \subset f^{-1}(\mathrm{Cl}(B))$ for every subset B of Y;*
5. *$f^{-1}(\mathrm{Int}(B)) \subset \omega\,\mathrm{Int}(f^{-1}(\omega\,\mathrm{Cl}(\mathrm{Int}(B))))$ for every subset B of Y;*
6. *$\omega\,\mathrm{Cl}(f^{-1}(V)) \subset f^{-1}(\omega\,\mathrm{Cl}(V))$ for every open set V of Y.*

Proof. (1)$\Rightarrow$(2): Let V be an open subset of Y and $x \in f^{-1}(V)$. Then $f(x) \in V$. There exists $U \in \omega O(X, x)$ such that $f(U) \subset \omega\,\mathrm{Cl}(V)$. Thus, $x \in U \subset f^{-1}(\omega\,\mathrm{Cl}(V))$. Hence $x \in \omega\,\mathrm{Int}(f^{-1}(\omega\,\mathrm{Cl}(V)))$. Then $f^{-1}(V) \subset \omega\,\mathrm{Int}(f^{-1}(\omega\,\mathrm{Cl}(V)))$. (2)$\Rightarrow$(3): Let F be any closed set of Y. Then $Y \backslash F$ is open in Y. By (2), $\omega\,\mathrm{Cl}(f^{-1}(\omega\,\mathrm{Int}(F))) \subset f^{-1}(F)$. (3)$\Rightarrow$(4): Let B be any subset of Y. Then $\mathrm{Cl}(B)$ is closed in Y and by (3), we obtain $\omega\,\mathrm{Cl}(f^{-1}(\omega\,\mathrm{Int}(\mathrm{Cl}(B)))) \subset f^{-1}(\mathrm{Cl}(B))$. (4)$\Rightarrow$(5): Let B be any subset of Y. Then we have $f^{-1}(\mathrm{Int}(B)) = X \setminus f^{-1}(\mathrm{Cl}(Y \backslash B)) \subset \omega\,\mathrm{Int}(f^{-1}(\omega\,\mathrm{Cl}(\mathrm{Int}(B))))$. (5)$\Rightarrow$(6): Let V be any open subset of Y. Suppose that $x \notin f^{-1}(\omega\,\mathrm{Cl}(V))$. Then $f(x) \notin \omega\,\mathrm{Cl}(V)$ and there exists $U \in \omega O(Y, f(x))$ such that $U \cap V = \emptyset$; hence $\omega\,\mathrm{Cl}(U) \cap V = \emptyset$. By (5), $x \in f^{-1}(U) \subset \omega\,\mathrm{Int}(f^{-1}(\omega\,\mathrm{Cl}(U)))$ and hence there exists $W \in \omega O(X, x)$ such that $W \subset f^{-1}(\omega\,\mathrm{Cl}(U))$. Since $\omega\,\mathrm{Cl}(U) \cap V = \emptyset$, $W \cap f^{-1}(V) = \emptyset$ and by Lemma 3 $x \notin \omega\,\mathrm{Cl}(f^{-1}(V))$. Therefore, $\omega\,\mathrm{Cl}(f^{-1}(V)) \subset f^{-1}(\omega\,\mathrm{Cl}(V))$. (6)$\Rightarrow$(1): Let $x \in X$ and V any open subset of Y containing $f(x)$. By (6), $x \in f^{-1}(V) \subset f^{-1}(\mathrm{Int}(\omega\,\mathrm{Cl}(V))) \subset f^{-1}(\omega\,\mathrm{Int}(\omega\,\mathrm{Cl}(V))) \subset X$

$\setminus \omega \operatorname{Cl}(f^{-1}(Y \setminus \omega \operatorname{Cl}(V))) = \omega \operatorname{Int}(f^{-1}(\omega \operatorname{Cl}(V)))$. Therefore, there exists $U \in \omega O(X, x)$ such that $U \subset \omega \operatorname{Cl}(V)$. This shows that f is weakly ω-continuous. $\square$

Definition 11. A function $f : (X, \tau) \to (Y, \sigma)$ is said to have a strongly ω-closed graph if for $(x, y) \in (X \times Y) \setminus G(f)$, there exists $U \in \omega O(X, x)$ and an open set V of Y containing y such that $(U \times V) \cap G(f) = \emptyset$.

The following Lemma is an immediate consequence of Definition 11.

Lemma 12. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function. Then its graph $G(f)$ is strongly ω-closed in $X \times Y$ if and only if for each point $(x, y) \in (X \times Y) \setminus G(f)$, there exist an ω-open set U of X and an open set V of Y, containing x and y, respectively, such that $f(U) \cap V = \emptyset$.*

Theorem 13. *If $f : (X, \tau) \to (Y, \sigma)$ is a weakly ω-continuous function and (Y, σ) is a Hausdorff space, then the graph $G(f)$ is an ω-closed set of $X \times Y$.*

Proof. Let $(x, y) \in (X \times Y) \setminus G(f)$. Then, we have $y \neq f(x)$. Since (Y, σ) is Hausdorff, there exist disjoint open sets W and V such that $f(x) \in W$ and $y \in V$. Since f is weakly ω-continuous, there exists an ω-open set U containing x such that $f(U) \subseteq \omega \operatorname{Cl}(W)$. Since W and V are disjoint subsets of Y, we have $V \cap \omega \operatorname{Cl}(W) = \emptyset$. This shows that $(U \times V) \cap G(f) = \emptyset$ and hence by Lemma 12 $G(f)$ is ω-closed. $\square$

Definition 14. By a weakly ω-continuous retraction, we mean a weakly ω-continuous function $f : X \to A$, where $A \subset X$ and $f|A$ is the identity function on A.

Theorem 15. *Let A be a subset of X and $f : (X, \tau) \to (Y, \sigma)$ be a weakly ω-continuous restraction of X onto A. If (X, τ) is a Hausdorff space, then A is an ω-closed set in X.*

Proof. Supoose that A is not ω-closed in X. Then there exists a point $x \in \omega \operatorname{Cl}(A) \setminus A$. Since f is weakly ω-continuous restraction, we have $f(x) \neq x$. Since X is Hausdorff, there exist disjoint open sets U and V of X such that $x \in U$ and $f(x) \in V$. Thus, we get $U \cap \omega \operatorname{Cl}(V) = \emptyset$. Now, let $W \in \omega O(X, x)$. Then $U \cap W \in \omega O(X, x)$ and hence $(U \cap W) \cap A \neq \emptyset$, because $x \in \omega \operatorname{Cl}(A)$. Let $y \in (U \cap W) \cap A$. Since $y \in A$, $f(y) \in U$ and hence $f(y) \notin \omega \operatorname{Cl}(V)$. This gives that $f(W)$ is not a subset of $\omega \operatorname{Cl}(V)$. This contradicts that f is weakly ω-continuous. Hence A is ω-closed in X. $\square$

Definition 16. A topological space (X, τ) is called ω-connected [6] if X cannot be written as the disjoint union of two nonempty ω-open sets.

Theorem 17. *Let $f : (X, \tau) \to (Y, \sigma)$ be a weakly ω-continuous surjective function. If X is ω-connected, then Y is connected.*

Proof. Suppose that (Y, σ) is not connected. Then there exist nonempty disjoint open sets V_1 and V_2 in Y such that $V_1 \cup V_2 = Y$. Since f is surjective, $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are nonempty disjoint subsets of X such that $f^{-1}(V_1) \cup f^{-1}(V_2) = X$. By Theorem 10, we have $f^{-1}(V_i) \subseteq \omega\,\mathrm{Int}(f^{-1}(\omega\,\mathrm{Cl}(V_i)))$, $i =$ 1,2. Since V_i is open and closed and every closed set is ω-closed, we obtain $f^{-1}(V_i) \subseteq \omega\,\mathrm{Int}(f^{-1}(V_i))$ and hence $f^{-1}(V_i)$ is ω-open for $i =$ 1,2. This implies that (X, τ) is not ω-connected. □

Definition 18. A topological space (X, τ) is said to be ultran ω-Urysohn if for each pair of distinct points x and y in X, there exist open sets U, V containing x, y respectively such that $\omega\,\mathrm{Cl}(U) \cap \omega\,\mathrm{Cl}(V) = \emptyset$.

Definition 19. A topologicalspace (X, τ) is said to be ω-T_2 [3] if for each pair of distinct points x and y in X, there exist $U \in \omega O(X, x)$ and $V \in \omega O(X, y)$ such that $U \cap V = \emptyset$.

Theorem 20. *Let $f : (X, \tau) \to (Y, \sigma)$ be a weakly ω-continuous injective function. If Y is ultran ω-Urysohn, then X is ω-T_2.*

Proof. Let x_1 and x_2 be any two distinct points of X. Since f is injective, $f(x_1) \neq f(x_2)$. Since (Y, σ) is ultran ω-Urysohn, there exist $V_1, V_2 \in \sigma$ such that $f(x_1) \in V_1$, $f(x_2) \in V_2$ and $\omega\,\mathrm{Cl}\,(V_1) \cap \omega\,\mathrm{Cl}(V_2) = \emptyset$. This gives $f^{-1}(\omega\,\mathrm{Cl}(V_1))$ $\cap\, f^{-1}(\omega\,\mathrm{Cl}(V_2)) = \emptyset$ and hence $\omega\,\mathrm{Int}(f^{-1}(\omega\,\mathrm{Cl}(V_1))) \cap \omega\,\mathrm{Int}(f^{-1}(\omega\,\mathrm{Cl}(V_2))) = \emptyset$. Since f is weakly ω-continuous, $x_i \in f^{-1}(V_i) \subset \omega\,\mathrm{Int}(f^{-1}(\omega\,\mathrm{Cl}(V_i)))$, $i = 1, 2$. By Theorem 10 and this indicates that the space (X, τ) is ω-T_2. □

4. Additional Properties

Definition 21. For a function $f : (X, \tau) \to (Y, \sigma)$, the graph $G(f)$ is said to be ultran ω-closed if for each $(x, y) \in (X \times Y) \backslash G(f)$, there exist $U \in \omega O(X, x)$, $V \in \omega O(Y, y)$ such that $(U \times \omega\,\mathrm{Cl}(V)) \cap G(f) = \emptyset$.

Lemma 22. *The function $f : (X, \tau) \to (Y, \sigma)$ has a ultran ω-closed graph if and only if for every $(x, y) \in (X \times Y) \backslash G(f)$ there exist $U \in \omega O(X, x)$, $V \in \omega O(Y, y)$ and $f(U) \cap \omega\,\mathrm{Cl}(V) = \emptyset$.*

Proof. It is an immediate consequence of Definition 21. □

Theorem 23. *Let $f : (X, \tau) \to (Y, \sigma)$ be a weakly ω-continuous function. If (Y, σ) is ultran ω-Urysohn, then the graph $G(f)$ is ultran ω-closed.*

Proof. Let $(x, y) \in (X \times Y) \backslash G(f)$. Then $y \neq f(x)$. Since Y is ultran ω-Urysohn, there exist open sets V and W containing x and y, repectively such that $\omega\,\mathrm{Cl}(V) \cap \omega\,\mathrm{Cl}(W) = \emptyset$. Since f is weakly ω-continuous, there exist $U \in \omega O(X, x)$ such that $f(U) \subset \omega\,\mathrm{Cl}(U)$. This implies that $f(U) \cap \omega\,\mathrm{Cl}(W) = \emptyset$. So, by Lemma 22 $G(f)$ is ultran ω-closed. □

Theorem 24. *If $f : (X, \tau) \to (Y, \sigma)$ is an injective weakly ω-continuous function with a ultran ω-closed graph, then the space (X, τ) is ω-T_2.*

Proof. Let x and y be any distinct points of X. Then, since f is injective, we have $f(x) \neq f(y)$. Then we have $(x, f(y)) \in (X \times Y) \setminus G(f)$. Since $G(f)$ is ultran ω-closed, by Lemma 22 there exist $U \in \omega O(X, x)$ and an open set V of Y containing $f(y)$ such that $f(U) \cap \omega\,\mathrm{Cl}(V) = \emptyset$. Since f is weakly ω-continuous, there exists $W \in \omega O(X, y)$ such that $f(W) \subset \omega\,\mathrm{Cl}(V)$. Therefore, we have $f(U) \cap G(f) = \emptyset$. Since f is injective, we obtain $U \cap W = \emptyset$. This shows that (X, τ) is an ω-T_2 space. □

Theorem 25. *If $f : (X, \tau) \to (Y, \sigma)$ is an ω-continuous function and (Y, σ) is a T_2 space, then the graph $G(f)$ is ultra-ω-closed.*

Proof. Let $(x, y) \in (X \times Y) \backslash G(f)$. The T_2 ness of Y gives the existence of an open set V containing y such that $f(x) \notin \mathrm{Cl}(V)$. Now $\mathrm{Cl}(V)$ is a closed set in Y. So, $Y \backslash \mathrm{Cl}(V)$ is an open set in Y containing $f(x)$. Therefore, by the ω-continuity of f there exist $U \in \omega O(X, x)$ such that $f(U) \subseteq Y \backslash \mathrm{Cl}(V)$, hence $f(U) \cap \mathrm{Cl}(V) = \emptyset$. Since $\omega\,\mathrm{Cl}(A) \subseteq \mathrm{Cl}(A)$ for every subset A of X, once obtain $f(U) \cap \omega\,\mathrm{Cl}(V) = \emptyset$. By Lemma 22, $G(f)$ is ultran ω-closed. □

Theorem 26. *If $f : (X, \tau) \to (Y, \sigma)$ is an ω-irresolute function and (Y, σ) is an ω-T_2 space, then the graph $G(f)$ is ultran ω-closed.*

Proof. Similar proof of Theorem 25. □

Definition 27. A topological space (X, τ) is said to be

(i) ω-compact [6] if every cover of X by ω-open sets has a finite subcover;

(ii) ω-closed if every cover of X by ω-open sets has a finite subcover whose ω-closure cover X.

Definition 28. A subset A of a topological space $(X, \tau$ is said to be ω-closed relative to X if for every cover $\{V_\alpha\colon \alpha \in \Lambda\}$ of A by ω-open sets of X, there exists a finite subset Λ_0 of Λ such that $A \subset \cup \{\omega \operatorname{Cl}(V_\alpha) \mid \alpha \in \Lambda_0\}$.

Theorem 29. *If $f : (X, \tau) \to (Y, \sigma)$ is a weakly ω-continuous function and A is an ω-compact subset of (X, τ), then $f(A)$ is ω-closed relative to (Y, σ).*

Proof. Let $\{V_i | i \in \Lambda\}$ be any cover of $f(K)$ by open sets of (Y, σ). For each $x \in X$, there exists $\alpha(x) \in \Lambda$ such that $f(x) \in V_{\alpha(x)}$. Since f is weakly ω-continuous, there exists $U(x) \in \omega O(X, x)$ such that $f(U(x)) \subset \omega \operatorname{Cl}(V_{\alpha(x)})$. The family $\{U(x)|x \in A\}$ is a cover of A by ω-open sets of X. Since A is ω-compact, there exists a finite number of points, say, x_1, x_2, ... x_n in A such that $A \subset \cup\{U(x_k) \mid x_k \in A,\ 1 \leq K \leq n\}$. Therefore, we obtain $f(A) \subset \cup \{f(U(x_k)) \mid x_k \in A,\ 1 \leq K \leq n\} \subset \cup \{\omega \operatorname{Cl}(V_{\alpha(x_k)}) \mid x_k \in A,\ 1 \leq K \leq n\}$. This shows that $f(A)$ is ω-closed relative to (Y, σ). □

Corollary 30. *If $f : (X, \tau) \to (Y, \sigma)$ is a weakly ω-continuous surjective function and the space (X, τ) is ω-compact, then (Y, σ) is an ω-closed space.*

Definition 31. Let A be a subset of a topological space (X, τ). Then the ω-frontier of A, denoted by $\omega Fr(A)$ is defined as $\omega Fr(A) = \omega \operatorname{Cl}(A) \cap \omega \operatorname{Cl}(X \backslash A)$.

Theorem 32. *The set of all points $x \in X$ at which a function $f : (X, \tau) \to (Y, \sigma)$ is not weakly ω-continuous if and only if the union of ω-frontier of the inverse images of the closure of open sets containing $f(x)$.*

Proof. Suppose that f is not weakly ω-continuous at $x \in X$. Then there exists an open set V of Y containing $f(x)$ such that $f(U)$ is not a subset of $\omega \operatorname{Cl}(V)$ for every $U \in \omega O(X, x)$. Then $U \cap (X \backslash f^{-1}(\omega \operatorname{Cl}(V))) \neq \emptyset$ for every $U \in \omega O(X, x)$ and hence by Lemma 3 $x \in \omega \operatorname{Cl}(X \backslash f^{-1}(\omega \operatorname{Cl}(V)))$. On the other hand, we have $x \in f^{-1}(V) \subset \omega \operatorname{Cl}(f^{-1}(\omega \operatorname{Cl}(V)))$ and hence $x \in \omega Fr(f^{-1}(\omega \operatorname{Cl}(V)))$. Conversely, Suppose that f is weakly ω-continuous at $x \in X$ and let V be any open set of Y containing $f(x)$. Then by Theorem 10, we have $x \in f^{-1}(V) \subset \omega \operatorname{Int}(f^{-1}(\omega \operatorname{Cl}(V)))$. Therefore, $x \in \omega Fr(f^{-1}(\operatorname{Cl}(V)))$ for each open set V of Y containing $f(x)$. □

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